Logic, Algebra, and Implication – Lesson 1

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Logic(s)

Logic studies the notion of logical consequence. There are many kinds of logical consequence, i.e. many different logics:

1. Classical logic
2. Non-classical logics:
    - Modal logics
    - Intuitionistic logic
    - Superintuitionistic logics
    - Linear logics
    - Fuzzy logics
    - Relevance logics
    - Substructural logics
    - Paraconsistent logics
    - Dynamic logics
    - Non-monotonic logics

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Algebraic Logic

Algebraic Logic is the subdiscipline of Mathematical Logic which studies logical systems (classical and non-classical) by using tools from Universal Algebra.

Logic	Algebraic counterpart
Classical logic	Boolean algebras
Modal logics	Modal algebras
Intuitionistic logic	Heyting algebras
Linear logics	Commutative residuated lattices
Fuzzy logics	Semilinear residuated lattices
Relevance logics	Commutative contractive residuated lattices
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Universal Algebra is the field of Mathematics which studies algebraic structures.

Abstract Algebraic Logic

AAL is the evolution of Algebraic Logic that wants to:

- understand the several ways by which a logic can be given an algebraic semantics
- build a general and abstract theory of non-classical logics based on their relation to algebras
- understand the rôle of connectives in (non-)classical logics.
- classify non-classical logics
- find general results connecting logical and algebraic properties (bridge theorems)
- generalize properties from syntax to semantics (transfer theorems)
- advance the study of particular (families of) non-classical logics by using the abstract notions and results

It works best, by far, when restricted to propositional logics.

Outline of the course

- L1: Basic notions of algebraic logic. Weakly implicative logics. Completeness theorem w.r.t. the class of all models and w.r.t. reduced models. [C.N. and P.C.]
- L2: Advanced semantic notions. Completeness w.r.t. RFSI models. Algebraically implicative logics. [P.C.]
- L3: Core theory of AAL. Leibniz hierarchy. Protoalgebraic, equivalential, and (weakly) algebraizable logics. Bridge theorems. [C.N.]

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Basic syntactical notions – 1

Propositional language: a countable type $\mathcal{L}$, i.e. a function $ar\colon C_{\mathcal{L}} \to \mathrm{N}$, where $C_{\mathcal{L}}$ is a countable set of symbols called connectives, giving for each one its arity. Nullary connectives are also called truth-constants. We write $\langle c, n\rangle \in \mathcal{L}$ whenever $c \in C_{\mathcal{L}}$ and $ar(c) = n$.

Formulae: Let Var be a fixed infinite countable set of symbols called variables. The set $Fm_{\mathcal{L}}$ of formulae in $\mathcal{L}$ is the least set containing Var and closed under connectives of $\mathcal{L}$, i.e. for each $\langle c, n\rangle \in \mathcal{L}$ and every $\varphi_1, \dots, \varphi_n \in Fm_{\mathcal{L}}$, $c(\varphi_1, \dots, \varphi_n)$ is a formula.

Substitution: a mapping $\sigma\colon Fm_{\mathcal{L}} \to Fm_{\mathcal{L}}$, such that $\sigma(c(\varphi_1, \dots, \varphi_n)) = c(\sigma(\varphi_1), \dots, \sigma(\varphi_n))$ holds for each $\langle c, n\rangle \in \mathcal{L}$ and every $\varphi_1, \dots, \varphi_n \in Fm_{\mathcal{L}}$.

Consecution: a pair $\Gamma \rhd \varphi$, where $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$.

Basic syntactical notions – 2

A set L of consecutions can be seen as relation between sets of formulae and formulae. We write '$\Gamma \vdash_{\mathrm{L}} \varphi$' instead of '$\Gamma \rhd \varphi \in \mathrm{L}$'.

Definition 1.1

A set L of consecutions in $\mathcal{L}$ is called a logic in $\mathcal{L}$ whenever

- If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathrm{L}} \varphi$. (Reflexivity)
- If $\Delta \vdash_{\mathrm{L}} \psi$ for each $\psi \in \Gamma$ and $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Cut)
- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each substitution σ. (Structurality)

Observe that reflexivity and cut entail:

- If $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Monotonicity)

The least logic Dumb is described as:

$$\Gamma \vdash_{\mathrm{Dumb}} \varphi \qquad \text{iff} \qquad \varphi \in \Gamma.$$

Basic syntactical notions – 3

Theorem: a consequence of the empty set (note that Dumb has no theorems).

