Logic, Algebra, and Implication – Lesson 2

Petr Cintula[1] Carles Noguera[2]

[1]Institute of Computer Science
Academy of Sciences of the Czech Republic

[2]Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic

Completeness theorem for classical logic

- Suppose that $T \in \mathrm{Th}(\mathrm{CPC})$ and $\varphi \notin T$ ($T \nvdash_{\mathrm{CPC}} \varphi$). We want to show that $T \not\models \varphi$ in some meaningful semantics.
- $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle} \varphi$. 1st completeness theorem
- $\langle \alpha, \beta\rangle \in \Omega(T)$ iff $\alpha \leftrightarrow \beta \in T$ (congruence relation on $\boldsymbol{Fm}_{\mathcal{L}}$ compatible with T: if $\alpha \in T$ and $\langle \alpha, \beta\rangle \in \Omega(T)$, then $\beta \in T$).
- Lindenbaum-Tarski algebra: $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T)$ is a Boolean algebra and $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}/\Omega(T), T/\Omega(T)\rangle} \varphi$. 2nd completeness theorem
- Lindenbaum Lemma: If $\varphi \notin T$, then there is a maximal consistent $T' \in \mathrm{Th}(\mathrm{CPC})$ such that $T \subseteq T'$ and $\varphi \notin T'$.
- $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T') \cong \mathbf{2}$ (subdirectly irreducible Boolean algebra) and $T \not\models_{\langle \mathbf{2}, \{1\}\rangle} \varphi$. 3rd completeness theorem

Closure systems and closure operators

Closure system over a set A: a collection of subsets $\mathcal{C} \subseteq \mathcal{P}(A)$ closed under arbitrary intersections and such that $A \in \mathcal{C}$. The elements of $\mathcal{C}$ are called closed sets.

Closure operator over a set A: a mapping $C\colon \mathcal{P}(A) \to \mathcal{P}(A)$ such that for every $X, Y \subseteq A$:

1. $X \subseteq C(X)$,
2. $C(X) = C(C(X))$, and
3. if $X \subseteq Y$, then $C(X) \subseteq C(Y)$.

If C is a closure operator, $\{X \subseteq A \mid C(X) = X\}$ is a closure system.

If $\mathcal{C}$ is closure system, $C(X) = \bigcap\{Y \in \mathcal{C} \mid X \subseteq Y\}$ is a closure operator.

Closure systems and closure operators and logics

Each logic L determines a closure system $\mathrm{Th}(\mathrm{L})$ and a closure operator Th_{L}.

Conversely, given a structural closure operator C over $Fm_{\mathcal{L}}$ (for every σ, if $\varphi \in C(\Gamma)$, then $\sigma(\varphi) \in C(\sigma[\Gamma])$), there is a logic L such that $C = \mathrm{Th}_{\mathrm{L}}$.

The set of all L-filters over a given algebra $\boldsymbol{A}$, $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is a closure system over A. Its associated closure operator is $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}$.

Transfer theorem for finitarity

A closure operator C is finitary if for every $X \subseteq A$,
$C(X) = \bigcup\{C(Y) \mid Y \subseteq X$ and Y is finite$\}$.

L is a finitary logic iff Th_{L} is a finitary closure operator.

Theorem 2.1 (Transfer theorem for finitarity)

Given a logic L, the following conditions are equivalent:

1. L *is finitary (i.e.,* Th_{L} *is a finitary closure operator).*
2. $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}$ *is a finitary closure operator for any* $\mathcal{L}$*-algebra* $\boldsymbol{A}$*.*

Abstract Lindenbaum Lemma

An element X of a closure system $\mathcal{C}$ over A is called maximal w.r.t. an element a if it is a maximal element of the set $\{Y \in \mathcal{C} \mid a \notin Y\}$ w.r.t. the order given by inclusion.

Lemma 2.2

Let $\mathcal{C}$ be a closure system corresponding to a finitary closure operator. If $T \in \mathcal{C}$ and $a \notin T$, then there is $T' \in \mathcal{C}$ such that $T \subseteq T'$ and T' is maximal with respect to a.

An element X of a closure system $\mathcal{C}$ over A is saturated if it is maximal w.r.t. some element a.

Thus Abstract Lindenbaum Lemma actually says that saturated sets form a base of $\mathcal{C}$.

Operations on matrices

$\langle \boldsymbol{A}, F \rangle$: first-order structure in the equality-free predicate language with function symbols from $\mathcal{L}$ and a unique unary predicate symbol interpreted by F.

Submatrix: $\langle \boldsymbol{A}, F \rangle \subseteq \langle \boldsymbol{B}, G \rangle$ if $\boldsymbol{A} \subseteq \boldsymbol{B}$ and $F = A \cap G$.

