

Abstract Algebraic Logic: Theory and Applications – Lesson 1

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Logic studies the notion of **logical consequence**. There are many kinds of logical consequence, i.e. many different logics:

- ① Classical logic
- ② Non-classical logics:
 - Modal logics
 - Intuitionistic logic
 - Superintuitionistic logics
 - Linear logics
 - Fuzzy logics
 - Relevance logics
 - Substructural logics
 - Paraconsistent logics
 - Dynamic logics
 - Non-monotonic logics
 - \vdots

Algebraic Logic

Algebraic Logic is the subdiscipline of Mathematical Logic which studies logical systems (classical and non-classical) by using tools from Universal Algebra.

Logic	Algebraic counterpart
Classical logic	Boolean algebras
Modal logics	Modal algebras
Intuitionistic logic	Heyting algebras
Linear logics	Commutative residuated lattices
Fuzzy logics	Semilinear residuated lattices
Relevance logics	Commutative contractive residuated lattices
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Universal Algebra is the field of Mathematics which studies algebraic structures.

Abstract Algebraic Logic

AAL is the evolution of Algebraic Logic that wants to:

- understand the several ways by which a logic can be given an algebraic semantics
- build a **general** and **abstract** theory of non-classical logics based on their relation to algebras
- understand the rôle of connectives in (non-)classical logics.
- classify non-classical logics
- find general results connecting logical and algebraic properties (**bridge theorems**)
- generalize properties from syntax to semantics (**transfer theorems**)
- advance the study of particular (families of) non-classical logics by using the abstract notions and results

It works best, by far, when restricted to **propositional logics**.

A little history of (Abstract) Algebraic Logic – 1

- 1847 George Boole, *Mathematical Analysis of Logic*.
Augustus De Morgan, *Formal Logic*.
- 1854 George Boole, *The Laws of Thought*.
- 1880 Charles Sanders Peirce, *On the Algebra of Logic*.
- 1890 Ernst Schröder, *Algebra der Logik* (in three volumes).
- 1920 Jan Łukasiewicz, *O logice trojwartosciowej*.
Three-valued logic.
- 1930 Jan Łukasiewicz, Alfred Tarski, *Untersuchungen über den Aussagenkalkül*. **Infinitely-valued logic.**
- 1930 Alfred Tarski, *Über einige fundamentale Begriffe der Metamathematik*. **Consequence operators.**
- 1931 Alfred Tarski, *Grundzüge der Systemenkalküls*.
Precise connection between classical logic and Boolean algebras. Lindenbaum–Tarski method.

A little history of (Abstract) Algebraic Logic – 2

- 1935 Garrett Birkhoff, *On the Structure of Abstract Algebras*. **Universal Algebra, equational classes, equational logic.**
- 1958 Jerzy Łoś and Roman Suszko, *Remarks on sentential logics*. **Structural consequence operators.**
- 1973 Ryszard Wójcicki, *Matrix approach in the methodology of sentential calculi*.
- 1974 Helena Rasiowa, *An algebraic approach to non-classical logics*.
- 1975 S.L. Bloom, *Some theorems on structural consequence operations*.
- 1981 Janusz Czelakowski, *Equivalential logics I and II*.
- 1986 Willem J. Blok, Don L. Pigozzi, *Protoalgebraic logics*.
- 1989 Willem J. Blok, Don L. Pigozzi, *Algebraizable logics*.
- 1996 Josep Maria Font, Ramon Jansana, *A general algebraic semantics for sentential logics*.
- 2000 Janusz Czelakowski, *Protoalgebraic logics*.

Outline of the course

- L1:** Basic notions of algebraic logic. Weakly implicative logics. Completeness theorem w.r.t. the class of all models and w.r.t. reduced models. [C.N. and P.C.]
- L2:** Advanced semantic notions. Completeness w.r.t. RFSI models. Algebraically implicative logics. [C.N.]
- L3:** Core theory of AAL. Leibniz hierarchy. Protoalgebraic logics. Algebraizable logics. [P.C.]
- L4:** Bridge theorems. Abstract study of disjunction. [C.N.]
- L5:** Applications of AAL to Mathematical Fuzzy Logic: the theory of semilinear logics. [P.C.]

Slides available at

www.carlesnoguera.cat/?q=en/teaching

- W.J. Blok and D. Pigozzi. *Algebraizable logics*. Memoirs of the American Mathematical Society 396, vol. 77, 1989.
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- P. Cintula, C. Noguera. A General Framework for Mathematical Fuzzy Logic. In *Handbook of Mathematical Fuzzy Logic – Volume 1*. Studies in Logic, Mathematical Logic and Foundations, vol. 37, London, College Publications, pp. 103-207, 2011.
- P. Cintula, C. Noguera. The proof by cases property and its variants in structural consequence relations. To appear in *Studia Logica*.

