

Abstract Algebraic Logic: Theory and Applications – Lesson 4

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AAL is the evolution of Algebraic Logic that wants to:

- understand the several ways by which a logic can be given an algebraic semantics
- build a general and abstract theory of non-classical logics based on their relation to algebras
- understand the rôle of connectives in (non-)classical logics
- classify non-classical logics
- find general results connecting logical and algebraic properties (bridge theorems)
- generalize properties from syntax to semantics (transfer theorems)
- advance the study of particular (families of) non-classical logics by using the abstract notions and results

What have we done so far?

- understand the several ways by which a logic can be given an algebraic semantics
- build a general and abstract theory of non-classical logics based on their relation to algebras
- understand the rôle of connectives in (non-)classical logics: implication, equivalence, disjunction,...
- classify non-classical logics
- find general results connecting logical and algebraic properties (bridge theorems)
- generalize properties from syntax to semantics (transfer theorems)
- advance the study of particular (families of) non-classical logics by using the abstract notions and results

Bridge theorems vs. transfer theorems

Theorem 4.1 (Bloom)

Let L be a logic. Then: $\mathbf{P}_U(\mathbf{MOD}(L)) = \mathbf{MOD}(L)$ iff L is finitary.

It is a **bridge theorem**, relating a logical property with an algebraic (or matricial) one.

Theorem 4.2

Given a logic L in a language \mathcal{L} , the following conditions are equivalent:

- 1 L is finitary, i.e. Th_L is a finitary closure operator.
- 2 Fi_L^A is a finitary closure operator for any \mathcal{L} -algebra A .

It is a **transfer theorem**, transferring a property of $\mathbf{Fm}_{\mathcal{L}}$ to a formally equal property of all \mathcal{L} -algebras.

A logic L has the **parameterized local deduction-detachment theorem** if there is a family of sets of formulae $\Sigma \subseteq \mathcal{P}(Fm_{\mathcal{L}})$ in two variables (and possible parameters) such that for all $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$,

$\Gamma, \varphi \vdash_L \psi$ iff $\exists \Delta(x, y, \vec{z}) \in \Sigma$ such that $\Gamma \vdash_L \bigcup_{\vec{z} \in Fm_{\mathcal{L}}} \Delta(\varphi, \psi, \vec{z})$.

Theorem 4.3

A logic L is protoalgebraic iff it has the parameterized local deduction-detachment theorem.

A logic L has the **local deduction-detachment theorem (LDDT)** if it has the parameterized local deduction-detachment theorem with an empty set of parameters, i.e. there is a family of sets of formulae $\Sigma \subseteq \mathcal{P}(Fm_{\mathcal{L}})$ in two variables such that for all $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$,

$\Gamma, \varphi \vdash_L \psi$ iff $\exists \Delta(x, y) \in \Sigma$ such that $\Gamma \vdash_L \Delta(\varphi, \psi)$.

Logic	Σ
\mathbb{L} (infinitely-valued Łukasiewicz logic)	$\{p \rightarrow^n q \mid n \geq 0\}$
global modal logic \mathbb{T}	$\{\Box^n p \rightarrow q \mid n \geq 0\}$

A class of models of a logic $\mathbb{K} \subseteq \mathbf{MOD}(\mathbf{L})$ has the **L-filter-extension-property** iff for all $\langle \mathbf{A}, F \rangle, \langle \mathbf{B}, G \rangle \in \mathbb{K}$ such that $\langle \mathbf{A}, F \rangle \subseteq \langle \mathbf{B}, G \rangle$ and every $F' \in \mathcal{F}i_{\mathbf{L}}(\mathbf{A})$ such $F \subseteq F'$ and $\langle \mathbf{A}, F' \rangle \in \mathbb{K}$, there exists a $G' \in \mathcal{F}i_{\mathbf{L}}(\mathbf{B})$ such that $G \subseteq G'$, $\langle \mathbf{B}, G' \rangle \in \mathbb{K}$, and $G' \cap A = F'$.