Inconsistent logic Inc: the set all consecutions (equivalently: a logic where all formulae are theorems).

Almost Inconsistent logic AInc: the maximum logic without theorems (note that $\Gamma, \varphi \vdash_{\mathrm{AInc}} \psi$).

Theory: a set of formulae T such that if $T \vdash_{\mathrm{L}} \varphi$ then $\varphi \in T$. By $\mathrm{Th}(\mathrm{L})$ we denote the set of all theories of L.

Note that

- $\mathrm{Th}(\mathrm{L})$ can be seen as a closure system. By $\mathrm{Th}_{\mathrm{L}}(\Gamma)$ we denote the theory generated in $\mathrm{Th}(\mathrm{L})$ by Γ (i.e., the intersection of all theories containing Γ).
- $\mathrm{Th}_{\mathrm{L}}(\Gamma) = \{\varphi \in Fm_{\mathcal{L}} \mid \Gamma \vdash_{\mathrm{L}} \varphi\}$.
- The set of all theorems is the least theory and it is generated by the empty set.

Basic syntactical notions – 4

Axiomatic system: a set $\mathcal{AS}$ of consecutions closed under substitutions. An element $\Gamma \rhd \varphi$ is an

- axiom if $\Gamma = \emptyset$,
- finitary deduction rule if Γ is a finite,
- infinitary deduction rule otherwise.

An axiomatic system is finitary if all its rules are finitary.

Proof: a proof of a formula φ from a set of formulae Γ in $\mathcal{AS}$ is a well-founded tree labeled by formulae such that

- its root is labeled by φ and leaves by axioms of $\mathcal{AS}$ or elements of Γ and
- if a node is labeled by ψ and $\Delta \neq \emptyset$ is the set of labels of its preceding nodes, then $\Delta \rhd \psi \in \mathcal{AS}$.

We write $\Gamma \vdash_{\mathcal{AS}} \varphi$ if there is a proof of φ from Γ in $\mathcal{AS}$.

Basic syntactical notions – 5

Lemma 1.2

Let $\mathcal{AS}$ be an axiomatic system. Then $\vdash_{\mathcal{AS}}$ is the least logic containing $\mathcal{AS}$.

Presentation: We say that $\mathcal{AS}$ is an axiomatic system for (or a presentation of) the logic L if $\mathrm{L} = \vdash_{\mathcal{AS}}$. A logic is said to be finitary if it has some finitary presentation.

Lemma 1.3

A logic L is finitary iff for each set of formulae $\Gamma \cup \{\varphi\}$ we have: if $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathrm{L}} \varphi$.

Note that $\mathrm{Inc}, \mathrm{AInc}, \mathrm{Dumb}$ are finitary because:

Inc	is axiomatized by	axioms $\{\varphi \mid \varphi \in Fm_{\mathcal{L}}\}$
AInc	is axiomatized by	unary rules $\{\varphi \rhd \psi \mid \varphi, \psi \in Fm_{\mathcal{L}}\}$
Dumb	is axiomatized by	by the empty set

Examples: classical logic CPC and logic BCI

Finitary axiomatic system for CPC in $\mathcal{L}_{\mathrm{CPC}} = \{\to, \neg\}$

A1 $\varphi \to (\psi \to \varphi)$

A2 $(\chi \to (\varphi \to \psi)) \to ((\chi \to \varphi) \to (\chi \to \psi))$

A3 $(\neg\psi \to \neg\varphi)) \to (\varphi \to \psi)$

MP $\varphi, \varphi \to \psi \rhd \psi$

Finitary axiomatic system for BCI in $\mathcal{L}_{\mathrm{BCI}} = \{\to\}$

B $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

C $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$

I $\varphi \to \varphi$

MP $\varphi, \varphi \to \psi \rhd \psi$

$\mathcal{L}$-algebra: $\boldsymbol{A} = \langle A, \langle c^{\boldsymbol{A}} \mid c \in C_{\mathcal{L}} \rangle\rangle$, where $A \neq \emptyset$ (universe) and $c^{\boldsymbol{A}} \colon A^n \to A$ for each $\langle c, n\rangle \in \mathcal{L}$.

Algebra of formulae: the algebra $\boldsymbol{Fm}_{\mathcal{L}}$ with domain $Fm_{\mathcal{L}}$ and operations $c^{\boldsymbol{Fm}_{\mathcal{L}}}$ for each $\langle c, n\rangle \in \mathcal{L}$ defined as:

$$c^{\boldsymbol{Fm}_{\mathcal{L}}}(\varphi_1, \dots, \varphi_n) = c(\varphi_1, \dots, \varphi_n).$$

$\boldsymbol{Fm}_{\mathcal{L}}$ if the absolutely free algebra in language $\mathcal{L}$ with generators Var.