Our works are downloadable from www.carlesnoguera.cat

- J. Czelakowski. *Protoalgebraic Logics*. Trends in Logic, vol 10, Dordrecht, Kluwer, 2001.
- J.M. Font and R. Jansana. *A General Algebraic Semantics for Sentential Logics*. Springer-Verlag, 1996.
- J.M. Font, R. Jansana, and D. Pigozzi. A survey of Abstract Algebraic Logic. *Studia Logica*, 74(1–2), Special Issue on Abstract Algebraic Logic II):13–97, 2003.
- N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, Studies in Logic and the Foundations of Mathematics, vol. 151, Amsterdam, Elsevier, 2007.
- J. Raftery. Order algebraizable logics. Submitted.
- H. Rasiowa. *An Algebraic Approach to Non-Classical Logics*. North-Holland, Amsterdam, 1974.

Basic syntactical notions – 1

Propositional language: a **countable** type \mathcal{L} , i.e. a function $ar: C_{\mathcal{L}} \rightarrow \mathbb{N}$, where $C_{\mathcal{L}}$ is a countable set of symbols called **connectives**, giving for each one its **arity**. Nullary connectives are also called **truth-constants**. We write $\langle c, n \rangle \in \mathcal{L}$ whenever $c \in C_{\mathcal{L}}$ and $ar(c) = n$.

Formulae: Let Var be a fixed **infinite countable** set of symbols called **variables**. The set $Fm_{\mathcal{L}}$ of formulae in \mathcal{L} is the least set containing Var and closed under connectives of \mathcal{L} , i.e. for each $\langle c, n \rangle \in \mathcal{L}$ and every $\varphi_1, \dots, \varphi_n \in Fm_{\mathcal{L}}$, $c(\varphi_1, \dots, \varphi_n)$ is a formula.

Substitution: a mapping $\sigma: Fm_{\mathcal{L}} \rightarrow Fm_{\mathcal{L}}$, such that $\sigma(c(\varphi_1, \dots, \varphi_n)) = c(\sigma(\varphi_1), \dots, \sigma(\varphi_n))$ holds for each $\langle c, n \rangle \in \mathcal{L}$ and every $\varphi_1, \dots, \varphi_n \in Fm_{\mathcal{L}}$.

Consecution: a pair $\Gamma \triangleright \varphi$, where $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$.

Basic syntactical notions – 2

A set L of consecutions can be seen as relation between sets of formulae and formulae. We write ' $\Gamma \vdash_L \varphi$ ' instead of ' $\Gamma \triangleright \varphi \in L$ '.

Definition 1.1

A set L of consecutions in \mathcal{L} is called a **logic** in \mathcal{L} whenever

- If $\varphi \in \Gamma$, then $\Gamma \vdash_L \varphi$. (Reflexivity)
- If $\Delta \vdash_L \psi$ for each $\psi \in \Gamma$ and $\Gamma \vdash_L \varphi$, then $\Delta \vdash_L \varphi$. (Cut)
- If $\Gamma \vdash_L \varphi$, then $\sigma[\Gamma] \vdash_L \sigma(\varphi)$ for each substitution σ . (Structurality)

Observe that reflexivity and cut entail:

- If $\Gamma \vdash_L \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_L \varphi$. (Monotonicity)

The least logic Dumb is described as:

$$\Gamma \vdash_{\text{Dumb}} \varphi \quad \text{iff} \quad \varphi \in \Gamma.$$

Basic syntactical notions – 3

Theorem: a consequence of the empty set
(note that Dumb has no theorems).

Inconsistent logic Inc: the set all consecutions
(equivalently: a logic where all formulae are theorems).

Almost Inconsistent logic AInc: the maximum logic without theorems
(note that $\Gamma, \varphi \vdash_{\text{AInc}} \psi$).

Theory: a set of formulae T such that if $T \vdash_{\text{L}} \varphi$ then $\varphi \in T$. By $\text{Th}(\text{L})$ we denote the set of all theories of L.

Note that

- $\text{Th}(\text{L})$ can be seen as a closure system. By $\text{Th}_{\text{L}}(\Gamma)$ we denote the theory generated in $\text{Th}(\text{L})$ by Γ (i.e., the intersection of all theories containing Γ).
- $\text{Th}_{\text{L}}(\Gamma) = \{\varphi \in \text{Fm}_{\mathcal{L}} \mid \Gamma \vdash_{\text{L}} \varphi\}$.
- The set of all theorems is the least theory and it is generated by the empty set.

Basic syntactical notions – 4

Axiomatic system: a set \mathcal{AS} of consecutions closed under substitutions. An element $\Gamma \triangleright \varphi$ is an

- **axiom** if $\Gamma = \emptyset$,
- **finitary deduction rule** if Γ is a finite,
- **infinitary deduction rule** otherwise.

An axiomatic system is **finitary** if all its rules are finitary.