Theorem 4.4 (Czelakowski, Blok-Pigozzi)

Let \mathbf{L} be a finitary protoalgebraic logic. TFAE:

- 1 \mathbf{L} has the LDDT.
- 2 $\mathbf{MOD}(\mathbf{L})$ has the L-filter-extension-property.
- 3 $\mathbf{MOD}^*(\mathbf{L})$ has the L-filter-extension-property.

Deduction theorems – 4

A logic L has the **global deduction-detachment theorem (GDDT)** if it has the local deduction-detachment theorem with a set Σ consisting of just one finite set of formulae i.e. there is a finite $\Delta(x, y) \subseteq Fm_{\mathcal{L}}$ in two variables such that for all $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$,

$$\Gamma, \varphi \vdash_L \psi \text{ iff } \Gamma \vdash_L \Delta(\varphi, \psi).$$

Logic	Δ
CPC, IPC, local modal logics	$\{p \rightarrow q\}$
\mathbb{L}_n (n -valued Łukasiewicz logic)	$\{p \rightarrow^n q\}$
global S4 and S5	$\{\Box p \rightarrow q\}$

A class of models of a logic $\mathbb{K} \subseteq \mathbf{MOD}(\mathbf{L})$ has **formula-definable principal \mathbf{L} -filters** if there is a finite set of formulae $\Delta(x, y) = \{\delta_i(x, y) \mid i < n\}$ of formulae in two variables such that, for every $\langle \mathbf{A}, F \rangle \in \mathbb{K}$ and every $a \in A$,

$$\mathbf{Fi}_{\mathbf{L}}^{\mathbf{A}}(F \cup \{a\}) = \{b \in A \mid \forall \delta \in \Delta, \delta^{\mathbf{A}}(a, b) \in F\}.$$

Theorem 4.5 (Blok-Pigozzi)

Let \mathbf{L} be a finitary protoalgebraic logic. TFAE:

- 1 \mathbf{L} has the GDDT.
- 2 $\mathbf{MOD}(\mathbf{L})$ has formula-definable principal \mathbf{L} -filters.
- 3 $\mathbf{MOD}^*(\mathbf{L})$ has formula-definable principal \mathbf{L} -filters.

A **dual Brouwerian semilattice** is an algebra $\mathbf{A} = \langle A, *^A, \vee^A, \top^A \rangle$ such that $\langle A, \vee^A, \top^A \rangle$ is a bounded join-semilattice and, for $a, b \in A$, there exists $a *^A b$, the smallest element c such that $a \leq b \vee^A c$. Hence for every $a, b, c \in A$:

$$a *^A b \leq c \text{ iff } a \leq b \vee^A c.$$

Theorem 4.6 (Czelakowski)

Let L be a finitary protoalgebraic logic. TFAE:

- 1 L has the GDDT.
- 2 The join-semilattice of finitely axiomatizable theories of L is dually Brouwerian.
- 3 For every A , the join-semilattice of finitely generated L -filters of A is dually Brouwerian.

A quasivariety \mathbb{K} has **equationally definable principal relative congruences (EDPRC)** if there is a finite set of equations in at most four variables $\{\varepsilon_i(x_0, x_1, y_0, y_1) \approx \delta_i(x_0, x_1, y_0, y_1) \mid i < n\}$ such that for every algebra $A \in \mathbb{K}$ and all $a, b, c, d \in A$,

$$\langle c, d \rangle \in \Theta_{\mathbb{K}}^A(a, b) \text{ iff } \forall i < n \varepsilon_i^A(a, b, c, d) = \delta_i^A(a, b, c, d),$$

where $\Theta_{\mathbb{K}}^A(a, b)$ denotes the relative congruence generated by $\langle a, b \rangle$.

Theorem 4.7 (Blok-Pigozzi)

Let L be a finitary and finitely algebraizable logic. TFAE:

- 1 L has the GDDT.
- 2 $\text{ALG}^*(L)$ has EDPRC.

A quasivariety \mathbb{K} has the **relative congruence extension property (RCEP)** if, and only if, for every $A, B \in \mathbb{K}$ such that $B \subseteq A$ and every $\theta \in \mathbf{Con}_{\mathbb{K}}(B)$, there exists $\theta' \in \mathbf{Con}_{\mathbb{K}}(A)$ such that $\theta' \cap B^2 = \theta$.

