Abstract Algebraic Logic: Theory and Applications – Lesson 5

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Completeness theorem for classical logic

- Suppose that $T \in \mathrm{Th}(\mathrm{CPC})$ and $\varphi \notin T$ ($T \nvdash_{\mathrm{CPC}} \varphi$). We want to show that $T \not\models \varphi$ in some meaningful semantics.
- $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}, T \rangle} \varphi$. 1st completeness theorem
- $\langle \alpha, \beta \rangle \in \Omega(T)$ iff $\alpha \leftrightarrow \beta \in T$ (congruence relation on $\boldsymbol{Fm}_{\mathcal{L}}$ compatible with T: if $\alpha \in T$ and $\langle \alpha, \beta \rangle \in \Omega(T)$, then $\beta \in T$).
- Lindenbaum-Tarski algebra: $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T)$ is a Boolean algebra and $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}/\Omega(T), T/\Omega(T) \rangle} \varphi$.
  2nd completeness theorem
- Lindenbaum Lemma: If $\varphi \notin T$, then there is a maximal consistent $T' \in \mathrm{Th}(\mathrm{CPC})$ such that $T \subseteq T'$ and $\varphi \notin T'$.
- $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T') \cong \mathbf{2}$ (subdirectly irreducible Boolean algebra) and $T \not\models_{\langle \mathbf{2}, \{1\} \rangle} \varphi$. 3rd completeness theorem

The scope restriction for this lecture

Unless said otherwise, any logic L is weakly implicative in a language $\mathcal{L}$ with an implication $\rightarrow$.

Order and Leibniz congruence

Recall

Let $\mathbf{A} = \langle A, F \rangle$ be an L-matrix. We define:

- the matrix preorder $\leq_{\mathbf{A}}$ of $\mathbf{A}$ as

$$a \leq_{\mathbf{A}} b \qquad \text{iff} \qquad a \to^{A} b \in F$$

- the Leibniz congruence $\Omega_{\mathbf{A}}(F)$ of $\mathbf{A}$ as

$$\langle a, b \rangle \in \Omega_{\mathbf{A}}(F) \qquad \text{iff} \qquad a \leq_{\mathbf{A}} b \text{ and } b \leq_{\mathbf{A}} a.$$

Observation

The Leibniz congruence of $\mathbf{A}$ is the identity iff $\leq_{\mathbf{A}}$ is an order.

Thus all reduced matrices of L are ordered by $\leq_{\mathbf{A}}$.

Weakly implicative logics are the logics of ordered matrices.

Linear filters

Definition 5.1

Let $\mathbf{A} = \langle A, F \rangle \in \mathbf{MOD}(\mathrm{L})$. Then

- F is *linear* if $\leq_{\mathbf{A}}$ is a total preorder, i.e. for every $a, b \in A$, $a \to^{\boldsymbol{A}} b \in F$ or $b \to^{\boldsymbol{A}} a \in F$
- $\mathbf{A}$ is a *linearly ordered model* (or just a *linear model*) if $\leq_{\mathbf{A}}$ is a linear order (equivalently: F is linear and $\mathbf{A}$ is reduced).

We denote the class of all linear models as $\mathbf{MOD}^{\ell}(\mathrm{L})$.

A theory T is linear in L if $T \vdash_{\mathrm{L}} \varphi \to \psi$ or $T \vdash_{\mathrm{L}} \psi \to \varphi$, for all φ, ψ

Lemma 5.2

Let $\mathbf{A} \in \mathbf{MOD}(\mathrm{L})$. Then F is linear iff $\mathbf{A}^* \in \mathbf{MOD}^{\ell}(\mathrm{L})$. In particular: a theory T is linear iff $\mathbf{LindT}_T \in \mathbf{MOD}^{\ell}(\mathrm{L})$

For proof just recall that: $[a]_F \leq_{\mathbf{A}^*} [b]_F$ iff $a \to^{\boldsymbol{A}} b \in F$.

Semilinear implications and semilinear logics

Definition 5.3

We say that $\to$ is *semilinear* if

$$\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^{\ell}(\mathrm{L})}.$$

We say that L is *semilinear* if it has a semilinear implication.

(Weakly implicative) *semilinear* logics are the logics of *linearly* ordered matrices.

Characterization of semilinearity via the Linear Extension Property LEP

Definition 5.4

We say that a L has the *Linear Extension Property* LEP if linear theories form a base of $\mathrm{Th}(\mathrm{L})$, i.e. for every theory $T \in Th(\mathrm{L})$ and every formula $\varphi \in Fm_{\mathcal{L}} \setminus T$, there is a linear theory $T' \supseteq T$ such that $\varphi \notin T'$.

Theorem 5.5

Let L *be a weakly implicative logic. TFAE:*

1. L *is semilinear.*
2. L *has the* LEP*.*

The proof

$1 \to 2$: If $T \nvdash_{\mathrm{L}} \chi$, then there is a $\mathbf{B} = \langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}^{\ell}(\mathrm{L})$ and a $\mathbf{B}$-evaluation e s.t. $e[T] \subseteq F$ and $e(\chi) \notin F$. We define $T' = e^{-1}[F]$: it is a theory (due to Lemma 1.5), $T \subseteq T'$, and $T' \nvdash_{\mathrm{L}} \chi$. Take φ, ψ and assume w.l.o.g. that $e(\varphi) \leq_{\mathbf{B}} e(\psi)$, thus $e(\varphi \to \psi) \in F$, i.e. $\varphi \to \psi \in T'$.