Proof: a proof of a formula φ from a set of formulae Γ in \mathcal{AS} is a well-founded tree labeled by formulae such that

- its root is labeled by φ and leaves by axioms of \mathcal{AS} or elements of Γ and
- if a node is labeled by ψ and $\Delta \neq \emptyset$ is the set of labels of its preceding nodes, then $\Delta \triangleright \psi \in \mathcal{AS}$.

We write $\Gamma \vdash_{\mathcal{AS}} \varphi$ if there is a proof of φ from Γ in \mathcal{AS} .

Lemma 1.2

Let \mathcal{AS} be an axiomatic system. Then $\vdash_{\mathcal{AS}}$ is the least logic containing \mathcal{AS} .

Presentation: We say that \mathcal{AS} is an axiomatic system for (or a presentation of) the logic L if $L = \vdash_{\mathcal{AS}}$. A logic is said to be **finitary** if it has some finitary presentation.

Lemma 1.3

A logic L is finitary iff for each set of formulae $\Gamma \cup \{\varphi\}$ we have: if $\Gamma \vdash_L \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_L \varphi$.

Note that Inc, AInc, Dumb are finitary because:

Inc	is axiomatized by	axioms $\{\varphi \mid \varphi \in Fm_{\mathcal{L}}\}$
AInc	is axiomatized by	unary rules $\{\varphi \triangleright \psi \mid \varphi, \psi \in Fm_{\mathcal{L}}\}$
Dumb	is axiomatized by	by the empty set

Examples: classical logic CPC and logic BCI

Finitary axiomatic system for CPC in $\mathcal{L}_{\text{CPC}} = \{\rightarrow, \neg\}$

A1 $\varphi \rightarrow (\psi \rightarrow \varphi)$

A2 $(\chi \rightarrow (\varphi \rightarrow \psi)) \rightarrow ((\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi))$

A3 $(\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi)$

MP $\varphi, \varphi \rightarrow \psi \triangleright \psi$

Finitary axiomatic system for BCI in $\mathcal{L}_{\text{BCI}} = \{\rightarrow\}$

B $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$

C $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$

I $\varphi \rightarrow \varphi$

MP $\varphi, \varphi \rightarrow \psi \triangleright \psi$

Basic syntactical notions – 6

Let $\mathcal{L}_1 \subseteq \mathcal{L}_2$ be propositional languages, L_i a logic in \mathcal{L}_i , and \mathcal{S} a set of consecutions in \mathcal{L}_2 .

- L_2 is the **expansion of L_1 by \mathcal{S}** if it is the weakest logic in \mathcal{L}_2 containing L_1 and \mathcal{S} , i.e. the logic axiomatized by all \mathcal{L}_2 -substitutional instances of consecutions from $\mathcal{S} \cup \mathcal{AS}$, for any presentation \mathcal{AS} of L_1 .
- L_2 is an **expansion of L_1** if $L_1 \subseteq L_2$, i.e. it is the expansion of L_1 by \mathcal{S} , for some set of consecutions \mathcal{S} .
- L_2 is an **axiomatic expansion of L_1** if it is an expansion obtained by adding a set of *axioms*.
- L_2 is a **conservative expansion of L_1** if it is an expansion and for each consecution $\Gamma \triangleright \varphi$ in \mathcal{L}_1 we have that $\Gamma \vdash_{L_2} \varphi$ entails $\Gamma \vdash_{L_1} \varphi$.

If $\mathcal{L}_1 = \mathcal{L}_2$, we use ‘extension’ instead ‘expansion’.

Note that CPC is the axiomatic expansion of BCI by A1–A3.

Basic semantical notions – 1

\mathcal{L} -algebra: $A = \langle A, \langle c^A \mid c \in C_{\mathcal{L}} \rangle \rangle$, where $A \neq \emptyset$ (universe) and $c^A : A^n \rightarrow A$ for each $\langle c, n \rangle \in \mathcal{L}$.

Algebra of formulae: the algebra $Fm_{\mathcal{L}}$ with domain $Fm_{\mathcal{L}}$ and operations $c^{Fm_{\mathcal{L}}}$ for each $\langle c, n \rangle \in \mathcal{L}$ defined as:

$$c^{Fm_{\mathcal{L}}}(\varphi_1, \dots, \varphi_n) = c(\varphi_1, \dots, \varphi_n).$$

$Fm_{\mathcal{L}}$ is the **absolutely free algebra in language \mathcal{L} with generators Var** .

Homomorphism of algebras: a mapping $f : A \rightarrow B$ such that for every $\langle c, n \rangle \in \mathcal{L}$ and every $a_1, \dots, a_n \in A$,

$$f(c^A(a_1, \dots, a_n)) = c^B(f(a_1), \dots, f(a_n)).$$

Note that substitutions are exactly endomorphisms of $Fm_{\mathcal{L}}$.