Theorem 4.8 (Blok-Pigozzi, Czelakowski-Dziobiak)

Let L be a finitary and finitely algebraizable logic. TFAE:

- 1 L has the LDDT.
- 2 $\mathbf{MOD}^*(L)$ has the RCEP.

Let L be a logic and $P, R \subseteq \text{Var}$, $P \cap R = \emptyset$, $\Gamma(\vec{p}, \vec{r}) \subseteq \text{Fm}_L$, $\vec{p} \in P$, $\vec{r} \in R$. We say that $\Gamma(\vec{p}, \vec{r})$ **defines R explicitly in terms of P** if for every $r \in R$ there is $\varphi_r \in \text{Fm}_L$ with variables in P such that $\langle r, \varphi_r \rangle \in \Omega(\text{Fi}_L^{P \cup R}(\Gamma))$ (filter generated in the subalgebra of formulae in variables $P \cup R$).

We say that $\Gamma(\vec{p}, \vec{r})$ **defines R implicitly in terms of P** if for every $R' \subseteq \text{Var}$, $R' \cap (P \cup R) = \emptyset$, $|R'| = |R|$, and every bijection f between R and R' , we have that for every $r \in R$, $\langle r, f(r) \rangle \in \Omega(\text{Fi}_L^{P \cup R \cup R'}(\Gamma))$.

L has the **Beth property** if for all disjoint sets of variables P and R , each set $\Gamma(\vec{p}, \vec{r}) \subseteq \text{Fm}_L$ that defines R implicitly in terms of P , defines also R explicitly in terms of P .

Beth property – 2

Let \mathbb{K} be a class of algebras of the same type, $\mathbf{A}, \mathbf{B} \in \mathbb{K}$, and $h: \mathbf{A} \rightarrow \mathbf{B}$ a homomorphism. h is an **epimorphism** in \mathbb{K} if for every $\mathbf{C} \in \mathbb{K}$ and each $g, g': \mathbf{B} \rightarrow \mathbf{C}$, if $g \circ h = g' \circ h$, then $g = g'$.

A class \mathbb{K} of algebras has the property that **epimorphisms are surjective (ES)** if every epimorphism between algebras of \mathbb{K} is a surjective mapping.

Theorem 4.9 (Hoogland)

Let L be an algebraizable logic. TFAE:

- 1 L has the Beth property.
- 2 $\mathbf{ALG}^*(L)$ has the ES.

A logic L has the **Craig interpolation property for consequence** if for every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ such that $\Gamma \vdash_L \varphi$, there is $\Gamma' \subseteq Fm_{\mathcal{L}}$ with variables in $Var(\Gamma) \cap Var(\varphi)$ such that $\Gamma \vdash_L \Gamma'$ and $\Gamma' \vdash_L \varphi$.

A class of algebras \mathbb{K} has the **amalgamation property** if for any $A, B, C \in \mathbb{K}$ and any embeddings $f: C \rightarrow A$ and $g: C \rightarrow B$, there is $D \in \mathbb{K}$ and embeddings $h: A \rightarrow D$ and $t: B \rightarrow D$ such that $h \circ f = t \circ g$.

Theorem 4.10 (Czelakowski)

Let L be an algebraizable logic with GDDT. TFAE:

- 1 *L has the Craig interpolation property for consequence.*
- 2 *$ALG^*(L)$ has the amalgamation property.*

A non-protoalgebraic logic – 1

$\text{CPC}_{\wedge\vee}$ is defined as the $\{\wedge, \vee\}$ -fragment of classical logic.