(Strict) Homomorphism from a matrix $\langle \boldsymbol{A}, F \rangle$ to a matrix $\langle \boldsymbol{B}, G \rangle$: an algebraic homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{B}$ such that $h[F] \subseteq G$. We say that h is strict if also $h[A \setminus F] \subseteq B \setminus G$.

Isomorphism: bijective strict homomorphisms.

Direct product: Given matrices $\{\langle \boldsymbol{A}_i, F_i \rangle \mid i \in I\}$, their direct product is $\langle \boldsymbol{A}, F \rangle$, where $\boldsymbol{A} = \prod_{i \in I} \boldsymbol{A}_i$ and $F = \prod_{i \in I} F_i$.
The j-projection $\pi_j(a) = a(j)$ is a strict surjective homomorphism from $\boldsymbol{A}$ onto $\boldsymbol{A}_j$.

Subdirect products and subdirect irreducibility

A matrix $\mathbf{A}$ is said to be representable as a subdirect product of the family of matrices $\{\mathbf{A}_i \mid i \in I\}$ if there is an embedding homomorphism α from $\mathbf{A}$ into the direct product $\prod_{i\in I} \mathbf{A}_i$ such that for every $i \in I$, the composition of α with the i-th projection, $\pi_i \circ \alpha$, is a surjective homomorphism. In this case, α is called a subdirect representation of $\mathbf{A}$

A matrix $\mathbf{A} \in \mathbb{K}$ is subdirectly irreducible relative to $\mathbb{K}$ if for every subdirect representation α of $\mathbf{A}$ with a family $\{\mathbf{A}_i \mid i \in I\} \subseteq \mathbb{K}$ there is $i \in I$ such that $\pi_i \circ \alpha$ is an isomorphism. The class of all subdirectly irreducible matrices relative to $\mathbb{K}$ is denoted as $\mathbb{K}_{\mathrm{RSI}}$.

Characterization of RSI reduced models

Theorem 2.3

Let L *be a weakly implicative logic and* $\mathbf{A} = \langle A, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$*.*
Then $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ *iff* F *is saturated in* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$*.*

Corollary 2.4

Let L *be a weakly implicative logic and* $\mathbf{A} = \langle A, F\rangle \in \mathbf{MOD}(\mathrm{L})$*.*
Then $\mathbf{A}^* \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ *iff* F *is saturated in* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$*.*

Corollary 2.5

If L *is a finitary weakly implicative logic, then every matrix in* $\mathbf{MOD}^*(\mathrm{L})$ *is representable as a subdirect product of matrices in* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$*.*

The second completeness theorem

Theorem 2.6

Let L *be a weakly implicative logic. Then for any set* Γ *of formulae and any formula* φ *the following holds:*

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi.$$

Proof.

Using just the soundness part of the FCT it remains to prove:

$$\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi \quad \text{implies} \quad \Gamma \vdash_{\mathrm{L}} \varphi.$$

Assume that $\Gamma \nvdash_{\mathrm{L}} \varphi$ then there is a theory T s.t. $T = \mathrm{Th}_{\mathrm{L}}(\Gamma)$ and $\varphi \notin T$. Then

- $\mathbf{LindT}_T = \langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle^* \in \mathbf{MOD}^*(\mathrm{L})$ and for the $\mathbf{LindT}_T$-evaluation $e(\psi) = [\psi]_T$ holds $e(\psi) \in [T]_T$ iff $\psi \in T$
- Thus $e[\Gamma] \subseteq [T]_T$ and $e(\varphi) \notin [T]_T$ □

The third completeness theorem

Theorem 2.7

Let L be a finitary weakly implicative logic. Then for any set Γ of formulae and any formula φ the following holds:

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \text{iff} \quad \Gamma \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}} \varphi.$$

Proof.

Using just the soundness part of the FCT it remains to prove:

$$\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}} \varphi \quad \text{implies} \quad \Gamma \vdash_{\mathrm{L}} \varphi.$$

Assume that $\Gamma \nvdash_{\mathrm{L}} \varphi$, then there is a saturated theory T s.t. $T \supseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$ and $\varphi \notin T$. Then

- $\mathbf{LindT}_T = \langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle^* \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ and for the $\mathbf{LindT}_T$-evaluation $e(\psi) = [\psi]_T$ holds $e(\psi) \in [T]_T$ iff $\psi \in T$
- Thus $e[\Gamma] \subseteq [T]_T$ and $e(\varphi) \notin [T]_T$ □

Leibniz operator

Leibniz operator: the function giving for each $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ the Leibniz congruence $\Omega_{\boldsymbol{A}}(F)$.