\mathcal{L} -matrix: a pair $\mathbf{A} = \langle A, F \rangle$ where A is an \mathcal{L} -algebra called the **algebraic reduct of \mathbf{A}** , and F is a subset of A called the **filter** of \mathbf{A} . The elements of F are called **designated elements** of \mathbf{A} .

A matrix $\mathbf{A} = \langle A, F \rangle$ is

- **trivial** if $F = A$.
- **finite** if A is finite.
- **Lindenbaum** if $A = Fm_{\mathcal{L}}$.

A -evaluation: a homomorphism from $Fm_{\mathcal{L}}$ to A , i.e. a mapping $e: Fm_{\mathcal{L}} \rightarrow A$, such that for each $\langle c, n \rangle \in \mathcal{L}$ and each n -tuple of formulae $\varphi_1, \dots, \varphi_n$ we have:

$$e(c(\varphi_1, \dots, \varphi_n)) = c^{\mathbf{A}}(e(\varphi_1), \dots, e(\varphi_n)).$$

Semantical consequence: A formula φ is a semantical consequence of a set Γ of formulae w.r.t. a class \mathbb{K} of \mathcal{L} -matrices if for each $\langle \mathbf{A}, F \rangle \in \mathbb{K}$ and each \mathbf{A} -evaluation e , we have $e(\varphi) \in F$ whenever $e[\Gamma] \subseteq F$; we denote it by $\Gamma \models_{\mathbb{K}} \varphi$.

Lemma 1.4

Let \mathbb{K} a class of \mathcal{L} -matrices. Then $\models_{\mathbb{K}}$ is a logic in \mathcal{L} . Furthermore if \mathbb{K} is a finite class of finite matrices, then the logic $\models_{\mathbb{K}}$ is finitary.

L-matrix: Let L be a logic in \mathcal{L} and \mathbf{A} an \mathcal{L} -matrix. We say that \mathbf{A} is an L -matrix if $L \subseteq \models_{\mathbf{A}}$. We denote the class of L -matrices by $\mathbf{MOD}(L)$.

Lemma 1.5

Let L be a logic in \mathcal{L} and a mapping $g: A \rightarrow B$ be a homomorphism of \mathcal{L} -algebras A, B . Then:

- $\langle A, g^{-1}[G] \rangle \in \mathbf{MOD}(L)$, whenever $\langle B, G \rangle \in \mathbf{MOD}(L)$.
- $\langle B, g[F] \rangle \in \mathbf{MOD}(L)$, whenever $\langle A, F \rangle \in \mathbf{MOD}(L)$ and g is surjective and $g(x) \in g[F]$ implies $x \in F$.

Logical filter: Given a logic L in \mathcal{L} and an \mathcal{L} -algebra A , a subset $F \subseteq A$ is an L -filter if $\langle A, F \rangle \in \mathbf{MOD}(L)$. By $\mathcal{F}i_L(A)$ we denote the set of all L -filters over A .

$\mathcal{F}i_L(A)$ is a closure system and can be given a lattice structure by defining for any $F, G \in \mathcal{F}i_L(A)$, $F \wedge G = F \cap G$ and $F \vee G = \text{Fi}_L^A(F \cup G)$.

Generated filter: Given a set $X \subseteq A$, the logical filter generated by X is $\text{Fi}_L^A(X) = \bigcap \{F \in \mathcal{F}i_L(A) \mid X \subseteq F\}$.

$$\mathcal{F}i_{\text{Dumb}}(A) = \mathcal{P}(A) \quad \mathcal{F}i_{A\text{Inc}}(A) = \{\emptyset, A\} \quad \mathcal{F}i_{\text{Inc}}(A) = \{A\}$$

Examples: classical logic CPC and logic BCI

1. Classical logic: Let A be a Boolean algebra. Then $\mathcal{F}i_{\text{CPC}}(A)$ is the class of lattice filters on A , in particular for the two-valued Boolean algebra $\mathbf{2}$:

$$\mathcal{F}i_{\text{CPC}}(\mathbf{2}) = \{\{1\}, \{0, 1\}\}.$$

2. The logic BCI: By M we denote the \mathcal{L}_{BCI} -algebra with domain $\{\perp, \top, t, f\}$ and:

\rightarrow^M	\top	t	f	\perp
\top	\top	\perp	\perp	\perp
t	\top	t	f	\perp
f	\top	\perp	t	\perp
\perp	\top	\top	\top	\top

We can easily compute that:

$$\mathcal{F}i_{\text{BCI}}(M) = \{\{t, \top\}, \{t, f, \top\}, M\}.$$

The first completeness theorem

Proposition 1.6

For any logic L in a language \mathcal{L} , $\mathcal{F}i_L(\mathbf{Fm}_{\mathcal{L}}) = \text{Th}(L)$.

Theorem 1.7

Let L be a logic. Then for each set Γ of formulae and each formula φ the following holds: $\Gamma \vdash_L \varphi$ iff $\Gamma \models_{\text{MOD}(L)} \varphi$.