$2 \to 1$: assume that $\Gamma \nvdash_{\mathrm{L}} \varphi$ and set $T = \mathrm{Th}_{\mathrm{L}}(\Gamma)$. Then there is a linear theory $T' \supseteq T$ such that $T' \nvdash_{\mathrm{L}} \varphi$.

Take Lindenbaum–Tarski matrix $\mathbf{LindT}_{T'}$ and note that $\mathbf{LindT}_{T'} \in \mathbf{MOD}^{\ell}(\mathrm{L})$ (due to Lemma 5.2). Then take evaluation $e(v) = [v]_{T'}$ and observe that $e[\Gamma] \subseteq e[T'] = [T']_{T'}$ and as $\varphi \notin T'$ we get $e(\varphi) \notin [T']_{T'}$ (due to Lemma 1.15).

Semilinearity Property SLP and its transfer

Definition 5.6

We say that a L has the *Semilinearity Property* SLP if the following meta-rule is valid:

$$\frac{\Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \to \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}.$$

Theorem 5.7

Assume that L *satisfies the* SLP. *Then for each* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *and each set* $X \cup \{a, b\} \subseteq A$ *we have:*

$$\mathrm{Fi}(X, a \to b) \cap \mathrm{Fi}(X, b \to a) = \mathrm{Fi}(X).$$

To prove the non-trivial direction we show that for each $t \notin \mathrm{Fi}(X)$ we have $t \notin \mathrm{Fi}(X, a \to b)$ or $t \notin \mathrm{Fi}(X, b \to a)$. We distinguish two cases:

1. proof of the transfer when A is countable.

Assume, w.l.o.g. that Var contains $\{v_z \mid z \in A\}$ and define:

$$\Gamma = \{v_z \mid z \in \mathrm{Fi}(X)\} \cup \bigcup_{\langle c,n\rangle \in \mathcal{L}} \{c(v_{z_1}, \dots, v_{z_n}) \leftrightarrow v_{c^{A}(z_1,\dots,z_n)} \mid z_i \in A\}.$$

Clearly, $\Gamma \nvdash_{\mathrm{L}} v_t$ (because for the $\boldsymbol{A}$-evaluation $e(v_z) = z$: $e[\Gamma] \subseteq \mathrm{Fi}(X)$ and $e(v_t) \notin \mathrm{Fi}(X)$). Thus by the SLP (w.l.o.g.): $\Gamma, v_a \to v_b \nvdash_{\mathrm{L}} v_t$. We define a theory $T' = \mathrm{Th}_{\mathrm{L}}(\Gamma, v_a \to v_b)$ and a mapping $h\colon A \to Fm_{\mathcal{L}}/\Omega T'$ as $h(z) = [v_z]_{T'}$. We show that h is a homomorphism:

$$\begin{aligned} h(c^{\boldsymbol{A}}(z_1, \dots, z_n)) &= [v_{c^{\boldsymbol{A}}(z_1,\dots,z_n)}]_{T'} = [c(v_{z_1}, \dots, v_{z_n})]_{T'} \\ &= c^{\boldsymbol{Fm}_{\mathcal{L}}/\Omega T'}([v_{z_1}]_{T'}, \dots, [v_{z_n}]_{T'}) \\ &= c^{\boldsymbol{Fm}_{\mathcal{L}}/\Omega T'}(h(z_1), \dots, h(z_n)). \end{aligned}$$

Thus $F = h^{-1}([T']_{T'}) \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ (via Lemma 1.5) and $X \cup \{a \to b\} \subseteq F$ and $t \notin F$, i.e. $t \notin \mathrm{Fi}(X, a \to b)$.

2. proof of the transfer when A is uncountable – 1

Set $Var' = \{v_z \mid z \in A\} \supseteq Var$; we define a logic L' in $\mathcal{L}'$ with the same connectives as $\mathcal{L}$ and variables from Var'. If we show that L' has the SLP we can repeat the constructions from the first part of this proof to complete the proof.

Let $\mathcal{AS}$ be a presentation of L (note that each rule of $\mathcal{AS}$ has countably many premises) and define:

$$\mathcal{AS}' = \{\sigma[X] \rhd \sigma(\varphi) \mid X \rhd \varphi \in \mathcal{AS} \text{ and } \sigma \text{ is an } \mathcal{L}'\text{-subst.}\} \quad \mathrm{L}' = \vdash_{\mathcal{AS}'}$$

Observe that $\Gamma \vdash_{\mathrm{L}'} \varphi$ iff there is a countable set $\Gamma' \subseteq \Gamma$ st. $\Gamma' \vdash_{\mathrm{L}'} \varphi$ (clearly any proof in $\mathcal{AS}'$ has countably many leaves, because all of its rules have countably many premises). Next observe that L' is a conservative expansion of L (consider the substitution σ sending all variables from Var to themselves and the rest to a fixed $p \in Var$, take any proof of φ from Γ in $\mathcal{AS}'$ and observe that the same tree with labels ψ replaced by $\sigma\psi$ is a proof of φ from Γ in L).