Homomorphism of algebras: a mapping $f \colon A \to B$ such that for every $\langle c, n\rangle \in \mathcal{L}$ and every $a_1, \dots, a_n \in A$,

$$f(c^{\boldsymbol{A}}(a_1, \dots, a_n)) = c^{\boldsymbol{B}}(f(a_1), \dots, f(a_n)).$$

Note that substitutions are exactly endomorphisms of $\boldsymbol{Fm}_{\mathcal{L}}$.

Basic semantical notions – 2

$\mathcal{L}$-matrix: a pair $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra called the algebraic reduct of $\mathbf{A}$, and F is a subset of A called the filter of $\mathbf{A}$. The elements of F are called designated elements of $\mathbf{A}$.

A matrix $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ is

- trivial if $F = A$.
- finite if A is finite.
- Lindenbaum if $\boldsymbol{A} = \boldsymbol{Fm}_{\mathcal{L}}$.

$\boldsymbol{A}$-evaluation: a homomorphism from $\boldsymbol{Fm}_{\mathcal{L}}$ to $\boldsymbol{A}$, i.e. a mapping $e\colon Fm_{\mathcal{L}} \to A$, such that for each $\langle c, n \rangle \in \mathcal{L}$ and each n-tuple of formulae $\varphi_1, \ldots, \varphi_n$ we have:

$$e(c(\varphi_1, \ldots, \varphi_n)) = c^{\boldsymbol{A}}(e(\varphi_1), \ldots, e(\varphi_n)).$$

Basic semantical notions – 3

Semantical consequence: A formula φ is a semantical consequence of a set Γ of formulae w.r.t. a class $\mathbb{K}$ of $\mathcal{L}$-matrices if for each $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e, we have $e(\varphi) \in F$ whenever $e[\Gamma] \subseteq F$; we denote it by $\Gamma \models_{\mathbb{K}} \varphi$.

Lemma 1.4

Let $\mathbb{K}$ a class of $\mathcal{L}$-matrices. Then $\models_{\mathbb{K}}$ is a logic in $\mathcal{L}$. Furthermore if $\mathbb{K}$ is a finite class of finite matrices, then the logic $\models_{\mathbb{K}}$ is finitary.

L-matrix: Let L be a logic in $\mathcal{L}$ and $\mathbf{A}$ an $\mathcal{L}$-matrix. We say that $\mathbf{A}$ is an L-matrix if $\mathrm{L} \subseteq \models_{\mathbf{A}}$. We denote the class of L-matrices by $\mathbf{MOD}(\mathrm{L})$.

Basic semantical notions – 4

Logical filter: Given a logic L in $\mathcal{L}$ and an $\mathcal{L}$-algebra $\boldsymbol{A}$, a subset $F \subseteq A$ is an L-*filter* if $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$. By $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ we denote the set of all L-filters over $\boldsymbol{A}$.

$\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is a closure system and can be given a lattice structure by defining for any $F, G \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, $F \wedge G = F \cap G$ and $F \vee G = \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(F \cup G)$.

Generated filter: Given a set $X \subseteq A$, the logical filter generated by X is $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X) = \bigcap\{F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A}) \mid X \subseteq F\}$.

$$\mathcal{F}i_{\mathrm{Dumb}}(\boldsymbol{A}) = \mathcal{P}(A) \qquad \mathcal{F}i_{\mathrm{AInc}}(\boldsymbol{A}) = \{\emptyset, A\} \qquad \mathcal{F}i_{\mathrm{Inc}}(\boldsymbol{A}) = \{A\}$$

Examples: classical logic CPC and logic BCI

1. Classical logic: Let $\boldsymbol{A}$ be a Boolean algebra. Then $\mathcal{F}i_{\mathrm{CPC}}(\boldsymbol{A})$ is the class of lattice filters on $\boldsymbol{A}$, in particular for the two-valued Boolean algebra $\mathbf{2}$:

$$\mathcal{F}i_{\mathrm{CPC}}(\mathbf{2}) = \{\{1\}, \{0, 1\}\}.$$

2. The logic BCI: By $\boldsymbol{M}$ we denote the $\mathcal{L}_{\mathrm{BCI}}$-algebra with domain $\{\bot, \top, t, f\}$ and:

$\to^{\boldsymbol{M}}$	$\top$	t	f	$\bot$
$\top$	$\top$	$\bot$	$\bot$	$\bot$
t	$\top$	t	f	$\bot$
f	$\top$	$\bot$	t	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$

We can easily compute that:

$$\mathcal{F}i_{\mathrm{BCI}}(\boldsymbol{M}) = \{\{t, \top\}, \{t, f, \top\}, M\}.$$

The first completeness theorem

Proposition 1.5

For any logic L *in a language* $\mathcal{L}$, $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{Fm}_{\mathcal{L}}) = \mathrm{Th}(\mathrm{L})$.

