Logic, Algebra, and Implication – Lesson 1

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Logic studies the notion of **logical consequence**. There are many kinds of logical consequence, i.e. many different logics:

1. **Classical logic**
2. **Non-classical logics:**
   - Modal logics
   - Intuitionistic logic
   - Superintuitionistic logics
   - Linear logics
   - Fuzzy logics
   - Relevance logics
   - Substructural logics
   - Paraconsistent logics
   - Dynamic logics
   - Non-monotonic logics
   - ...
**Algebraic Logic** is the subdiscipline of Mathematical Logic which studies logical systems (classical and non-classical) by using tools from Universal Algebra.

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**Universal Algebra** is the field of Mathematics which studies algebraic structures.
Abstract Algebraic Logic

AAL is the evolution of Algebraic Logic that wants to:

- understand the several ways by which a logic can be given an algebraic semantics
- build a general and abstract theory of non-classical logics based on their relation to algebras
- understand the rôle of connectives in (non-)classical logics.
- classify non-classical logics
- find general results connecting logical and algebraic properties (bridge theorems)
- generalize properties from syntax to semantics (transfer theorems)
- advance the study of particular (families of) non-classical logics by using the abstract notions and results

It works best, by far, when restricted to propositional logics.
Outline of the course

L1: Basic notions of algebraic logic. Weakly implicative logics. Completeness theorem w.r.t. the class of all models and w.r.t. reduced models. [C.N. and P.C.]

L2: Advanced semantic notions. Completeness w.r.t. RFSI models. Algebraically implicative logics. [P.C.]

L3: Core theory of AAL. Leibniz hierarchy. Protoalgebraic, equivalential, and (weakly) algebraizable logics. Bridge theorems. [C.N.]

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Bibliography – 1


- P. Cintula, C. Noguera. The proof by cases property and its variants in structural consequence relations. To appear in *Studia Logica*.

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Propositional language: a countable type $\mathcal{L}$, i.e. a function $ar: C_{\mathcal{L}} \to \mathbb{N}$, where $C_{\mathcal{L}}$ is a countable set of symbols called connectives, giving for each one its arity. Nullary connectives are also called truth-constants. We write $\langle c, n \rangle \in \mathcal{L}$ whenever $c \in C_{\mathcal{L}}$ and $ar(c) = n$.

Formulae: Let $\text{Var}$ be a fixed infinite countable set of symbols called variables. The set $Fm_{\mathcal{L}}$ of formulae in $\mathcal{L}$ is the least set containing $\text{Var}$ and closed under connectives of $\mathcal{L}$, i.e. for each $\langle c, n \rangle \in \mathcal{L}$ and every $\varphi_1, \ldots, \varphi_n \in Fm_{\mathcal{L}}$, $c(\varphi_1, \ldots, \varphi_n)$ is a formula.

Substitution: a mapping $\sigma: Fm_{\mathcal{L}} \to Fm_{\mathcal{L}}$, such that $\sigma(c(\varphi_1, \ldots, \varphi_n)) = c(\sigma(\varphi_1), \ldots, \sigma(\varphi_n))$ holds for each $\langle c, n \rangle \in \mathcal{L}$ and every $\varphi_1, \ldots, \varphi_n \in Fm_{\mathcal{L}}$.

Consecution: a pair $\Gamma \triangleright \varphi$, where $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$. 
A set $L$ of consecutions can be seen as relation between sets of formulae and formulae. We write ‘$\Gamma \vdash_L \varphi$’ instead of ‘$\Gamma \supset \varphi \in L$’.

**Definition 1.1**

A set $L$ of consecutions in $\mathcal{L}$ is called a logic in $\mathcal{L}$ whenever

- If $\varphi \in \Gamma$, then $\Gamma \vdash_L \varphi$. (Reflexivity)
- If $\Delta \vdash_L \psi$ for each $\psi \in \Gamma$ and $\Gamma \vdash_L \varphi$, then $\Delta \vdash_L \varphi$. (Cut)
- If $\Gamma \vdash_L \varphi$, then $\sigma[\Gamma] \vdash_L \sigma(\varphi)$ for each substitution $\sigma$. (Structurality)

Observe that reflexivity and cut entail:

- If $\Gamma \vdash_L \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_L \varphi$. (Monotonicity)

The least logic $\text{Dumb}$ is described as:

$$\Gamma \vdash_{\text{Dumb}} \varphi \quad \text{iff} \quad \varphi \in \Gamma.$$
Theorem: a consequence of the empty set
(note that Dumb has no theorems).