Completeness theorem for classical logic

- Suppose that $T \in \text{Th}(\text{CPC})$ and $\varphi \notin T$ ($T \not\vdash_{\text{CPC}} \varphi$). We want to show that $T \not\models \varphi$ in some meaningful semantics.
- $T \not\models_{\langle \mathbf{Fm}_{\mathcal{L}}, T \rangle} \varphi$. 1st completeness theorem
- $\langle \alpha, \beta \rangle \in \Omega(T)$ iff $\alpha \leftrightarrow \beta \in T$ (congruence relation on $\mathbf{Fm}_{\mathcal{L}}$ compatible with T : if $\alpha \in T$ and $\langle \alpha, \beta \rangle \in \Omega(T)$, then $\beta \in T$).
- Lindenbaum–Tarski algebra: $\mathbf{Fm}_{\mathcal{L}}/\Omega(T)$ is a Boolean algebra and $T \not\models_{\langle \mathbf{Fm}_{\mathcal{L}}/\Omega(T), T/\Omega(T) \rangle} \varphi$. 2nd completeness theorem
- Lindenbaum Lemma: If $\varphi \notin T$, then there is a maximal consistent $T' \in \text{Th}(\text{CPC})$ such that $T \subseteq T'$ and $\varphi \notin T'$.
- $\mathbf{Fm}_{\mathcal{L}}/\Omega(T') \cong \mathbf{2}$ (subdirectly irreducible Boolean algebra) and $T \not\models_{\langle \mathbf{2}, \{1\} \rangle} \varphi$. 3rd completeness theorem

Definition 1.8

A logic L in a language \mathcal{L} is **weakly implicative** if there is a binary connective \rightarrow (primitive or definable) such that:

$$(R) \quad \vdash_L \varphi \rightarrow \varphi$$

$$(MP) \quad \varphi, \varphi \rightarrow \psi \vdash_L \psi$$

$$(T) \quad \varphi \rightarrow \psi, \psi \rightarrow \chi \vdash_L \varphi \rightarrow \chi$$

$$(sCng) \quad \varphi \rightarrow \psi, \psi \rightarrow \varphi \vdash_L c(\chi_1, \dots, \chi_i, \varphi, \dots, \chi_n) \rightarrow \\ c(\chi_1, \dots, \chi_i, \psi, \dots, \chi_n)$$

for each $\langle c, n \rangle \in \mathcal{L}$ and each $0 \leq i < n$.

Examples

The following logics **are** weakly implicative:

- CPC, BCI, and Inc
- **global** modal logics
- intuitionistic and superintuitionistic logic
- linear logic and its variants
- (the most of) fuzzy logics
- substructural logics

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The following logics **are not** weakly implicative:

- **local** modal logics we will see why later today
- Dumb, AInc, and the conjunction-disjunction fragment of classical logic as they have no theorems
- logics of ortholattices lesson 3

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Congruence Property

Conventions

Unless said otherwise, L is a weakly implicative in a language \mathcal{L} with an implication \rightarrow . We write:

- $\varphi \leftrightarrow \psi$ instead of $\{\varphi \rightarrow \psi, \psi, \rightarrow \varphi\}$
- $\Gamma \vdash \Delta$ whenever $\Gamma \vdash \chi$ for each $\chi \in \Delta$
- $\Gamma \dashv\vdash \Delta$ whenever $\Gamma \vdash \Delta$ and $\Delta \vdash \Gamma$.

Theorem 1.9

Let φ, ψ, χ be formulae. Then:

- $\vdash_L \varphi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \psi \vdash_L \psi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \delta, \delta \leftrightarrow \psi \vdash_L \varphi \leftrightarrow \psi$
- $\varphi \leftrightarrow \psi \vdash_L \chi \leftrightarrow \hat{\chi}$, *where $\hat{\chi}$ is obtained from χ by replacing some occurrences of φ in χ by ψ .*

Corollary 1.10

Let \rightarrow' be a connective satisfying (R), (MP), (T), (sCng). Then

$$\varphi \leftrightarrow \psi \dashv\vdash_{\mathbf{L}} \varphi \leftrightarrow' \psi.$$

Proof.

Consider formulas $\chi = \varphi \rightarrow' \varphi$ and $\hat{\chi} = \varphi \rightarrow' \psi$ and the proof

$$\varphi \leftrightarrow \psi, \dots, (\varphi \rightarrow' \varphi) \rightarrow (\varphi \rightarrow' \psi), \varphi \rightarrow' \varphi, \varphi \rightarrow' \psi.$$

Analogously for $\hat{\chi} = \psi \rightarrow' \varphi$ we can write

$$\varphi \leftrightarrow \psi, \dots, (\varphi \rightarrow' \varphi) \rightarrow (\psi \rightarrow' \varphi), \varphi \rightarrow' \varphi, \psi \rightarrow' \varphi. \quad \square$$

So we have shown $\varphi \leftrightarrow \psi \vdash_{\mathbf{L}} \varphi \leftrightarrow' \psi$. The reverse direction is fully analogous.