Theorem 1.6

Let L *be a logic. Then for each set* Γ *of formulae and each formula* φ *the following holds:* $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\Gamma \models_{\mathbf{MOD}(\mathrm{L})} \varphi$.

Completeness theorem for classical logic

- Suppose that $T \in \mathrm{Th}(\mathrm{CPC})$ and $\varphi \notin T$ ($T \nvdash_{\mathrm{CPC}} \varphi$). We want to show that $T \not\models \varphi$ in some meaningful semantics.
- $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle} \varphi$. 1st completeness theorem
- $\langle \alpha, \beta\rangle \in \Omega(T)$ iff $\alpha \leftrightarrow \beta \in T$ (congruence relation on $\boldsymbol{Fm}_{\mathcal{L}}$ compatible with T: if $\alpha \in T$ and $\langle \alpha, \beta\rangle \in \Omega(T)$, then $\beta \in T$).
- Lindenbaum–Tarski algebra: $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T)$ is a Boolean algebra and $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}/\Omega(T), T/\Omega(T)\rangle} \varphi$. 2nd completeness theorem
- Lindenbaum Lemma: If $\varphi \notin T$, then there is a maximal consistent $T' \in \mathrm{Th}(\mathrm{CPC})$ such that $T \subseteq T'$ and $\varphi \notin T'$.
- $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T') \cong \mathbf{2}$ (subdirectly irreducible Boolean algebra) and $T \not\models_{\langle \mathbf{2}, \{1\}\rangle} \varphi$. 3rd completeness theorem

Definition 1.7

A logic L in a language $\mathcal{L}$ is weakly implicative if there is a binary connective $\to$ (primitive or definable) such that:

(R)	$\vdash_{\mathrm{L}} \varphi \to \varphi$
(MP)	$\varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi$
(T)	$\varphi \to \psi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \to \chi$
(sCng)	$\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} c(\chi_1, \dots, \chi_i, \varphi, \dots, \chi_n) \to c(\chi_1, \dots, \chi_i, \psi, \dots, \chi_n)$

for each $\langle c, n\rangle \in \mathcal{L}$ and each $0 \leq i < n$.

Examples

The following logics are weakly implicative:

- CPC, BCI, and Inc
- global modal logics
- intuitionistic and superintuitionistic logic
- linear logic and its variants
- (the most of) fuzzy logics
- substructural logics

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The following logics are not weakly implicative:

- local modal logics
- Dumb, AInc, and the conjunction-disjunction fragment of classical logic as they have no theorems
- logics of ortholattices

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Congruence Property

Conventions

Unless said otherwise, L is a weakly implicative in a language $\mathcal{L}$ with an implication $\to$. We write:

- $\varphi \leftrightarrow \psi$ instead of $\{\varphi \to \psi, \psi, \to \varphi\}$
- $\Gamma \vdash \Delta$ whenever $\Gamma \vdash \chi$ for each $\chi \in \Delta$
- $\Gamma \dashv\vdash \Delta$ whenever $\Gamma \vdash \Delta$ and $\Delta \vdash \Gamma$.

Theorem 1.8

Let φ, ψ, χ be formulae. Then:

- $\vdash_{\mathrm{L}} \varphi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \psi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \delta, \delta \leftrightarrow \psi \vdash_{\mathrm{L}} \varphi \leftrightarrow \psi$
- $\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \chi \leftrightarrow \hat{\chi}$, *where $\hat{\chi}$ is obtained from χ by replacing some occurrences of φ in χ by ψ.*

Corollaries

Corollary 1.9

Let $\to'$ *be a connective satisfying* (R), (MP), (T), (sCng). *Then*

$$\varphi \leftrightarrow \psi \dashv\vdash_{\mathrm{L}} \varphi \leftrightarrow' \psi.$$

Proof.