Inconsistent logic $\text{Inc}$: the set all consecutions
(equivalently: a logic where all formulae are theorems).

Almost Inconsistent logic $\text{AInc}$: the maximum logic without
theorems
(note that $\Gamma, \varphi \vdash_{\text{AInc}} \psi$).

Theory: a set of formulae $T$ such that if $T \vdash_L \varphi$ then $\varphi \in T$. By
$\text{Th}(L)$ we denote the set of all theories of $L$.

Note that

- $\text{Th}(L)$ can be seen as a closure system. By $\text{Th}_L(\Gamma)$ we
denote the theory generated in $\text{Th}(L)$ by $\Gamma$ (i.e., the
intersection of all theories containing $\Gamma$).
- $\text{Th}_L(\Gamma) = \{ \varphi \in Fm_L \mid \Gamma \vdash_L \varphi \}$.
- The set of all theorems is the least theory and it is
generated by the empty set.
Axiomatic system: a set \( \mathcal{AS} \) of consecutions closed under substitutions. An element \( \Gamma \vdash \varphi \) is an
- axiom if \( \Gamma = \emptyset \),
- finitary deduction rule if \( \Gamma \) is a finite,
- infinitary deduction rule otherwise.

An axiomatic system is finitary if all its rules are finitary.

Proof: a proof of a formula \( \varphi \) from a set of formulae \( \Gamma \) in \( \mathcal{AS} \) is a well-founded tree labeled by formulae such that
- its root is labeled by \( \varphi \) and leaves by axioms of \( \mathcal{AS} \) or elements of \( \Gamma \) and
- if a node is labeled by \( \psi \) and \( \Delta \neq \emptyset \) is the set of labels of its preceding nodes, then \( \Delta \vdash \psi \in \mathcal{AS} \).

We write \( \Gamma \vdash_{\mathcal{AS}} \varphi \) if there is a proof of \( \varphi \) from \( \Gamma \) in \( \mathcal{AS} \).
Lemma 1.2

Let $\mathcal{AS}$ be an axiomatic system. Then $\vdash_{\mathcal{AS}}$ is the least logic containing $\mathcal{AS}$.

**Presentation:** We say that $\mathcal{AS}$ is an axiomatic system for (or a presentation of) the logic $L$ if $L = \vdash_{\mathcal{AS}}$. A logic is said to be finitary if it has some finitary presentation.

Lemma 1.3

A logic $L$ is finitary iff for each set of formulae $\Gamma \cup \{\varphi\}$ we have: if $\Gamma \vdash_L \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_L \varphi$.

Note that Inc, AInc, Dumb are finitary because:

- Inc is axiomatized by axioms $\{\varphi \mid \varphi \in Fm_L\}$
- AInc is axiomatized by unary rules $\{\varphi \triangleright \psi \mid \varphi, \psi \in Fm_L\}$
- Dumb is axiomatized by by the empty set
### Finitary axiomatic system for CPC in $\mathcal{L}_{\text{CPC}} = \{\to, \neg\}$

| A1 | $\varphi \to (\psi \to \varphi)$ |
| A2 | $(\chi \to (\varphi \to \psi)) \to ((\chi \to \varphi) \to (\chi \to \psi))$ |
| A3 | $(\neg \psi \to \neg \varphi)) \to (\varphi \to \psi)$ |
| MP | $\varphi, \varphi \to \psi \triangleright \psi$ |

### Finitary axiomatic system for BCI in $\mathcal{L}_{\text{BCI}} = \{\to\}$

| B | $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$ |
| C | $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$ |
| I | $\varphi \to \varphi$ |
| MP | $\varphi, \varphi \to \psi \triangleright \psi$ |
\[ \mathcal{L}\text{-algebra}: A = \langle A, \langle c^A \mid c \in C_\mathcal{L} \rangle \rangle, \] where \( A \neq \emptyset \) (universe) and 
\( c^A : A^n \to A \) for each \( \langle c, n \rangle \in \mathcal{L} \).

**Algebra of formulae:** the algebra \( Fm_\mathcal{L} \) with domain \( Fm_\mathcal{L} \) and operations \( c^{Fm_\mathcal{L}} \) for each \( \langle c, n \rangle \in \mathcal{L} \) defined as:
\[
 c^{Fm_\mathcal{L}}(\varphi_1, \ldots, \varphi_n) = c(\varphi_1, \ldots, \varphi_n).
\]

\( Fm_\mathcal{L} \) if the absolutely free algebra in language \( \mathcal{L} \) with generators \( \text{Var} \).