Now we show that L' has the SLP: assume that $\Gamma, \varphi \to \psi \vdash_{\mathrm{L}'} \chi$ and $\Gamma, \psi \to \varphi \vdash_{\mathrm{L}'} \chi$.

Then there is a countable subset $\Gamma' \subseteq \Gamma$ st. $\Gamma', \varphi \to \psi \vdash_{\mathrm{L}'} \chi$ and $\Gamma', \psi \to \varphi \vdash_{\mathrm{L}'} \chi$. Let Var_0 be the variables occurring in $\Gamma' \cup \{\varphi, \psi, \chi\}$ and g a bijection on Var' st. $g[Var_0] \subseteq Var$

Let σ be the $\mathcal{L}'$-substitution induced by g and σ^{-1} its inverse. Note that: $\sigma[\Gamma'] \cup \{\sigma\varphi, \sigma\psi, \sigma\chi\} \subseteq Fm_{\mathcal{L}}$, $\sigma[\Gamma'], \sigma\varphi \to \sigma\psi \vdash_{\mathrm{L}'} \sigma\chi$ and $\sigma[\Gamma'], \sigma\psi \to \sigma\varphi \vdash_{\mathrm{L}'} \sigma\chi$.

As L' expands L conservatively, we have $\sigma[\Gamma'], \sigma\varphi \to \sigma\psi \vdash_{\mathrm{L}} \sigma\chi$ and $\sigma[\Gamma'], \sigma\psi \to \sigma\varphi \vdash_{\mathrm{L}} \sigma\chi$. Thus $\sigma[\Gamma'] \vdash_{\mathrm{L}} \sigma\chi$ (by SLP of L).

Thus also $\sigma[\Gamma'] \vdash_{\mathrm{L}'} \sigma\chi$; $\sigma^{-1}[\sigma[\Gamma']] \vdash_{\mathrm{L}'} \sigma^{-1}(\sigma\chi)$ i.e., $\Gamma' \vdash_{\mathrm{L}'} \chi$.

Properties of linear filters

Lemma 5.8

Let $\boldsymbol{A}$ an $\mathcal{L}$-algebra and F a linear filter. Then the set $[F,A] = \{G \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A}) \mid F \subseteq G\}$ is linearly ordered by inclusion.

Proof.

Take $G_1, G_2 \in [F,A]$ and elements $a_1 \in G_1 \setminus G_2$ and $a_2 \in G_2 \setminus G_1$. Assume w.l.o.g. that $a_1 \leq_{\langle \boldsymbol{A},F\rangle} a_2$. Thus also $a_1 \to^{\boldsymbol{A}} a_2 \in F \subseteq G_1$ and so by (MP) also $a_2 \in G_1$—a contradiction. □

Lemma 5.9

Linear filters are finitely $\cap$-irred. i.e. $\mathbf{MOD}^{\ell}(\mathrm{L}) \subseteq \mathbf{MOD}^{*}(\mathrm{L})_{\mathrm{RFSI}}$.

Proof.

Let $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ be a linear filter and $F = G_1 \cap G_2$. Then $G_1, G_2 \in [F,A]$ which is linearly ordered by inclusion, therefore $F = G_1$ or $F = G_2$. The second claim follows from Theorem 2.6. □

Characterization of semilinear logics

Theorem 5.10

Let L *be a weakly implicative logic. TFAE:*

1. L *is semilinear.*
2. L *has the* LEP.

If L *is finitary the list can be expanded by:*

3. L *has the* SLP.
4. L *has the transferred* SLP.
5. *Linear filters coincide with finitely* $\cap$*-irreducible ones in each* $\mathcal{L}$*-algebra.*
6. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}} = \mathbf{MOD}^\ell(\mathrm{L})$.
7. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}} \subseteq \mathbf{MOD}^\ell(\mathrm{L})$.

(Every semilinear logic enjoys properties 3.–7.)

The proof

$1 \leftrightarrow 2$: Theorem 5.5

$2 \to 3$: assume that $T \nvdash_{\mathrm{L}} \chi$, let $T' \supseteq T$ be a linear theory s.t. $T' \nvdash_{\mathrm{L}} \chi$. Assume w.l.o.g. that $T' \vdash_{\mathrm{L}} \varphi \to \psi$, then obviously $T, \varphi \to \psi \nvdash_{\mathrm{L}} \chi$.

$3 \to 4$: Theorem 5.7.

$4 \to 5$: let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra. One direction is Lemma 5.9. Converse one: assume that F is not linear, i.e., there are $a, b \in A$ st. $a \to b \notin F$ and $b \to a \notin F$. Thus $F \subsetneq \mathrm{Fi}(F, a \to b)$ and $F \subsetneq \mathrm{Fi}(F, b \to a)$ and so $\mathrm{Fi}(F, a \to b) \cap \mathrm{Fi}(F, b \to a) = \mathrm{Fi}(F) = F$, i.e., F is finitely $\cap$-reducible.

$5 \to 6$: due to Theorem 2.6.

$6 \to 7$: trivial consequence.

$7 \to 1$: due to Theorem 2.8. Note only here we need finitarity

Classes of semilinear logics

Corollary 5.11

Every regularly implicative semilinear logic is also Rasiowa-implicative.

Proof.