Gentzen presentation [Font and Verdú, 1991]

Hilbert presentation [Dyrda and Prucnal, 1980]:

$$\begin{array}{ll} \varphi \wedge \psi \triangleright \varphi & \varphi \vee (\psi \vee \xi) \triangleright (\varphi \vee \psi) \vee \xi \\ \varphi \wedge \psi \triangleright \psi \wedge \varphi & (\varphi \vee \psi) \vee \xi \triangleright \varphi \vee (\psi \vee \xi) \\ \varphi, \psi \triangleright \varphi \wedge \psi & \varphi \vee (\psi \wedge \xi) \triangleright (\varphi \vee \psi) \wedge (\varphi \vee \xi) \\ \varphi \triangleright \varphi \vee \psi & (\varphi \vee \psi) \wedge (\varphi \vee \xi) \triangleright \varphi \vee (\psi \wedge \xi) \\ \varphi \vee \psi \triangleright \psi \vee \varphi & \varphi \wedge (\psi \vee \xi) \triangleright (\varphi \wedge \psi) \vee (\varphi \wedge \xi) \\ \varphi \vee (\varphi \vee \psi) \triangleright \varphi \vee \psi & \varphi \vee \varphi \triangleright \varphi \end{array}$$

It is a logic **without theorems**, not almost inconsistent, and hence **not protoalgebraic**.

$\mathbf{2}_{\wedge, \vee}$: $\{\wedge, \vee\}$ -reduct of the two-element Boolean algebra $\mathbf{2}$

$$\text{CPC}_{\wedge, \vee} = \models_{\mathbf{2}_{\wedge, \vee}}$$

$\mathbf{V}(\mathbf{2}_{\wedge, \vee}) = \mathbb{D}$ (variety of distributive lattices)

Is \mathbb{D} the algebraic semantics of $\text{CPC}_{\wedge, \vee}$?

Theorem 4.11

$\mathbf{ALG}^*(\mathbf{CPC}_{\wedge\vee}) = \{A \in \mathbb{D} \mid A \text{ has a maximum element } 1 \text{ and for every } a, b \in A \text{ if } a < b \text{ then there is } c \in A \text{ such that } a \vee c \neq 1 \text{ and } b \vee c = 1\}$ (a proper subclass of \mathbb{D} , not even quasivariety).

Theorem 4.12

\mathbb{D} is not the equivalent algebraic semantics of any algebraizable logic.

$\mathbf{ALG}(\mathbf{CPC}_{\wedge\vee}) = \mathbb{D}$ [Font-Jansana] (alternative AAL theory based on generalized models)

- $\mathbf{ALG}(L) = \mathbf{P}_{\text{SD}}(\mathbf{ALG}^*(L))$.
- If L is protoalgebraic, then $\mathbf{ALG}(L) = \mathbf{ALG}^*(L)$.

Proposition 4.13

A logic L in a language \mathcal{L} is protoalgebraic iff for every $T \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$

$\langle \alpha, \beta \rangle \in \Omega_{Fm_{\mathcal{L}}}(T)$ implies $\text{Th}_L(T, \alpha) = \text{Th}_L(T, \beta)$.

Frege relation: $\langle \varphi, \psi \rangle \in \Lambda_L$ iff $\varphi \vdash_L \psi$ and $\psi \vdash_L \varphi$.

Selfextensional logic: L is selfextensional iff $\Lambda_L \in \mathbf{Con}(Fm_{\mathcal{L}})$.

Frege relation w.r.t. a theory: $\langle \varphi, \psi \rangle \in \Lambda_L(T)$ iff $T, \varphi \vdash_L \psi$ and $T, \psi \vdash_L \varphi$.

Fregean logic: L is Fregean iff $\Lambda_L(T) \in \mathbf{Con}(Fm_{\mathcal{L}})$ for every $T \in \text{Th}(L)$.

Frege hierarchy – 2

Inc, AInc, CPC, IPC, $\text{CPC}_{\wedge\vee}$ are Fregean.

Dumb is selfextensional but not Fregean.

\mathbb{L}_3 is not selfextensional ($\varphi \dashv\vdash \psi$ does not imply $\neg\varphi \dashv\vdash \neg\psi$; take $\varphi = p$ and $\psi = \neg(p \rightarrow \neg p)$, $e(p) = \frac{1}{2}$).

Theorem 4.14

- *Every protoalgebraic Fregean logic with theorems is regularly algebraizable.*
- *Every finitary and protoalgebraic Fregean logic with theorems is regularly, finitely algebraizable.*

Linear logic is not Fregean.