**Homomorphism of algebras:** a mapping \( f : A \to B \) such that for every \( \langle c, n \rangle \in \mathcal{L} \) and every \( a_1, \ldots, a_n \in A \),
\[
 f(\langle c^A(a_1, \ldots, a_n) \rangle) = c^B(\langle f(a_1), \ldots, f(a_n) \rangle).
\]

Note that substitutions are exactly endomorphisms of \( Fm_\mathcal{L} \).
\( \mathcal{L} \)-matrix: a pair \( A = \langle A, F \rangle \) where \( A \) is an \( \mathcal{L} \)-algebra called the algebraic reduct of \( A \), and \( F \) is a subset of \( A \) called the filter of \( A \). The elements of \( F \) are called designated elements of \( A \).

A matrix \( A = \langle A, F \rangle \) is

- **trivial** if \( F = A \).
- **finite** if \( A \) is finite.
- **Lindenbaum** if \( A = Fm_{\mathcal{L}} \).

\( A \)-evaluation: a homomorphism from \( Fm_{\mathcal{L}} \) to \( A \), i.e. a mapping \( e : Fm_{\mathcal{L}} \rightarrow A \), such that for each \( \langle c, n \rangle \in \mathcal{L} \) and each \( n \)-tuple of formulae \( \varphi_1, \ldots, \varphi_n \) we have:

\[
e(c(\varphi_1, \ldots, \varphi_n)) = c^A(e(\varphi_1), \ldots, e(\varphi_n)).
\]
Semantical consequence: A formula \( \varphi \) is a semantical consequence of a set \( \Gamma \) of formulae w.r.t. a class \( \mathbb{K} \) of \( \mathcal{L} \)-matrices if for each \( \langle A, F \rangle \in \mathbb{K} \) and each \( A \)-evaluation \( e \), we have \( e(\varphi) \in F \) whenever \( e[\Gamma] \subseteq F \); we denote it by \( \Gamma \models_{\mathbb{K}} \varphi \).

Lemma 1.4

Let \( \mathbb{K} \) a class of \( \mathcal{L} \)-matrices. Then \( \models_{\mathbb{K}} \) is a logic in \( \mathcal{L} \). Furthermore if \( \mathbb{K} \) is a finite class of finite matrices, then the logic \( \models_{\mathbb{K}} \) is finitary.

\( \mathcal{L} \)-matrix: Let \( \mathcal{L} \) be a logic in \( \mathcal{L} \) and \( A \) an \( \mathcal{L} \)-matrix. We say that \( A \) is an \( \mathcal{L} \)-matrix if \( L \subseteq \models_{A} \). We denote the class of \( \mathcal{L} \)-matrices by \( \text{MOD}(L) \).
Logical filter: Given a logic $L$ in $\mathcal{L}$ and an $\mathcal{L}$-algebra $A$, a subset $F \subseteq A$ is an $L$-filter if $\langle A, F \rangle \in \text{MOD}(L)$. By $\mathcal{F}_L(A)$ we denote the set of all $L$-filters over $A$.

$\mathcal{F}_L(A)$ is a closure system and can be given a lattice structure by defining for any $F, G \in \mathcal{F}_L(A)$, $F \land G = F \cap G$ and $F \lor G = \mathcal{F}_L^A(F \cup G)$.

Generated filter: Given a set $X \subseteq A$, the logical filter generated by $X$ is $\mathcal{F}_L^A(X) = \bigcap\{F \in \mathcal{F}_L(A) \mid X \subseteq F\}$.

$\mathcal{F}_\text{Dumb}(A) = \mathcal{P}(A)$, $\mathcal{F}_\text{AInc}(A) = \{\emptyset, A\}$, $\mathcal{F}_\text{Inc}(A) = \{A\}$
1. Classical logic: Let $A$ be a Boolean algebra. Then $\mathcal{F}_i_{\text{CPC}}(A)$ is the class of lattice filters on $A$, in particular for the two-valued Boolean algebra $2$:

$$\mathcal{F}_i_{\text{CPC}}(2) = \{\{1\}, \{0, 1\}\}.$$

2. The logic BCI: By $M$ we denote the $\mathcal{L}_{\text{BCI}}$-algebra with domain \{\bot, \top, t, f\} and:

\[
\begin{array}{cccc}
\rightarrow^M & \top & t & f & \bot \\
\top & \top & \bot & \bot & \bot \\
 t & \top & t & f & \bot \\
 f & \top & \bot & t & \bot \\
 \bot & \top & \top & \top & \top \\
\end{array}
\]