Proposition 2.8

Let L *be a weakly implicative logic* L *and* $\boldsymbol{A}$ *an* $\mathcal{L}$*-algebra. Then*

1. $\Omega_{\boldsymbol{A}}$ *is monotone: if* $F \subseteq G$ *then* $\Omega_{\boldsymbol{A}}(F) \subseteq \Omega_{\boldsymbol{A}}(G)$*.*
2. $\Omega_{\boldsymbol{A}}$ *commutes with inverse images by homomorphisms: for every* $\mathcal{L}$*-algebra* $\boldsymbol{B}$*, homomorphism* $h\colon \boldsymbol{A} \to \boldsymbol{B}$*, and* $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{B})$*:*

$$\Omega_{\boldsymbol{A}}(h^{-1}[F]) = h^{-1}[\Omega_{\boldsymbol{B}}(F)] = \{\langle a, b\rangle \mid \langle h(a), h(b)\rangle \in \Omega_{\boldsymbol{B}}(F)\}.$$

3. $\Omega_{\boldsymbol{A}}[\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})] = \boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$*.*

$\boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$ is the set ordered by inclusion of congruences of $\boldsymbol{A}$ giving a quotient in $\mathbf{ALG}^*(\mathrm{L})$.

An example

Recall that for the algebra $\boldsymbol{M} \in \mathbf{ALG}^*(\mathrm{BCI})$ defined via:

$\to^{\boldsymbol{M}}$	$\top$	t	f	$\bot$
$\top$	$\top$	$\bot$	$\bot$	$\bot$
t	$\top$	t	f	$\bot$
f	$\top$	$\bot$	t	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$

we have

$\Omega_{\boldsymbol{M}}(\{t, \top\}) = \Omega_{\boldsymbol{M}}(\{t, f, \top\}) = \mathrm{Id}_{\boldsymbol{M}}$ i.e., $\Omega_{\boldsymbol{M}}$ is not injective

Interesting equivalence

Theorem 2.9

Given any weakly implicative logic L*, TFAE:*

1. *For every* $\mathcal{L}$*-algebra* $\boldsymbol{A}$*, the Leibniz operator* $\Omega_{\boldsymbol{A}}$ *is a lattice isomorphism from* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ *to* $\boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$*.*
2. *For every* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$*,* F *is the least* L*-filter on* $\boldsymbol{A}$*.*
3. *The Leibniz operator* $\Omega_{\boldsymbol{Fm}_{\mathcal{L}}}$ *is a lattice isomorphism from* $\mathrm{Th}(\mathrm{L})$ *to* $\boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{Fm}_{\mathcal{L}})$*.*
4. *There is a set of equations* $\mathcal{T}$ *in one variable such that for each* $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$ *and each* $a \in A$ *holds:*
   $a \in F$ *if, and only if,* $\mu^{\boldsymbol{A}}(a) = \nu^{\boldsymbol{A}}(a)$ *for every* $\mu \approx \nu \in \mathcal{T}$*.*
5. *There is a set of equations* $\mathcal{T}$ *in one variable such that*
   $$(\mathrm{Alg}) \quad p \dashv\vdash_{\mathrm{L}} \{\mu(p) \leftrightarrow \nu(p) \mid \mu \approx \nu \in \mathcal{T}\}.$$

In the last two items the sets $\mathcal{T}$ *can be taken the same.*

Algebraically implicative logics

Definition 2.10

We say that a logic L is algebraically implicative if it is weakly implicative and satisfies one of the equivalent conditions from the previous theorem.
In this case, $\mathbf{ALG}^*(\mathrm{L})$ is called an equivalent algebraic semantics for L and the set $\mathcal{T}$ is called a truth definition.

Example 2.11

In many cases, one equation is enough for the truth definition. For instance, in classical logic, intuitionism, t-norm based fuzzy logics, etc. the truth definition is $\{p \approx \overline{1}\}$. Linear logic is algebraically implicative with $\mathcal{T} = \{p \wedge \overline{1} \approx \overline{1}\}$.

Different logics with the same algebras

$\mathcal{L} = \{\neg, \to\}$. Algebra $\boldsymbol{A}$ with domain $\{0, \frac{1}{2}, 1\}$ and operations:

	$\neg$
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

$\to$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

$Ł_3 = \models_{\langle \boldsymbol{A}, \{1\}\rangle}$ [three-valued Łukasiewicz logic]

$J_3 = \models_{\langle \boldsymbol{A}, \{\frac{1}{2}, 1\}\rangle}$ [Da Costa, D'Ottaviano]

$Ł_3$ and J_3 are both algebraically implicative with

L	$\mathbf{ALG}^*(\mathrm{L})$	$\mathcal{T}(p)$
$Ł_3$	$\mathbf{Q}(\boldsymbol{A})$	$\{p \approx p \to p\}$
J_3	$\mathbf{Q}(\boldsymbol{A})$	$\{\neg p \to p \approx p \to p\}$

Equational consequence

An equation in the language $\mathcal{L}$ is a formal expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in Fm_{\mathcal{L}}$.