Infinitely-valued Łukasiewicz logics

$\mathbf{A} = \langle [0, 1], \rightarrow, \neg \rangle$, $a \rightarrow b = \min\{1, 1 - a + b\}$ and $\neg a = 1 - a$.

- **Infinitary version** \mathbb{L}_∞ : $\models_{\langle \mathbf{A}, \{1\} \rangle}$
- **Finitary version** \mathbb{L} : finitary companion of \mathbb{L}_∞
 $\Gamma \vdash_{\mathbb{L}} \varphi$ iff there is a finite $\Gamma_0 \subseteq \Gamma$ s.t. $\Gamma_0 \models_{\langle \mathbf{A}, \{1\} \rangle} \varphi$.
- **Degree-preserving version** \mathbb{L}^\leq : $\varphi_1, \dots, \varphi_n \vdash_{\mathbb{L}^\leq} \varphi$ iff for each \mathbf{A} -evaluation e , $\min\{e(\varphi_1), \dots, e(\varphi_n)\} \leq e(\varphi)$.

They all have **the same theorems**.

\mathbb{L}_∞ is **Rasiowa-implicative** (but **$\mathbf{ALG}^*(\mathbb{L}_\infty)$ is not quasivariety**) and **not selfextensional** (counterexample as in \mathbb{L}_3).

\mathbb{L} is **Rasiowa-implicative** (and **strongly BP-algebraizable**) and **not selfextensional** (counterexample as in \mathbb{L}_3).

\mathbb{L}^\leq is **selfextensional** (not Fregean) and **not protoalgebraic**.

Disjunction in Classical Logic

(PD) $\varphi \vdash_{\text{CPC}} \varphi \vee \psi$ and $\psi \vdash_{\text{CPC}} \varphi \vee \psi$

PCP **If $\Gamma, \varphi \vdash_{\text{CPC}} \chi$ and $\Gamma, \psi \vdash_{\text{CPC}} \chi$, then $\Gamma, \varphi \vee \psi \vdash_{\text{CPC}} \chi$.**

The same holds for many other logics: IPC, \mathbb{L} , FL_{ew} , BL, ...

(PD) and PCP could be equivalently formulated as:

$\Gamma, \varphi \vdash_{\text{CPC}} \chi$ and $\Gamma, \psi \vdash_{\text{CPC}} \chi$, **if and only if**, $\Gamma, \varphi \vee \psi \vdash_{\text{CPC}} \chi$.

Dummett in '*The Logical Basis of Metaphysics*, HUP, 1991' says about (a weaker variant of) PCP:

If this law does not hold, the operator \vee could not legitimately be called disjunction operator.

Theorem 4.15

In FL_e , the lattice connective \vee does not satisfy the PCP (it would entail $\varphi \vee \psi \vdash (\varphi \wedge \bar{1}) \vee (\psi \wedge \bar{1})$).

A solution of this problem:

Theorem 4.16

The connective \vee' defined as $\varphi \vee' \psi = (\varphi \wedge \bar{1}) \vee (\psi \wedge \bar{1})$ satisfies

(PD) $\varphi \vdash (\varphi \wedge \bar{1}) \vee (\psi \wedge \bar{1})$ **and** $\psi \vdash (\varphi \wedge \bar{1}) \vee (\psi \wedge \bar{1})$

PCP **If** $\Gamma, \varphi \vdash \chi$ **and** $\Gamma, \psi \vdash \chi$, **then** $\Gamma, (\varphi \wedge \bar{1}) \vee (\psi \wedge \bar{1}) \vdash \chi$.

Theorem 4.17

In the implication fragment of Gödel-Dummett logic we cannot define any connective \vee satisfying (PD) and PCP.