Trivially: $\varphi, \psi \to \varphi \vdash \psi \to \varphi$ and from regularity also: $\varphi, \varphi \to \psi \vdash \psi \to \varphi$. Thus, by the SLP, we derive $\varphi \vdash \psi \to \varphi$. □

Example 5.12

$Ł_3^{\leq}$ (the degree-preserving version of $Ł_3$) is is weakly implicative semilinear logic but it is not algebraically implicative.

Example 5.13

Logic of linear residuated lattices is algebraically implicative semilinear logic but it is not regularly implicative.

Intuitionistic logic is not semilinear

Example 5.14

Intuitionistic logic is not semilinear w.r.t. any implication.

Corollary 5.15

All axiomatic extensions of a semilinear logic are semilinear too.

If L can be axiomatically extended to IPC, then it is not semilinear.

The least semilinear extension

Corollary 5.16

The intersection of a family of semilinear logics in the same language is a semilinear logic.

As Inc is trivially semilinear we can soundly define:

Definition 5.17 (Logic L^{ℓ})

Given a weakly implicative logic L, we denote by L^{ℓ} the least semilinear logic extending L.

Proposition 5.18

If L is a finitary weakly implicative logic, then so is L^{ℓ}.

The least semilinear extension—semantics

Proposition 5.19

Let L *be a weakly implicative logic. Then* $\mathrm{L}^\ell = \models_{\mathbf{MOD}^\ell(\mathrm{L})}$ *and* $\mathbf{MOD}^\ell(\mathrm{L}^\ell) = \mathbf{MOD}^\ell(\mathrm{L})$.

Proof.

Let L' be any extension of L, then $\mathbf{MOD}^\ell(\mathrm{L}') \subseteq \mathbf{MOD}^\ell(\mathrm{L})$. Thus in particular:

$$\mathbf{MOD}^\ell(\mathrm{L}^\ell) \subseteq \mathbf{MOD}^\ell(\mathrm{L}) \text{ and so } \models_{\mathbf{MOD}^\ell(\mathrm{L})} \subseteq \models_{\mathbf{MOD}^\ell(\mathrm{L}^\ell)} = \mathrm{L}^\ell$$

As $\models_{\mathbf{MOD}^\ell(\mathrm{L})}$ is clearly semilinear we have the first claim.
The second inclusion of the second claim is trivial

(as $\mathbb{K} \subseteq \mathbf{MOD}^*(\models_{\mathbb{K}})$) □

The least semilinear extension—axiomatization

Theorem 5.20 (Axiomatization of L^ℓ)

Let L be a finitary p-disjunctional weakly implicative logic. Then L^ℓ is the extension of L with the axiom(s):

$$(\mathrm{P}_\nabla) \quad \vdash_{\mathrm{L}} (\varphi \to \psi) \nabla (\psi \to \varphi).$$

Proof.

Using the previous proposition we know that $\mathrm{L}^\ell = \models_{\mathbf{MOD}^\ell(\mathrm{L})}$. The proof is completed by Theorem 4.37; we only need to observe that a matrix $\mathbf{A} \in \mathbf{MOD}^\ell(\mathrm{L})$ iff $\mathbf{A} \models P$, where P is the positive clause $F(\varphi \to \psi) \vee F(\psi \to \varphi)$. □

The axiom(s) (P_∇) is (are) called the *prelinearity axiom(s)*.

Semilinearity and (generalized) disjunction

How to proceed if we do not know any p-disjunction of L?

Idea: choose a *suitable* p-protodisjunction ∇, extend L to L^{∇}, and proceed as above.

Problem: what if $\mathrm{L}^{\nabla} \not\subseteq \mathrm{L}^{\ell}$? To overcome it, we define:

$(\mathrm{MP}_{\nabla}) \quad \varphi \to \psi, \varphi \nabla \psi \vdash_{\mathrm{L}} \psi$ and $\varphi \to \psi, \psi \nabla \varphi \vdash_{\mathrm{L}} \psi$.

Proposition 5.21

Let ∇ be a p-protodisjunction in L.

1. *If L is p-disjunctional, than (MP_{∇}) is satisfied.*
2. *If L is semilinear, than (P_{∇}) is satisfied.*

Proof.

1. Using PCP for $\varphi, \varphi \to \psi \vdash \psi$ and $\psi, \varphi \to \psi \vdash \psi$.
2. Using SLP for $\varphi \to \psi \vdash_{\mathrm{L}} (\varphi \to \psi) \nabla (\psi \to \varphi)$ and

$$\psi \to \varphi \vdash_{\mathrm{L}} (\varphi \to \psi) \nabla (\psi \to \varphi)).$$

□

(P_∇) and (MP_∇): natural binding conditions – 1

Lemma 5.22

Let ∇ be a p-protodisjunction and $\boldsymbol{A}$ an $\mathcal{L}$-algebra.

1. *If L fulfils (MP_∇), then each linear filter in $\boldsymbol{A}$ is ∇-prime.*
2. *If L fulfils (P_∇), then each ∇-prime filter in $\boldsymbol{A}$ is linear.*

Proof.

1. Assume that F is linear ($a \to^{\boldsymbol{A}} b \in F$ or $b \to^{\boldsymbol{A}} a \in F$) and $a \nabla^{\boldsymbol{A}} b \subseteq F$. Thus from (MP_∇) we obtain: $b \in F$ or $a \in F$.