We say that an equation $\varphi \approx \psi$ is a consequence of a set of equations Π w.r.t. a class $\mathbb{K}$ of $\mathcal{L}$-algebras if for each $\boldsymbol{A} \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e we have $e(\varphi) = e(\psi)$ whenever $e(\alpha) = e(\beta)$ for each $\alpha \approx \beta \in \Pi$; we denote it by $\Pi \models_{\mathbb{K}} \varphi \approx \psi$.

Proposition 2.12

Let L *be a weakly implicative logic and* $\Pi \cup \{\varphi \approx \psi\}$ *a set of equations. Then*

$$\Pi \models_{\mathbf{ALG}^*(\mathrm{L})} \varphi \approx \psi \quad \textit{iff} \quad \{\alpha \leftrightarrow \beta \mid \alpha \approx \beta \in \Pi\} \vdash_{\mathrm{L}} \varphi \leftrightarrow \psi.$$

Alternatively, using translation $\rho[\Pi] = \bigcup_{\alpha\approx\beta\in\Pi}(\alpha \leftrightarrow \beta)$*:*

$$\Pi \models_{\mathbf{ALG}^*(\mathrm{L})} \varphi \approx \psi \quad \textit{iff} \quad \rho[\Pi] \vdash_{\mathrm{L}} \rho(\varphi \approx \psi).$$

Characterizations of algebraically implicative logics

We have defined a translation ρ from (sets of) equations to sets of formulae using $\leftrightarrow$.

Analogously we define a translation τ from (sets of) formulae to sets of equations using the truth definition $\mathcal{T}$:

$$\tau[\Gamma] = \{\alpha(\varphi) \approx \beta(\varphi) \mid \varphi \in \Gamma \text{ and } \alpha \approx \beta \in \mathcal{T}\}$$

Theorem 2.13

Given any weakly implicative logic L*, TFAE:*

1. L *is algebraically implicative with the truth definition* $\mathcal{T}$*.*
2. *There is a set of equations* $\mathcal{T}$ *in one variable such that:*
   1. $\Pi \models_{\mathbf{ALG}^*(\mathrm{L})} \varphi \approx \psi$ *iff* $\rho[\Pi] \vdash_{\mathrm{L}} \rho(\varphi \approx \psi)$
   2. $p \dashv\vdash_{\mathrm{L}} \rho[\tau(p)]$
3. *There is a set of equations* $\mathcal{T}$ *in one variable such that:*
   1. $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\tau[\Gamma] \models_{\mathbf{ALG}^*(\mathrm{L})} \tau(\varphi)$
   2. $p \approx q \mathrel{=\!\!|\!\models}_{\mathbf{ALG}^*(\mathrm{L})} \tau[\rho(p \approx q)]$

Finitary algebraically implicative logics and quasivarieties

A quasivariety is a class of algebras described by quasiequations, formal expressions of the form $\bigwedge_{i=1}^{n} \alpha_i \approx \beta_i \Rightarrow \varphi \approx \psi$, where $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \varphi, \psi \in Fm_{\mathcal{L}}$.

Proposition 2.14

If L *is a finitary algebraically implicative logic, then it has a finite truth definition and* $\mathbf{ALG}^*(\mathrm{L})$ *is a quasivariety.*

Rasiowa-implicative and regularly implicative logics

Definition 2.15

We say that a weakly implicative logic L is

- regularly implicative if:

  (Reg) $\varphi, \psi \vdash_{\mathrm{L}} \psi \to \varphi$.

- Rasiowa-implicative if:

  (W) $\varphi \vdash_{\mathrm{L}} \psi \to \varphi$.

Proposition 2.16

A weakly implicative logic L is regularly implicative iff all the filters of the matrices in $\mathbf{MOD}^*(\mathrm{L})$ are singletons.

Proposition 2.17

A regularly implicative logic L is Rasiowa-implicative iff for each $\mathbf{A} = \langle A, \{t\}\rangle \in \mathbf{MOD}^*(\mathrm{L})$ the element t is the maximum of $\leq_{\mathbf{A}}$.

Proposition 2.18

Each Rasiowa-implicative logic is regularly implicative and each regularly implicative logic is algebraically implicative.

Examples

The following logics are Rasiowa-implicative:

- classical logic
- global modal logics
- intuitionistic and superintuitionistic logics
- many fuzzy logics (Łukasiewicz, Gödel-Dummett, product logics, BL, MTL, ...)
- substructural logics with weakening
- inconsistent logic
- ...

Example 2.19

- The equivalence fragment of classical logic is a regularly implicative but not Rasiowa-implicative logic.
- Linear logic is algebraically, but not regularly, implicative.
- The logic BCI is weakly, but not algebraically, implicative.