A solution of this problem:

Theorem 4.18

The 'connective' $\{(\varphi \rightarrow \psi) \rightarrow \psi, (\psi \rightarrow \varphi) \rightarrow \varphi\}$ satisfies

(PD) _{φ} $\varphi \vdash (\varphi \rightarrow \psi) \rightarrow \psi$ *and* $\varphi \vdash (\psi \rightarrow \varphi) \rightarrow \varphi$

(PD) _{ψ} $\psi \vdash (\varphi \rightarrow \psi) \rightarrow \psi$ *and* $\psi \vdash (\psi \rightarrow \varphi) \rightarrow \varphi$

PCP *If $\Gamma, \varphi \vdash \chi$ and $\Gamma, \psi \vdash \chi$, then*
 $\Gamma, (\varphi \rightarrow \psi) \rightarrow \psi, (\psi \rightarrow \varphi) \rightarrow \varphi \vdash \chi.$

An even bigger problem

Theorem 4.19

In FL no finite set of formulae of two variables defines any 'connective' satisfying (PD) and PCP.

BUT there is still a solution of this problem:

Theorem 4.20

*The following 'connective' satisfies both (PD) and PCP
 $\{\gamma_1(\varphi) \vee \gamma_2(\psi) \mid \text{where } \gamma_1, \gamma_2 \text{ are iterated conjugates}\}$.*

An **iterated conjugate** of φ is a formula $\gamma_{\alpha_1}(\gamma_{\alpha_2}(\dots \gamma_{\alpha_n}(\varphi) \dots))$ where $\gamma_{\alpha_i} = \lambda_{\alpha_i}(\varphi) = (\alpha_i \setminus \varphi \& \alpha_i) \wedge \bar{1}$ or $\gamma_{\alpha_i} = \rho_{\alpha_i}(\varphi) = (\alpha_i \& \varphi / \alpha_i) \wedge \bar{1}$ for some formulae α_i .

Let $\nabla(p, q, \vec{r})$ be a set of formulae. We write

$$\varphi \nabla \psi = \bigcup \{ \nabla(\varphi, \psi, \vec{\alpha}) \mid \vec{\alpha} \in \mathbf{Fm}^{\leq \omega} \}.$$

$$\Sigma_1 \nabla \Sigma_2 = \bigcup \{ \varphi \nabla \psi \mid \varphi \in \Sigma_1, \psi \in \Sigma_2 \}$$

Generalized disjunctions

A (parameterized) set of formulae ∇ is a (p-)protodisjunction if:

$$(PD) \quad \varphi \vdash_L \varphi \nabla \psi \quad \text{and} \quad \psi \vdash_L \varphi \nabla \psi$$

We will consider the following three properties:

$$\text{wPCP} \quad \varphi \vdash_L \chi \quad \text{and} \quad \psi \vdash_L \chi \quad \text{implies} \quad \varphi \nabla \psi \vdash_L \chi$$

$$\text{PCP} \quad \Gamma, \varphi \vdash_L \chi \quad \text{and} \quad \Gamma, \psi \vdash_L \chi \quad \text{implies} \quad \Gamma, \varphi \nabla \psi \vdash_L \chi$$

$$\text{sPCP} \quad \Gamma, \Sigma \vdash_L \chi \quad \text{and} \quad \Gamma, \Pi \vdash_L \chi \quad \text{implies} \quad \Gamma, \Sigma \nabla \Pi \vdash_L \chi$$

$$\text{Clearly:} \quad \text{sPCP} \Rightarrow \text{PCP} \Rightarrow \text{wPCP}$$

Theorem 4.21

For finitary logics: $\text{sPCP} \Leftrightarrow \text{PCP} \not\Leftrightarrow \text{wPCP}$

But in general: $\text{sPCP} \not\Leftrightarrow \text{PCP}$

We define also **transferred** variants of these notions.

Example 4.22

Consider the non-distributive lattice *diamond*, with the domain $\{\perp, a, b, t, \top\}$, with t as central element, and the finitary logic given by all matrices over this algebra with a lattice filter.

Observe: $\Gamma \vdash \varphi$ iff $\bigwedge e[\Gamma] \leq e(\varphi)$ for every evaluation e .

\vee is a protodisjunction with wPCP.