2. Assume that F is not linear, i.e. there are elements a, b st. $x = a \to^{\boldsymbol{A}} b \notin F$ and $y = b \to^{\boldsymbol{A}} a \notin F$. From (P_∇) we obtain $x \nabla^{\boldsymbol{A}} y = (a \to^{\boldsymbol{A}} b) \nabla^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} a) \subseteq F$, i.e., F is not ∇-prime. □

(P_∇) and (MP_∇): natural binding conditions – 2

Theorem 5.23 (Interplay of p-disjunctions and semilinearity)

Let L *be a finitary and* ∇ *a p-protodisjunction. TFAE:*

1. L *is p-disjunctional and satisfies* (P_∇).
2. L *is semilinear and satisfies* (MP_∇).

Thus in particular:

- *If* L *satisfies* (P_∇) *and* (MP_∇)*:* L *is semilinear iff it is p-disjunctional.*
- *If* L *is p-disjunctional:* L *is semilinear iff* L *satisfies* (P_∇).
- *If* L *is semilinear:* L *is p-disjunctional iff* L *satisfies* (MP_∇).

Proof.

(MP_∇) follows from Proposition 5.21. From (P_∇) we know that ∇-prime theories are linear and as we have PEP, we get LEP. The converse direction is analogous. □

Corollary 5.24

Let L *be a finitary logic and* ∇ *a p-protodisjunction* satisfying (MP_∇). *Then* L^ℓ *is the extension of* L^∇ *by* (P_∇).

Proof.

Since $\mathrm{L}^\nabla + (\mathrm{P}_\nabla)$ is an axiomatic extension of L^∇, ∇ remains a p-disjunction there. Thus, by Theorem 5.23, it is a semilinear logic.

Let L' be a finitary semilinear extension of L. Clearly L' satisfies (MP_∇) as well and thus by Theorem 5.23 it is a p-disjunctional logic and satisfies (P_∇). Thus $\mathrm{L}^\nabla \subseteq \mathrm{L}'$ and so

$$\mathrm{L}^\nabla + (\mathrm{P}_\nabla) \subseteq \mathrm{L}' + (\mathrm{P}_\nabla) = \mathrm{L}'. \quad \square$$

Corollary 5.25

Let L_1 *be a semilinear logic with a p-protodisjunction which satisfies* (MP_∇) *and* L_2 *its* finitary *weakly implicative expansion by a set of consecutions* $\mathcal{C}$. *TFAE:*

- L_2 *is semilinear.*
- $\Gamma \nabla \chi \vdash_{\mathrm{L}_2} \varphi \nabla \chi$ *for each consecution* $\Gamma \rhd \varphi \in \mathcal{C}$.

Corollary 5.26

Let L *be a semilinear logic with a p-protodisjunction which satisfies* (MP_∇). *Then all its weakly implicative axiomatic expansions are semilinear as well.*

Definition 5.27

We say that L with connective $\vee$ in its language is *lattice-disjunctive* if $\vee$ is a disjunction and:

$$
\begin{array}{ll}
(\vee 1) & \vdash_{\mathrm{L}} \varphi \to \varphi \vee \psi \\
(\vee 2) & \vdash_{\mathrm{L}} \psi \to \varphi \vee \psi \\
(\vee 3) & \varphi \to \chi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \vee \psi \to \chi.
\end{array}
$$

Proposition 5.28

Let L be a finitary lattice-disjunctive logic. Then: L^{ℓ} is the extension of $\mathrm{L}^{\vee}$ by any of these axioms:

$$
\begin{array}{ll}
(\mathrm{P}_{\vee}) & \vdash_{\mathrm{L}} (\varphi \to \psi) \vee (\psi \to \varphi) \\
(\mathrm{lin}_{\vee}) & \vdash_{\mathrm{L}} (\chi \to \varphi \vee \psi) \to (\chi \to \varphi) \vee (\chi \to \psi).
\end{array}
$$

Completeness w.r.t. densely ordered matrices

Definition 5.29 (Dense filter)

A filter F in $\mathbf{A}$ is *dense* if it is linear and for every $a, b \in A$ if $a <_{\mathbf{A}} b$ there is $z \in A$ st. $a <_{\mathbf{A}} z$ and $z <_{\mathbf{A}} b$.
A matrix $\mathbf{A}$ is *dense linear matrix*, $\mathbf{A} \in \mathbf{MOD}^{\delta}(\mathrm{L})$, if it is reduced and F is dense (equivalently: if $\leq_{\mathbf{A}}$ is a dense order).

Definition 5.30 (Density Property)

Logic L with has p-protodisjunction ∇ has

- *Density Property* DP w.r.t. ∇ if for any set of formulae $\Gamma \cup \{\varphi, \psi, \chi\}$ and any variable p not occurring them: $\Gamma \vdash_{\mathrm{L}} (\varphi \to p) \nabla (p \to \psi) \nabla \chi$ implies $\Gamma \vdash_{\mathrm{L}} (\varphi \to \psi) \nabla \chi$.
- *Dense Extension Property* DEP if every set of formulae Γ st. $\Gamma \nvdash_{\mathrm{L}} \varphi$ and there are infinitely many variables not occurring in Γ can be extended into a dense theory $T \supseteq \Gamma$ st. $T \nvdash_{\mathrm{L}} \varphi$.