Consider formulae $\chi = \varphi \to' \varphi$ and $\hat{\chi} = \varphi \to' \psi$ and the proof

$$\varphi \leftrightarrow \psi, \ldots, \ (\varphi \to' \varphi) \to (\varphi \to' \psi), \ \varphi \to' \varphi, \ \varphi \to' \psi.$$

Analogously for $\hat{\chi} = \psi \to' \varphi$ we can write

$$\varphi \leftrightarrow \psi, \ldots, \ (\varphi \to' \varphi) \to (\psi \to' \varphi), \ \varphi \to' \varphi, \ \psi \to' \varphi.$$

□

So we have shown $\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \varphi \leftrightarrow' \psi$. The reverse direction is fully analogous.

Corollaries

Corollary 1.9

Let $\to'$ *be a connective satisfying* (R), (MP), (T), (sCng). *Then*

$$\varphi \leftrightarrow \psi \dashv\vdash_{\mathrm{L}} \varphi \leftrightarrow' \psi.$$

Corollary 1.10

Local modal logic T^l *is not weakly implicative.*

Proof.

Let $\to'$ be a 'good' implication in T^l. Then $\to'$ (along with classical implication $\to$) is an implication in global T^g. Thus $\top \to \varphi, \varphi \to \top \vdash_{\mathrm{T}^g} \top \leftrightarrow' \varphi$ and so $\Box^n(\top \to \varphi) \vdash_{\mathrm{T}^l} \top \leftrightarrow' \varphi$ for some n. Consider the proof in T^l: $\Box^n\varphi, \ldots, \Box^n(\top \to \varphi), \ldots,$ $\top \leftrightarrow' \varphi, \ldots, \Box^{n+1}\top \leftrightarrow' \Box^{n+1}\varphi, \Box^{n+1}\top, \Box^{n+1}\varphi$, a contradiction. $\Box$

Lindenbaum–Tarski matrix

Let L be a weakly implicative logic in $\mathcal{L}$ and $T \in \mathit{Th}(\mathrm{L})$. For every formula φ, we define the set

$$[\varphi]_T = \{\psi \in \mathit{Fm}_\mathcal{L} \mid \varphi \leftrightarrow \psi \subseteq T\}.$$

The Lindenbaum–Tarski matrix with respect to L and T, $\mathbf{LindT}_T$, has the filter $\{[\varphi]_T \mid \varphi \in T\}$ and algebraic reduct with the domain $\{[\varphi]_T \mid \varphi \in \mathit{Fm}_\mathcal{L}\}$ and operations:

$$c^{\mathbf{LindT}_T}([\varphi_1]_T, \dots, [\varphi_n]_T) = [c(\varphi_1, \dots, \varphi_n)]_T$$

What are Lindenbaum–Tarski matrices in general?
Recall that Lindenbaum matrices have domain $\mathit{Fm}_\mathcal{L}$ and

$$\mathcal{F}i_{\mathrm{L}}(\boldsymbol{Fm}_{\mathcal{L}}) = \mathrm{Th}(\mathrm{L}).$$

Definition 1.11

Let $\mathbf{A} = \langle A, F \rangle$ be an L-matrix. We define:

- the Leibniz congruence $\Omega_{\boldsymbol{A}}(F)$ of $\mathbf{A}$ as

$$\langle a, b \rangle \in \Omega_{\boldsymbol{A}}(F) \qquad \text{iff} \qquad a \leftrightarrow^{\boldsymbol{A}} b \subseteq F$$

- the matrix preorder $\leq_{\mathbf{A}}$ of $\mathbf{A}$ as

$$a \leq_{\mathbf{A}} b \qquad \text{iff} \qquad a \to^{\boldsymbol{A}} b \in F$$

Note that

$$\langle a, b \rangle \in \Omega_{\boldsymbol{A}}(F) \qquad \text{iff} \qquad a \leq_{\mathbf{A}} b \;\text{ and }\; b \leq_{\mathbf{A}} a.$$

Leibniz congruence – 2

A congruence θ of $\boldsymbol{A}$ is logical in a matrix $\langle \boldsymbol{A}, F\rangle$ if for each $a, b \in A$ if $a \in F$ and $\langle a, b\rangle \in \theta$, then $b \in F$.