Assume now, for a contradiction, that it satisfies the PCP too. Then from $\varphi, \psi \vdash (\varphi \wedge \psi) \vee \chi$ and $\chi, \psi \vdash (\varphi \wedge \psi) \vee \chi$ we obtain $\varphi \vee \chi, \psi \vdash (\varphi \wedge \psi) \vee \chi$ and thus also (applying the PCP again) $\varphi \vee \chi, \psi \vee \chi \vdash (\varphi \wedge \psi) \vee \chi$ (a form of distributivity). Then, we reach a contradiction by observing that $a \vee b = t \vee b = \top$ while $(a \wedge t) \vee b = \perp \vee b = b$.

Example 4.23

Let A be a complete distributive lattice such that it is not a dual frame, i.e. there are elements $x_i \in A$ for $i \geq 0$ such that

$$\bigwedge_{i \geq 1} (x_0 \vee x_i) \not\leq x_0 \vee \bigwedge_{i \geq 1} x_i$$

expand the lattice language by constants $\{c_i \mid i \geq 0\} \cup \{c\}$ and define algebra A' in this language by setting $c_i^{A'} = x_i$ and $c = \bigwedge_{i \geq 1} x_i$. Then we define the logic L in this language semantically given by the class of matrices $\{\langle A', F \rangle \mid F \text{ is a principal lattice filter in } A\}$.

Observe: $\Gamma \vdash_L \varphi$ iff $\bigwedge_{\psi \in \Gamma} e(\psi) \leq e(\varphi)$ for each A -evaluation e .

Example 4.24

First we show that ∇ enjoys the PCP: assume that for each e evaluation holds $(\bigwedge_{\delta \in \Gamma} e(\delta)) \wedge e(\varphi) \leq e(\chi)$ and

$(\bigwedge_{\delta \in \Gamma} e(\delta)) \wedge e(\psi) \leq e(\chi)$, thus

$[(\bigwedge_{\delta \in \Gamma} e(\delta)) \wedge e(\varphi)] \vee [(\bigwedge_{\delta \in \Gamma} e(\delta)) \wedge e(\psi)] \leq e(\chi)$, the

distributivity of \mathbf{A} completes the proof. Finally, by the way of contradiction, assume that ∇ enjoys the sPCP. Observe that:

$c_0 \vdash_{\mathbf{L}} c_0 \vee c$ and $\{c_i \mid i \geq 1\} \vdash_{\mathbf{L}} c_0 \vee c$. Using the sPCP we obtain $\{c_0 \vee c_i \mid i \geq 1\} \vdash_{\mathbf{L}} c_0 \vee c$ —a contradiction.

Theorem 4.25

Let ∇ a *commutative and idempotent* p -protodisjunction. TFAE:

- 1 ∇ satisfies sPCP,
- 2 whenever $\Gamma \vdash_L \varphi$ we have also: $\Gamma \nabla \chi \vdash_L \varphi \nabla \chi$ for each χ .

This theorem was previously known for *finitary* logics and PCP.

Theorem 4.26

TFAE:

- 1 There is a (p -)protodisjunction satisfying wPCP.
- 2 For each (surjective) substitution σ and formulae φ, ψ :

$$\text{Th}_L(\sigma\varphi) \cap \text{Th}_L(\sigma\psi) = \text{Th}_L(\sigma[\text{Th}_L(\varphi) \cap \text{Th}_L(\psi)]).$$

If there is a (p -)protodisjunction satisfying wPCP, then $\text{Th}_L(p) \cap \text{Th}_L(q)$ is the largest.

$\text{Th}(\mathcal{L})$ is both a closure system and a complete lattice. A theory is **intersection-prime** if it is finitely \cap -irreducible in $\text{Th}(\mathcal{L})$.

Definition 4.27

We say that \mathcal{L} :

- is **distributive** if $\text{Th}(\mathcal{L})$ is a distributive lattice
- is **framal** if $\text{Th}(\mathcal{L})$ is a frame (meets distribute over arbitrary joins)
- has the **IPEP** (intersection-prime extension property) if intersection-prime theories form a base of $\text{Th}(\mathcal{L})$, i.e. if $T \in \text{Th}(\mathcal{L})$ and $\varphi \notin T$, there is an intersection-prime theory $T' \supseteq T$ such that $\varphi \notin T'$.