Corollary 1.10

Let \rightarrow' be a connective satisfying (R), (MP), (T), (sCng). Then

$$\varphi \leftrightarrow \psi \dashv\vdash_{\mathbf{L}} \varphi \leftrightarrow' \psi.$$

Corollary 1.11

Local modal logic \mathbf{T}^l is not weakly implicative.

Proof.

Let \rightarrow' be a 'good' implication in \mathbf{T}^l . Then \rightarrow' (along with classical implication \rightarrow) is an implication in global \mathbf{T}^g . Thus $\top \rightarrow \varphi, \varphi \rightarrow \top \vdash_{\mathbf{T}^g} \top \leftrightarrow' \varphi$ and so $\Box^n(\top \rightarrow \varphi) \vdash_{\mathbf{T}^l} \top \leftrightarrow' \varphi$ for some n . Consider the proof in \mathbf{T}^l : $\Box^n \varphi, \dots, \Box^n(\top \rightarrow \varphi), \dots, \top \leftrightarrow' \varphi, \dots, \Box^{n+1} \top \leftrightarrow' \Box^{n+1} \varphi, \Box^{n+1} \top, \Box^{n+1} \varphi$, a contradiction.



Definition 1.12

Let $\mathbf{A} = \langle A, F \rangle$ be an L-matrix. We define:

- the **matrix preorder** $\leq_{\mathbf{A}}$ of \mathbf{A} as

$$a \leq_{\mathbf{A}} b \quad \text{iff} \quad a \rightarrow^{\mathbf{A}} b \in F$$

- the **Leibniz congruence** $\Omega_{\mathbf{A}}(F)$ of \mathbf{A} as

$$\langle a, b \rangle \in \Omega_{\mathbf{A}}(F) \quad \text{iff} \quad a \leq_{\mathbf{A}} b \text{ and } b \leq_{\mathbf{A}} a.$$

A congruence θ of A is **logical** in a matrix $\langle A, F \rangle$ if for each $a, b \in A$ if $a \in F$ and $\langle a, b \rangle \in \theta$, then $b \in F$.

Theorem 1.13

Let $\mathbf{A} = \langle A, F \rangle$ be an L-matrix. Then:

- 1 $\leq_{\mathbf{A}}$ is a preorder.
- 2 $\Omega_{\mathbf{A}}(F)$ is the largest logical congruence of \mathbf{A} .
- 3 $\langle a, b \rangle \in \Omega_{\mathbf{A}}(F)$ iff for each $\chi \in Fm_{\mathcal{L}}$ and each \mathbf{A} -evaluation e :

$$e[p \rightarrow a](\chi) \in F \quad \text{iff} \quad e[p \rightarrow b](\chi) \in F.$$

Proof.

1. Take \mathbf{A} -evaluation e such that $e(p) = a$, $e(q) = b$, and $e(r) = c$. Recall that in \mathbf{L} we have: $\vdash_{\mathbf{L}} p \rightarrow p$ and $p \rightarrow q, q \rightarrow r \vdash_{\mathbf{L}} p \rightarrow r$. As $\mathbf{A} = \mathbf{MOD}(\mathbf{L})$ we have: $e(p \rightarrow p) \in F$, i.e., $a \leq_{\mathbf{A}} a$ and if $e(p \rightarrow q), e(q \rightarrow r) \in F$, then $e(p \rightarrow r) \in F$ i.e., if $a \leq_{\mathbf{A}} b$ and $b \leq_{\mathbf{A}} c$, then $a \leq_{\mathbf{A}} c$.

Theorem 1.13

Let $\mathbf{A} = \langle A, F \rangle$ be an L-matrix. Then:

- 1 $\leq_{\mathbf{A}}$ is a preorder.
- 2 $\Omega_{\mathbf{A}}(F)$ is the largest logical congruence of \mathbf{A} .
- 3 $\langle a, b \rangle \in \Omega_{\mathbf{A}}(F)$ iff for each $\chi \in Fm_{\mathcal{L}}$ and each \mathbf{A} -evaluation e :

$$e[p \rightarrow a](\chi) \in F \quad \text{iff} \quad e[p \rightarrow b](\chi) \in F.$$

Proof.

2. $\Omega_{\mathbf{A}}(F)$ is obviously a equivalence. It is a congruence due to (sCng) and logical due to (MP).

Take a logical congruence θ st. $\langle a, b \rangle \in \theta$. Since $\langle a, a \rangle \in \theta$, we have $\langle a \rightarrow^{\mathbf{A}} a, a \rightarrow^{\mathbf{A}} b \rangle \in \theta$. As $a \rightarrow^{\mathbf{A}} a \in F$ and θ is logical we get $a \rightarrow^{\mathbf{A}} b \in F$, i.e., $a \leq_{\mathbf{A}} b$. The proof of $b \leq_{\mathbf{A}} a$ is analogous.