Theorem 1.12

Let $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ be an L-matrix. Then:

1. $\leq_{\mathbf{A}}$ *is a preorder.*
2. $\Omega_{\boldsymbol{A}}(F)$ *is the largest logical congruence of* $\boldsymbol{A}$.
3. $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$ *iff for each* $\chi \in Fm_{\mathcal{L}}$ *and each* $\mathbf{A}$*-evaluation* e*:*

$$e[p{\to}a](\chi) \in F \qquad \textit{iff} \qquad e[p{\to}b](\chi) \in F.$$

Algebraic counterpart

Definition 1.13

A L-matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ is reduced, $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ in symbols, if $\Omega_{\boldsymbol{A}}(F)$ is the identity relation $\mathrm{Id}_{\boldsymbol{A}}$ (iff $\leq_{\mathbf{A}}$ is an order).

An algebra $\boldsymbol{A}$ is L-algebra, $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})$ in symbols, if there a set $F \subseteq A$ such that $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$.

Note that $\Omega_{\boldsymbol{A}}(A) = A^2$. Thus from $\mathcal{F}i_{\mathrm{Inc}}(\boldsymbol{A}) = \{A\}$ we obtain:

$$\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{Inc}) \qquad \text{iff} \qquad A \text{ is a singleton}$$

Examples: classical logic CPC and logic BCI

1. Classical logic: it is easy to see that

$$\Omega_{\mathbf{2}}(\{1\}) = \mathrm{Id}_{\mathbf{2}} \qquad \text{i.e., } \mathbf{2} \in \mathbf{ALG}^*(\mathrm{CPC}).$$

Actually for any Boolean algebra $\boldsymbol{A}$:

$$\Omega_{\boldsymbol{A}}(\{1\}) = \mathrm{Id}_{\boldsymbol{A}} \qquad \text{i.e., } \boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{CPC}).$$

But: $\Omega_{\mathbf{4}}(\{a,1\}) = \mathrm{Id}_{\boldsymbol{A}} \cup \{\langle 1, a\rangle, \langle 0, \neg a\rangle\}$ i.e. $\langle \mathbf{4}, \{a,1\}\rangle \notin \mathbf{MOD}^*(\mathrm{CPC})$.

2. The logic BCI: recall the algebra $\boldsymbol{M}$ defined via:

$\to^{\boldsymbol{M}}$	$\top$	t	f	$\bot$
$\top$	$\top$	$\bot$	$\bot$	$\bot$
t	$\top$	t	f	$\bot$
f	$\top$	$\bot$	t	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$

We can easily show that:

$$\Omega_{\boldsymbol{M}}(\{t,\top\}) = \Omega_{\boldsymbol{M}}(\{t,f,\top\}) = \mathrm{Id}_{\boldsymbol{M}} \qquad \text{i.e.} \quad \boldsymbol{M} \in \mathbf{ALG}^*(\mathrm{BCI}).$$

Factorizing matrices

Let us take $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$. We write:

- $\boldsymbol{A}^*$ for $\boldsymbol{A}/\Omega_{\boldsymbol{A}}(F)$
- $[\cdot]_F$ for the canonical epimorphism of $\boldsymbol{A}$ onto $\boldsymbol{A}^*$ defined as:

$$[a]_F = \{b \in A \mid \langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)\}$$

- $\mathbf{A}^*$ for $\langle \boldsymbol{A}^*, [F]_F\rangle$.

Theorem 1.14

Let T be a theory, $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$, and $a, b \in A$. Then:

1. $\mathbf{LindT}_T = \langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle^*$
2. $a \in F$ *iff* $[a]_F \in [F]_F$.
3. $[a]_F \leq_{\mathbf{A}^*} [b]_F$ *iff* $a \to^{\boldsymbol{A}} b \in F$.
4. $\mathbf{A}^* \in \mathbf{MOD}^*(\mathrm{L})$.

The second completeness theorem

Theorem 1.15

Let L *be a weakly implicative logic. Then for any set* Γ *of formulae and any formula* φ *the following holds:*

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi.$$

Proof.

Using just the soundness part of the FCT it remains to prove:

$$\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi \quad \text{implies} \quad \Gamma \vdash_{\mathrm{L}} \varphi.$$

Assume that $\Gamma \nvdash_{\mathrm{L}} \varphi$ and take the theory $T = \mathrm{Th}_{\mathrm{L}}(\Gamma)$. Then

- $\mathbf{LindT}_T = \langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle^* \in \mathbf{MOD}^*(\mathrm{L})$ and for $\mathbf{LindT}_T$-evaluation $e(\psi) = [\psi]_T$ holds $e(\psi) \in [T]_T$ iff $\Gamma \vdash \psi$
- Thus $e[\Gamma] \subseteq e[T] = [T]_T$ and $e(\varphi) \notin [T]_T$ □

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