We can easily compute that:

$$\mathcal{F}_i_{\text{BCI}}(M) = \{\{t, \top\}, \{t, f, \top\}, M\}.$$
Proposition 1.5
For any logic $L$ in a language $\mathcal{L}$, $\mathcal{F}_L(\text{Fm}_\mathcal{L}) = \text{Th}(L)$.

Theorem 1.6
Let $L$ be a logic. Then for each set $\Gamma$ of formulae and each formula $\varphi$ the following holds: $\Gamma \vdash_L \varphi$ iff $\Gamma \models_{\text{MOD}(L)} \varphi$. 
Suppose that $T \in \text{Th}(\text{CPC})$ and $\varphi \notin T$ ($T \nvdash_{\text{CPC}} \varphi$). We want to show that $T \nvdash \varphi$ in some meaningful semantics.

$T \nvdash \langle \text{Fm}_L, T \rangle \varphi$. 1st completeness theorem

$\langle \alpha, \beta \rangle \in \Omega(T)$ iff $\alpha \leftrightarrow \beta \in T$ (congruence relation on $\text{Fm}_L$ compatible with $T$: if $\alpha \in T$ and $\langle \alpha, \beta \rangle \in \Omega(T)$, then $\beta \in T$).

Lindenbaum–Tarski algebra: $\text{Fm}_L/\Omega(T)$ is a Boolean algebra and $T \nvdash \langle \text{Fm}_L/\Omega(T), T/\Omega(T) \rangle \varphi$. 2nd completeness theorem

Lindenbaum Lemma: If $\varphi \notin T$, then there is a maximal consistent $T' \in \text{Th}(\text{CPC})$ such that $T \subseteq T'$ and $\varphi \notin T'$.

$\text{Fm}_L/\Omega(T') \cong 2$ (subdirectly irreducible Boolean algebra) and $T \nvdash \langle 2, \{1\} \rangle \varphi$. 3rd completeness theorem
A logic $L$ in a language $\mathcal{L}$ is \textbf{weakly implicative} if there is a binary connective $\rightarrow$ (primitive or definable) such that:

(R) \[ \vdash_L \varphi \rightarrow \varphi \]

(MP) \[ \varphi, \varphi \rightarrow \psi \vdash_L \psi \]

(T) \[ \varphi \rightarrow \psi, \psi \rightarrow \chi \vdash_L \varphi \rightarrow \chi \]

(sCng) \[ \varphi \rightarrow \psi, \psi \rightarrow \varphi \vdash_L c(\chi_1, \ldots, \chi_i, \varphi, \ldots, \chi_n) \rightarrow c(\chi_1, \ldots, \chi_i, \psi, \ldots, \chi_n) \]

for each $\langle c, n \rangle \in \mathcal{L}$ and each $0 \leq i < n$. 
The following logics are weakly implicative:

- CPC, BCI, and Inc
- global modal logics
- intuitionistic and superintuitionistic logic
- linear logic and its variants
- (the most of) fuzzy logics
- substructural logics

The following logics are not weakly implicative:

- local modal logics
- Dumb, AInc, and the conjunction-disjunction fragment of classical logic
- logics of ortholattices

as they have no theorems
Conventions

Unless said otherwise, $L$ is a weakly implicative in a language $\mathcal{L}$ with an implication $\rightarrow$. We write:

- $\varphi \leftrightarrow \psi$ instead of $\{\varphi \rightarrow \psi, \psi, \psi \rightarrow \varphi\}$
- $\Gamma \vdash \Delta$ whenever $\Gamma \vdash \chi$ for each $\chi \in \Delta$
- $\Gamma \nvdash \Delta$ whenever $\Gamma \vdash \Delta$ and $\Delta \vdash \Gamma$.

Theorem 1.8

Let $\varphi, \psi, \chi$ be formulae. Then:

- $\vdash_L \varphi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \psi \vdash_L \psi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \delta, \delta \leftrightarrow \psi \vdash_L \varphi \leftrightarrow \psi$
- $\varphi \leftrightarrow \psi \vdash_L \chi \leftrightarrow \hat{\chi}$, where $\hat{\chi}$ is obtained from $\chi$ by replacing some occurrences of $\varphi$ in $\chi$ by $\psi$. 