Characterization of dense completeness

Proposition 5.31

Any L *with* DEP*:*

1. *is semilinear and*
2. *enjoys* DP *for any p-protodisjunction* ∇ *satisfying* (MP_{∇})

Theorem 5.32 (Characterization of dense completeness)

Let L *be a weakly implicative logic. TFAE*

1. $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^{\delta}(\mathrm{L})}$.
2. L *has the* DEP.

If furthermore L *is finitary semilinear disjunctional logic, then we can add:*

3. L *has the* DP.

Completeness w.r.t. arbitrary class of chains

Convention

From now on assume that L is an algebraically implicative semilinear logic and $\mathbb{K}$ a class of L-chains.

Definition 5.33 (Completeness properties)

We say that L has the property of:

- *Strong* $\mathbb{K}$*-completeness*, $\mathrm{S}\mathbb{K}\mathrm{C}$ for short, when for every set of formulae $\Gamma \cup \{\varphi\}$: $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$.
- *Finite strong* $\mathbb{K}$*-completeness*, $\mathrm{FS}\mathbb{K}\mathrm{C}$ for short, when for every *finite* set of formulae $\Gamma \cup \{\varphi\}$: $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$.
- $\mathbb{K}$*-completeness*, $\mathbb{K}\mathrm{C}$ for short, when for every formula φ: $\vdash_{\mathrm{L}} \varphi$ iff $\models_{\mathbb{K}} \varphi$.

Algebraic characterization of completeness properties

Theorem 5.34

1. L *has the* $\mathbb{K}\mathrm{C}$ *if, and only if,* $\mathbf{V}(\mathbf{ALG}^*(\mathrm{L})) = \mathbf{V}(\mathbb{K})$.
2. L *has the* $\mathrm{FS}\mathbb{K}\mathrm{C}$ *if, and only if,* $\mathbf{Q}(\mathbf{ALG}^*(\mathrm{L})) = \mathbf{Q}(\mathbb{K})$.
3. L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$ *if, and only if,* $\mathbf{ALG}^*(\mathrm{L}) = \mathbf{ISP}_{\sigma\text{-}f}(\mathbb{K})$.

Proof.

1. $\Rightarrow$: take an arbitrary equation $\varphi \approx \psi$: then $\models_{\mathbf{ALG}^*(\mathrm{L})} \varphi \approx \psi$ iff $\vdash_{\mathrm{L}} \varphi \leftrightarrow \psi$ iff $\models_{\mathbb{K}} \varphi \leftrightarrow \psi$ iff $\models_{\mathbb{K}} \varphi \approx \psi$. Therefore $\mathbf{ALG}^*(\mathrm{L})$ and $\mathbb{K}$ satisfy the same equations and hence they generate the same variety.

$\Leftarrow$: $\vdash_{\mathrm{L}} \varphi$ iff $\models_{\mathbf{ALG}^*(\mathrm{L})} \mu(\varphi) \approx \nu(\varphi)$ for each $\mu \approx \nu \in \mathcal{T}$ iff $\models_{\mathbb{K}} \mu(\varphi) \approx \nu(\varphi)$ for each $\mu \approx \nu \in \mathcal{T}$ iff $\models_{\mathbb{K}} \varphi$. $\square$

Theorem 5.34

1. L *has the* $\mathbb{K}\mathrm{C}$ *if, and only if,* $\mathbf{V}(\mathbf{ALG}^*(\mathrm{L})) = \mathbf{V}(\mathbb{K})$.
2. L *has the* $\mathrm{FS}\mathbb{K}\mathrm{C}$ *if, and only if,* $\mathbf{Q}(\mathbf{ALG}^*(\mathrm{L})) = \mathbf{Q}(\mathbb{K})$.
3. L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$ *if, and only if,* $\mathbf{ALG}^*(\mathrm{L}) = \mathbf{ISP}_{\sigma\text{-}f}(\mathbb{K})$.

Proof.

The remaining points are proved analogously using that quasivarieties are characterized by quasiequations, and the classes closed under the operator $\mathbf{ISP}_{\sigma\text{-}f}$ are characterized by generalized quasiequations with countably many premises (we can omit this operator on the left side of the equation because that $\mathbf{ALG}^*(\mathrm{L})$ is closed under $\mathbf{ISP}_{\sigma\text{-}f}$). □

Theorem 5.35 (Characterization of strong completeness)

Let L be a finitary lattice-disjunctive logic. TFAE:

1. L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$.
2. *Every non-trivial countable member of* $\mathbf{ALG}^{*}(\mathrm{L})_{\mathrm{RFSI}}$ *is embeddable into some member of* $\mathbb{K}$.
3. *Every countable member of* $\mathbf{ALG}^{*}(\mathrm{L})_{\mathrm{RSI}}$ *is embeddable into some member of* $\mathbb{K}$.

A technical lemma

Definition 5.36 (Directed set of formulae)

A set of formulae Ψ is *directed* if for each $\varphi, \psi \in \Psi$ there is $\chi \in \Psi$ such that both $\varphi \to \chi$ and $\psi \to \chi$ are provable in L (we call χ an *upper bound* of φ and ψ).