Theorem 1.13

Let $\mathbf{A} = \langle A, F \rangle$ be an L-matrix. Then:

- 1 $\leq_{\mathbf{A}}$ is a preorder.
- 2 $\Omega_{\mathbf{A}}(F)$ is the largest logical congruence of \mathbf{A} .
- 3 $\langle a, b \rangle \in \Omega_{\mathbf{A}}(F)$ iff for each $\chi \in Fm_{\mathcal{L}}$ and each \mathbf{A} -evaluation e :

$$e[p \rightarrow a](\chi) \in F \quad \text{iff} \quad e[p \rightarrow b](\chi) \in F.$$

Proof.

3. One direction is a corollary of Theorem 1.9 and (MP).

The converse one: set $\chi = p \rightarrow q$ and $e(q) = b$: then $a \rightarrow^{\mathbf{A}} b \in F$ iff $b \rightarrow^{\mathbf{A}} b \in F$, thus $a \leq_{\mathbf{A}} b$. The proof of $b \leq_{\mathbf{A}} a$ is analogous (using $e(q) = a$). \square

Definition 1.14

A L -matrix $\mathbf{A} = \langle A, F \rangle$ is **reduced**, $\mathbf{A} \in \mathbf{MOD}^*(L)$ in symbols, if $\Omega_A(F)$ is the identity relation Id_A .

An algebra A is **L -algebra**, $A \in \mathbf{ALG}^*(L)$ in symbols, if there a set $F \subseteq A$ such that $\langle A, F \rangle \in \mathbf{MOD}^*(L)$.

Note that $\Omega_A(A) = A^2$. Thus from $\mathcal{F}i_{\text{Inc}}(A) = \{A\}$ we obtain:

$$A \in \mathbf{ALG}^*(\text{Inc}) \quad \text{iff} \quad A \text{ is a singleton}$$

Examples: classical logic CPC and logic BCI

1. Classical logic: it is easy to see that

$$\Omega_2(\{1\}) = \text{Id}_2 \quad \text{i.e., } \mathbf{2} \in \mathbf{ALG}^*(\text{CPC}).$$

Actually for any Boolean algebra A :

$$\Omega_A(\{1\}) = \text{Id}_A \quad \text{i.e., } A \in \mathbf{ALG}^*(\text{CPC}).$$

But: $\Omega_4(\{a, 1\}) = \text{Id}_A \cup \{\langle 1, a \rangle, \langle 0, \neg a \rangle\}$ i.e. $\langle \mathbf{4}, \{a, 1\} \rangle \notin \mathbf{MOD}^*(\text{CPC})$.

2. The logic BCI: recall the algebra M defined via:

\rightarrow^M	\top	t	f	\perp
\top	\top	\perp	\perp	\perp
t	\top	t	f	\perp
f	\top	\perp	t	\perp
\perp	\top	\top	\top	\top

We can easily show that:

$$\Omega_M(\{t, \top\}) = \Omega_M(\{t, f, \top\}) = \text{Id}_M \quad \text{i.e., } M \in \mathbf{ALG}^*(\text{BCI}).$$

Factorizing matrices – 1

Let us take $\mathbf{A} = \langle \mathbf{A}, F \rangle \in \mathbf{MOD}(\mathbf{L})$. We write:

- \mathbf{A}^* for $\mathbf{A}/\Omega_{\mathbf{A}}(F)$
- $[\cdot]_F$ for the canonical epimorphism of \mathbf{A} onto \mathbf{A}^* defined as:

$$[a]_F = \{b \in A \mid \langle a, b \rangle \in \Omega_{\mathbf{A}}(F)\}$$

- \mathbf{A}^* for $\langle \mathbf{A}^*, [F]_F \rangle$.

Lemma 1.15

Let $\mathbf{A} = \langle \mathbf{A}, F \rangle \in \mathbf{MOD}(\mathbf{L})$ and $a, b \in A$. Then:

- 1 $a \in F$ iff $[a]_F \in [F]_F$.
- 2 $\mathbf{A}^* \in \mathbf{MOD}(\mathbf{L})$.
- 3 $[a]_F \leq_{\mathbf{A}^*} [b]_F$ iff $a \rightarrow^{\mathbf{A}} b \in F$.
- 4 $\mathbf{A}^* \in \mathbf{MOD}^*(\mathbf{L})$.

Proof.