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Corollary 1.9

Let $\rightarrow'$ be a connective satisfying (R), (MP), (T), (sCng). Then

$$\varphi \leftrightarrow \psi \vdash_L \varphi \leftrightarrow' \psi.$$ 

Proof.

Consider formulae $\chi = \varphi \rightarrow' \varphi$ and $\hat{\chi} = \varphi \rightarrow' \psi$ and the proof

$$\varphi \leftrightarrow \psi, \ldots, (\varphi \rightarrow' \varphi) \rightarrow (\varphi \rightarrow' \psi), \varphi \rightarrow' \varphi, \varphi \rightarrow' \psi.$$ 

Analogously for $\hat{\chi} = \psi \rightarrow' \varphi$ we can write

$$\varphi \leftrightarrow \psi, \ldots, (\varphi \rightarrow' \varphi) \rightarrow (\psi \rightarrow' \varphi), \varphi \rightarrow' \varphi, \psi \rightarrow' \varphi.$$ 

So we have shown $\varphi \leftrightarrow \psi \vdash_L \varphi \leftrightarrow' \psi$. The reverse direction is fully analogous.
Corollary 1.9

Let →′ be a connective satisfying (R), (MP), (T), (sCng). Then

\[ \varphi \leftrightarrow \psi \vdash_L \varphi \leftrightarrow' \psi. \]

Corollary 1.10

Local modal logic \( T^l \) is not weakly implicative.

Proof.

Let →′ be a ‘good’ implication in \( T^l \). Then →′ (along with classical implication →) is an implication in global \( T^g \). Thus

\[ T \rightarrow \varphi, \varphi \rightarrow T \vdash_{T^g} T \leftrightarrow' \varphi \quad \text{and so} \quad \square^n (T \rightarrow \varphi) \vdash_{T^l} T \leftrightarrow' \varphi \quad \text{for some } n. \]

Consider the proof in \( T^l \):

\[ \square^n \varphi, \ldots, \square^n (T \rightarrow \varphi), \ldots, T \leftrightarrow' \varphi, \ldots, \square^{n+1} T \leftrightarrow' \square^{n+1} \varphi, \square^{n+1} T, \square^{n+1} \varphi, \text{a contradiction}. \]
Let \( L \) be a weakly implicative logic in \( \mathcal{L} \) and \( T \in Th(L) \). For every formula \( \varphi \), we define the set

\[
[\varphi]_T = \{ \psi \in Fm_L \mid \varphi \leftrightarrow \psi \subseteq T \}.
\]

The **Lindenbaum–Tarski matrix** with respect to \( L \) and \( T \), \( LindT_T \), has the filter \( \{[\varphi]_T \mid \varphi \in T \} \) and algebraic reduct with the domain \( \{[\varphi]_T \mid \varphi \in Fm_L \} \) and operations:

\[
c^{LindT_T}([\varphi_1]_T, \ldots, [\varphi_n]_T) = [c(\varphi_1, \ldots, \varphi_n)]_T
\]

What are Lindenbaum–Tarski matrices in general?
Recall that Lindenbaum matrices have domain \( Fm_L \) and

\[
F_i_L(Fm_L) = Th(L).
\]
Definition 1.11

Let $\mathbf{A} = \langle A, F \rangle$ be an $L$-matrix. We define:

- the **Leibniz congruence** $\Omega_A(F)$ of $\mathbf{A}$ as
  
  $$\langle a, b \rangle \in \Omega_A(F) \quad \text{iff} \quad a \leftrightarrow^A b \subseteq F$$

- the **matrix preorder** $\leq_A$ of $\mathbf{A}$ as
  
  $$a \leq_A b \quad \text{iff} \quad a \rightarrow^A b \in F$$

Note that

$$\langle a, b \rangle \in \Omega_A(F) \quad \text{iff} \quad a \leq_A b \quad \text{and} \quad b \leq_A a.$$
A congruence $\theta$ of $A$ is logical in a matrix $\langle A, F \rangle$ if for each $a, b \in A$ if $a \in F$ and $\langle a, b \rangle \in \theta$, then $b \in F$.