Lemma 5.37

Assume that L *is finitary and has the* $\mathrm{S}\mathbb{K}\mathrm{C}$*. Then for every set of formulae* Γ *and every directed set of formulae* Ψ *the following are equivalent:*

- $\Gamma \nvdash_{\mathrm{L}} \psi$ *for each* $\psi \in \Psi$*.*
- *There is a algebra* $\boldsymbol{A} \in \mathbb{K}$ *and an* $\boldsymbol{A}$*-evaluation* e *such that* $e[\Gamma] \subseteq F$ *and* $e[\Psi] \cap F = \emptyset$*.*

Proof of $1 \to 2$

Take a countable $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})_{\mathrm{RFSI}}$ with filter F. Consider a set of variables $\{v_a \mid a \in A\}$ and sets of formulae:

$$\Gamma = \{c(v_{a_1}, \dots, v_{a_n}) \leftrightarrow v_{c^{\boldsymbol{A}}(a_1,\dots,a_n)} \mid \langle c, n\rangle \in \mathcal{L} \text{ and } a_1, \dots, a_n \in A\},$$

$$\Psi = \{v_{a_1} \vee \dots \vee v_{a_n} \mid n \in \mathrm{N} \text{ and } a_1, \dots, a_n \in A \setminus F\}.$$

Ψ is directed and $\Gamma \nvdash_{\mathrm{L}} \psi$ for each $\psi \in \Psi$ (set $e(v_a) = a$: clearly $e[\Gamma] \subseteq F$ and if $a_1 \vee \dots \vee a_n \in F$, then as F is prime we have: $a_i \in F$ for some i—a contradiction).

Using Lemma 5.37 we get an algebra $\boldsymbol{B} \in \mathbb{K}$ with filter G and a $\boldsymbol{B}$-evaluation e st. $e[\Gamma] \subseteq G$ and $e(\psi) \notin G$ for each $\psi \in \Psi$.

Define homomorphism $f \colon A \to B$ as $f(a) = e(v_a)$. We show it is one-one: take $a, b \in A$ st. $a \neq b$ and w.l.o.g. $a \to^{\boldsymbol{A}} b \notin F$. Thus $f(a) \to^{\boldsymbol{B}} f(b) = e(v_a) \to^{\boldsymbol{B}} e(v_b) = e(v_{a \to^{\boldsymbol{A}} b}) \notin G$, i.e. $f(a) \neq f(b)$.

Suppose that for some Γ and φ we have $\Gamma \nvdash_{\mathrm{L}} \varphi$. Then, since L is finitary, by Theorem 5.10, there are $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ and e such that $e[\Gamma] \subseteq F$ and $e(\varphi) \notin F$. Let $\boldsymbol{B}$ be the countable subalgebra of $\boldsymbol{A}$ generated by $e[Fm_{\mathcal{L}}]$. Consider the submatrix $\langle \boldsymbol{B}, B \cap F\rangle \in \mathbf{MOD}^\ell(\mathrm{L})$. $\boldsymbol{B}$ is not necessarily subdirectly irreducible but it is representable as a subdirect product of a family of $\{\boldsymbol{C}_i \mid i \in I\} \subseteq \mathbf{ALG}^*(\mathrm{L})_{\mathrm{RSI}}$; let G_i be their corresponding filters and let α be the representation homomorphism. It is clear that $e[\Gamma] \subseteq B \cap F$ and $e(\varphi) \notin B \cap F$. There is some $j \in I$ such that $(\pi_j \circ \alpha)(e(\varphi)) \notin G_j$. $\boldsymbol{C}_j$ is a countable member of $\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RSI}}$, so by the assumption there is a matrix $\langle \boldsymbol{C}, G\rangle \in \mathbf{MOD}^\ell(\mathrm{L})$ with $\boldsymbol{C} \in \mathbb{K}$ and an embedding $f\colon \boldsymbol{C}_j \hookrightarrow \boldsymbol{C}$, and hence, using this model and the evaluation $f \circ \pi_j \circ \alpha \circ e$, we obtain $\Gamma \not\models_{\mathbb{K}} \varphi$.

Theorem 5.38 (Characterization of finite strong completeness)

If L is finitary, then the following are equivalent:

1. L *satisfies the* $\mathrm{FS}\mathbb{K}\mathrm{C}$.
2. *Every* L*-chain in embeddable into* $\mathbf{P}_{\mathrm{U}}(\mathbb{K})$.

Corollary 5.39

Assume that L *is finitary and enjoys the* $\mathrm{FS}\mathbb{K}\mathrm{C}$. *Then* L *has the* $\mathbf{SP}_{\mathrm{U}}(\mathbb{K})\mathrm{C}$.

Characterization of finite strong completeness – 2

A finite subset X of an $\mathcal{L}$-algebra $\boldsymbol{A}$ is partially embeddable into an $\mathcal{L}$-algebra $\boldsymbol{B}$ if there is a one-to-one mapping $f\colon X \to B$ st. for each $\langle c, n\rangle \in \mathcal{L}$ and each $a_1, \dots, a_n \in X$ if $c^{\boldsymbol{A}}(a_1, \dots, a_n) \in X$, then $f(c^{\boldsymbol{A}}(a_1, \dots, a_n)) = c^{\boldsymbol{B}}(f(a_1), \dots, f(a_n))$.