- 1 One direction is trivial. Conversely: $[a]_F \in [F]_F$ implies that $[a]_F = [b]_F$ for some $b \in F$; thus $\langle a, b \rangle \in \Omega_A(F)$ and, since $\Omega_A(F)$ is a logical congruence, we obtain $a \in F$.
- 2 Recall that the second claim of Lemma 1.5 says that for an epimorphism $g: \mathbf{A} \rightarrow \mathbf{B}$ and $F \in \mathcal{F}i_L(\mathbf{A})$ we get $g[F] \in \mathcal{F}i_L(\mathbf{B})$, whenever $g(x) \in g[F]$ implies $x \in F$.
- 3 $[a]_F \leq_{\mathbf{A}^*} [b]_F$ iff $[a]_F \rightarrow^{\mathbf{A}^*} [b]_F \in [F]_F$ iff $[a \rightarrow^{\mathbf{A}} b]_F \in [F]_F$ iff $a \rightarrow^{\mathbf{A}} b \in F$.
- 4 Assume that $\langle [a]_F, [b]_F \rangle \in \Omega_{\mathbf{A}^*}([F]_F)$, i.e., $[a]_F \leq_{\mathbf{A}^*} [b]_F$ and $[b]_F \leq_{\mathbf{A}^*} [a]_F$. Therefore $a \rightarrow^{\mathbf{A}} b \in F$ and $b \rightarrow^{\mathbf{A}} a \in F$, i.e., $\langle a, b \rangle \in \Omega_A(F)$. Thus $[a]_F = [b]_F$. □

Lindenbaum–Tarski matrix

Let L be a weakly implicative logic in \mathcal{L} and $T \in Th(L)$. For every formula φ , we define the set

$$[\varphi]_T = \{\psi \in Fm_{\mathcal{L}} \mid \varphi \leftrightarrow \psi \subseteq T\}.$$

The **Lindenbaum–Tarski matrix** with respect to L and T , \mathbf{LindT}_T , has the filter $\{[\varphi]_T \mid \varphi \in T\}$ and algebraic reduct with the domain $\{[\varphi]_T \mid \varphi \in Fm_{\mathcal{L}}\}$ and operations:

$$c^{\mathbf{LindT}_T}([\varphi_1]_T, \dots, [\varphi_n]_T) = [c(\varphi_1, \dots, \varphi_n)]_T$$

Clearly, for every $T \in Th(L)$ we have:

$$\mathbf{LindT}_T = \langle Fm_{\mathcal{L}}, T \rangle^*.$$

The second completeness theorem

Theorem 1.16

Let L be a weakly implicative logic. Then for any set Γ of formulae and any formula φ the following holds:

$$\Gamma \vdash_L \varphi \quad \text{iff} \quad \Gamma \models_{\mathbf{MOD}^*(L)} \varphi.$$

Proof.

Using the first completeness theorem all we need to prove is:

$$\Gamma \models_{\mathbf{MOD}(L)} \varphi \quad \text{iff} \quad \Gamma \models_{\mathbf{MOD}^*(L)} \varphi.$$

One direction is trivial. Reverse one: assume that $\Gamma \not\models_{\mathbf{MOD}(L)} \varphi$, i.e., there is $\mathbf{A} = \langle A, F \rangle \in \mathbf{MOD}(L)$ and an A -evaluation e st. $e[\Gamma] \subseteq F$ and $e(\varphi) \notin F$. Take matrix $\mathbf{A}^* \in \mathbf{MOD}^*(L)$ and the A^* -evaluation $e^*(\psi) = [e(\psi)]_F$. Then clearly $e^*[\Gamma] \subseteq [F]_F$ and $e^*(\varphi) \notin [F]_F$.

The second completeness theorem

Theorem 1.16

Let L be a weakly implicative logic. Then for any set Γ of formulae and any formula φ the following holds:

$$\Gamma \vdash_L \varphi \quad \text{iff} \quad \Gamma \models_{\text{MOD}^*(L)} \varphi.$$

Proof.

Using just the soundness part of the FCT it remains to prove:

$$\Gamma \models_{\text{MOD}^*(L)} \varphi \quad \text{implies} \quad \Gamma \vdash_L \varphi.$$

Take Lindenbaum–Tarski matrix $\mathbf{LindT}_{\text{Th}_L(\Gamma)} = \langle \mathbf{Fm}_{\mathcal{L}}, \text{Th}_L(\Gamma) \rangle^*$ and evaluation $e(\psi) = [\psi]_{\text{Th}_L(\Gamma)}$. As clearly $e[\Gamma] \subseteq e[\text{Th}_L(\Gamma)] = [\text{Th}_L(\Gamma)]_{\text{Th}_L(\Gamma)}$, then, as $\mathbf{LindT}_{\text{Th}_L(\Gamma)}$ is an L -model, we have: $e(\varphi) = [\varphi]_{\text{Th}_L(\Gamma)} \in [\text{Th}_L(\Gamma)]_{\text{Th}_L(\Gamma)}$, and so $\varphi \in \text{Th}_L(\Gamma)$ i.e., $\Gamma \vdash_L \varphi$.

□