**Theorem 1.12**

Let $A = \langle A, F \rangle$ be an $L$-matrix. Then:

1. $\leq_A$ is a preorder.
2. $\Omega_A(F)$ is the largest logical congruence of $A$.
3. $\langle a, b \rangle \in \Omega_A(F)$ iff for each $\chi \in Fm_L$ and each $A$-evaluation $e$:
   
   $$e[p \rightarrow a](\chi) \in F \quad \text{iff} \quad e[p \rightarrow b](\chi) \in F.$$
**Definition 1.13**

A L-matrix $A = \langle A, F \rangle$ is **reduced**, $A \in \text{MOD}^*(L)$ in symbols, if $\Omega_A(F)$ is the identity relation $\text{Id}_A$ (iff $\leq_A$ is an order).

An algebra $A$ is **L-algebra**, $A \in \text{ALG}^*(L)$ in symbols, if there a set $F \subseteq A$ such that $\langle A, F \rangle \in \text{MOD}^*(L)$.

Note that $\Omega_A(A) = A^2$. Thus from $\mathcal{F}_\text{Inc}(A) = \{A\}$ we obtain:

$$A \in \text{ALG}^*(\text{Inc}) \quad \text{iff} \quad A \text{ is a singleton}$$
1. Classical logic: it is easy to see that
\[ \Omega_2(\{1\}) = \text{Id}_2 \quad \text{i.e.,} \quad 2 \in \text{ALG}^*(\text{CPC}). \]

Actually for any Boolean algebra \( A \):
\[ \Omega_A(\{1\}) = \text{Id}_A \quad \text{i.e.,} \quad A \in \text{ALG}^*(\text{CPC}). \]

But: \( \Omega_4(\{a, 1\}) = \text{Id}_A \cup \{\langle 1, a \rangle, \langle 0, \neg a \rangle\} \quad \text{i.e.} \quad \langle 4, \{a, 1\} \rangle \notin \text{MOD}^*(\text{CPC}). \)

2. The logic BCI: recall the algebra \( M \) defined via:
\[
\begin{array}{c|cccc}
\rightarrow^M & \top & t & f & \bot \\
\hline
\top & \top & \bot & \bot & \bot \\
t & \top & t & f & \bot \\
f & \top & \bot & t & \bot \\
\bot & \top & \top & \top & \top \\
\end{array}
\]

We can easily show that:
\[ \Omega_M(\{t, \top\}) = \Omega_M(\{t, f, \top\}) = \text{Id}_M \quad \text{i.e.} \quad M \in \text{ALG}^*(\text{BCI}). \]
Let us take $A = \langle A, F \rangle \in \text{MOD}(L)$. We write:

- $A^*$ for $A / \Omega_A(F)$
- $[\cdot]_F$ for the canonical epimorphism of $A$ onto $A^*$ defined as:

$$[a]_F = \{ b \in A \mid \langle a, b \rangle \in \Omega_A(F) \}$$

- $A^*$ for $\langle A^*, [F]_F \rangle$.

**Theorem 1.14**

*Let $T$ be a theory, $A = \langle A, F \rangle \in \text{MOD}(L)$, and $a, b \in A$. Then:*

1. $\text{Lind}_{T_T} = \langle Fm_L, T \rangle^*$
2. $a \in F$ iff $[a]_F \in [F]_F$.
3. $[a]_F \leq_{A^*} [b]_F$ iff $a \rightarrow^A b \in F$.
4. $A^* \in \text{MOD}^*(L)$. 
The second completeness theorem

**Theorem 1.15**

Let $L$ be a weakly implicative logic. Then for any set $\Gamma$ of formulae and any formula $\varphi$ the following holds:

$$\Gamma \vdash_L \varphi \iff \Gamma \models_{\text{MOD}^*(L)} \varphi.$$ 

**Proof.**

Using just the soundness part of the FCT it remains to prove:

$$\Gamma \models_{\text{MOD}^*(L)} \varphi \implies \Gamma \vdash_L \varphi.$$ 

Assume that $\Gamma \nvdash_L \varphi$ and take the theory $T = \text{Th}_L(\Gamma)$. Then

- $\text{Lind}_{T_T} = \langle \text{Fm}_L, T \rangle^* \in \text{MOD}^*(L)$ and for $\text{Lind}_{T_T}$-evaluation $e(\psi) = [\psi]_T$ holds $e(\psi) \in [T]_T$ iff $\Gamma \vdash \psi$
- Thus $e[\Gamma] \subseteq e[T] = [T]_T$ and $e(\varphi) \notin [T]_T$
Slides available at
www.carlesnoguera.cat/?q=en/teaching