A class $\mathbb{K}$ is *partially embeddable into* $\mathbb{K}'$ if every finite subset of every member of $\mathbb{K}$ is partially embeddable into a member of $\mathbb{K}'$

Theorem 5.40

Let L *be a finitary lattice-disjunctive logic with a* finite *language* $\mathcal{L}$*. Then the following are equivalent:*

1. L *has the* $\mathrm{FS}\mathbb{K}\mathrm{C}$*.*
2. *Every non-trivial member of* $\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RFSI}}$ *is partially embeddable into* $\mathbb{K}$*.*
3. *Every countable member of* $\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RSI}}$ *is partially embeddable into* $\mathbb{K}$*.*

Take a $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})_{\mathrm{RFSI}}$ with filter F and a finite set $B \subseteq A$ and define $B' = B \cup \{a \to^{\boldsymbol{A}} b \mid a, b \in B\}$.
Consider a set of variables $\{v_a \mid a \in B'\}$, a formula φ and set Γ:

$$\varphi = \bigvee_{a \in B' \setminus F} v_a$$

$$\Gamma = \{c(v_{a_1}, \dots, v_{a_n}) \leftrightarrow v_{c^{\boldsymbol{A}}(a_1,\dots,a_n)} \mid \langle c, n\rangle \in \mathcal{L} \text{ and } a_1, \dots, a_n, c^{\boldsymbol{A}}(a_1, \dots, a_n) \in B'\}.$$

Observe that Γ is finite and $\Gamma \nvdash_{\mathrm{L}} \varphi$.
Thus, by the $\mathrm{FS}\mathbb{K}\mathrm{C}$, there is $\boldsymbol{C} \in \mathbb{K}$, with filter G, and a $\boldsymbol{C}$-evaluation e such that $e[\Gamma] \subseteq G$ and $e(\varphi) \notin G$.
Define a partial homomorphism $f \colon B \to C$ as $f(a) = e(v_a)$. We show it is one-one in the same way as before.

Proposition 5.41

Assume that L *is finitary and lattice-disjunctive. TFAE:*

1. L *enjoys the* $\mathrm{S}\mathcal{F}\mathrm{C}$.
2. *All* L*-chains are finite.*
3. *There is* $n \in$ N *st. each* L*-chain has at most* n *elements.*
4. *There is* $n \in$ N *st.* $\vdash_{\mathrm{L}} \bigvee_{i<n}(x_i \to x_{i+1})$.

Proof.

$1 \to 2$: From Theorem 5.35 we know that every countable L-chain is embeddable into some member of $\mathcal{F}$, thus there are no infinite countable L-chains and so by the downward Löwenheim–Skolem Theorem there are no infinite chains.
$2 \to 3$: If all the algebras in $\mathbf{ALG}^*(\mathrm{L})$ are finite then there must a bound for their length, because otherwise by means of an ultraproduct we could build an infinite one. □

Proposition 5.41

Assume that L *is finitary and lattice-disjunctive. TFAE:*

1. L *enjoys the* $\mathrm{S}\mathcal{F}\mathrm{C}$.
2. *All* L*-chains are finite.*
3. *There is* $n \in$ N *st. each* L*-chain has at most* n *elements.*
4. *There is* $n \in$ N *st.* $\vdash_{\mathrm{L}} \bigvee_{i<n}(x_i \to x_{i+1})$.

Proof.

$3 \to 4$: Take an arbitrary L-chain $\boldsymbol{A}$, with filter F, and elements $a_0, \dots, a_n \in A$. Since A has at most n elements it is impossible that $a_0 > a_1 > \dots > a_n$, thus there is some k such that $a_k \leq a_{k+1}$, i.e. $a_k \to^{\boldsymbol{A}} a_{k+1} \in F$, and hence it satisfies the formula.
$4 \to 2$: Take an L-chain $\boldsymbol{A}$, with filter F and elements $a_0, \dots, a_n \in A$ st. $a_0 > a_1 > \dots > a_n$. Then $a_i \to^{\boldsymbol{A}} a_{i+1} \notin F$, for every $i < n$, and as F is $\vee$-prime we get $\not\models_{\boldsymbol{A}} \bigvee_{i<n}(x_i \to x_{i+1})$. □

Proposition 5.41

Assume that L *is finitary and lattice-disjunctive. TFAE:*

1. L *enjoys the* $\mathrm{S}\mathcal{F}\mathrm{C}$.
2. *All* L*-chains are finite.*
3. *There is* $n \in$ N *st. each* L*-chain has at most* n *elements.*
4. *There is* $n \in$ N *st.* $\vdash_{\mathrm{L}} \bigvee_{i<n}(x_i \to x_{i+1})$.

Corollary 5.42

For a finitary lattice-disjunctive logic L *and a natural number* n*, the axiomatic extension* $\mathrm{L}_{\leq n}$ *obtained by adding the schema* $\bigvee_{i<n}(x_i \to x_{i+1})$*, is a semilinear logic which is strongly complete with respect the* L*-chains of length less than or equal to* n.

Summary: Abstract Algebraic Logic

In this course we have tried to demonstrate that AAL provides powerful tools to:

- understand the several ways by which a logic can be given an algebraic semantics
- build a general and abstract theory of non-classical logics based on their relation to algebras
- understand the rôle of connectives in (non-)classical logics
- classify non-classical logics
- find general results connecting logical and algebraic properties (bridge theorems)
- generalize properties from syntax to semantics (transfer theorems)
- advance the study of particular (families of) non-classical logics by using the abstract notions and results