We define **filter-distributivity/framality** by demanding the defining conditions for $\mathcal{F}i_{\mathcal{L}}(\mathbf{A})$ for each \mathcal{L} -algebra \mathbf{A} .

Theorem 4.28

Every finitary logic has IPEP and **NOT** vice versa.

Example 4.29

Recall that \mathbb{L}_∞ . If $T \not\vdash_{\mathbb{L}_\infty} \chi$, then there is an evaluation e such that $e[T] = \{1\}$ and $e(\chi) \neq 1$. We define $T' = e^{-1}[\{1\}]$.

Obviously T' is a theory, $T \subseteq T'$ and $T' \not\vdash_{\mathbb{L}_\infty} \chi$. Assume that T' is not intersection-prime; thus there are formulae $\varphi, \psi \notin T'$ such that $T' = \text{Th}_{\mathbb{L}_\infty}(T, \varphi) \cap \text{Th}_{\mathbb{L}_\infty}(T, \psi)$. Assume without loss of generality that $e(\varphi) \leq e(\psi)$, so $e(\varphi \rightarrow \psi) = 1$ and so $\varphi \rightarrow \psi \in T'$. Thus $\psi \in \text{Th}_{\mathbb{L}_\infty}(T, \varphi)$ (because $\varphi, \varphi \rightarrow \psi \vdash_{\mathbb{L}_\infty} \psi$) and thus $\psi \in T'$ —a contradiction. Therefore, it has the IPEP.

Definition 4.30

A theory T is ∇ -prime if it is consistent and $T \vdash \varphi \nabla \psi$ implies
 $T \vdash \varphi$ or $T \vdash \psi$.

∇ has the PEP if ∇ -prime theories form a base of $\text{Th}(\mathbf{L})$.

Theorem 4.31

If ∇ has PCP, then ∇ -prime and intersection-prime theories coincide.

Theorem 4.32

Let \mathbf{L} be a logic satisfying the IPEP. TFAE:

- 1 ∇ has the sPCP.
- 2 ∇ has the PCP.
- 3 ∇ has the PEP.

Theorem 4.33 (Characterizations of sPCP)

The following are equivalent:

- 1 ∇ enjoys the sPCP,
- 2 ∇ enjoys the wPCP and the logic L is framal,
- 3 ∇ enjoys the wPCP and the logic L is filter-framal,
- 4 ∇ enjoys the transferred sPCP.

Theorem 4.34 (Characterizations of PCP)

Let L have IPEP. The following are equivalent:

- 1 ∇ enjoys the PCP,
- 2 ∇ enjoys the wPCP and the logic L is distributive,
- 3 ∇ enjoys the wPCP and the logic L is filter-distributive,
- 4 ∇ enjoys the transferred PCP.

Theorem 4.35

Let L be a protoalgebraic logic.

- *L is distributive/framal IFF there is a p -protodisjunction ∇ which has PCP/sPCP.*
- *If L has IPEP and is distributive, then it is filter-framal.*
- *If ∇ has PCP, then it has transferred PCP.*

Corollary 4.36

Let L be a logic with the IPEP, ∇ a p -protodisjunction with PCP, and let L_1, L_2 be axiomatic extensions of L by sets of axioms \mathcal{A}_1 and \mathcal{A}_2 , respectively. Then:

$$L_1 \cap L_2 = L + \{\varphi \nabla \psi \mid \varphi \in \mathcal{A}_1, \psi \in \mathcal{A}_2\}.$$

Note: we can safely always assume that \mathcal{A}_1 and \mathcal{A}_2 are written in disjoint sets of variables.

Theorem 4.37

Let L be a logic with the IPEP, ∇ a p -protodisjunction with PCP, and \mathcal{C} a set of positive clauses. Then:

$$\models_{\{\mathbf{A} \in \text{MOD}^*(L) \mid \mathbf{A} \models \mathcal{C}\}} = L + \{\nabla_{\psi \in \Sigma_{\mathcal{C}}} \psi \mid \mathcal{C} \in \mathcal{C}\}.$$