Non-associative substructural logics: alternative axiomatization, algebraic and logical properties

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(Associative) Substructural Logics

Substructural Logics: Non-classical logics not satisfying some structural rule.

- without weakening: relevance logics
- without contraction: monoidal logic ($\mathrm{FL_{ew}}$), fuzzy logics, ...
- without weakening and contraction: linear logic ($\mathrm{FL_e}$)
- without exchange, weakening and contraction: Full Lambek calculus (FL)

Uniform approach in Algebraic Logic: logics of residuated lattices (lattice-ordered residuated monoids).

Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2007.

Associativity is always assumed.

What about removing associativity?

Some works on non-associative substructural logics:

- Lambek (1961)
- Buszkowski and Farulewski (2009)
- Galatos and Ono. Cut elimination and strong separation for substructural logics: An algebraic approach, *Annals of Pure and Applied Logic*, 161(9):1097–1133, 2010.
- Botur (2011)

Hilbert-style calculus (Galatos-Ono) – 1

$$
\begin{array}{rl}
(\mathrm{R}) & \varphi \to \varphi \\
(\mathrm{As}_{\ell\ell}) & \varphi \to ((\varphi \rightsquigarrow \psi) \to \psi) \\
(\wedge 1) & \varphi \wedge \psi \to \varphi \\
(\wedge 2) & \varphi \wedge \psi \to \psi \\
(\wedge 3) & (\chi \to \varphi) \wedge (\chi \to \psi) \to (\chi \to \varphi \wedge \psi) \\
(\vee 1) & \varphi \to \varphi \vee \psi \\
(\vee 2) & \psi \to \varphi \vee \psi \\
(\vee 3) & (\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi) \\
(\vee 3_{\rightsquigarrow}) & (\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \to (\varphi \vee \psi \rightsquigarrow \chi) \\
(\mathrm{Adj}_{\&}) & \varphi \to (\psi \to \psi \,\&\, \varphi)
\end{array}
$$

Hilbert-style calculus (Galatos-Ono) – 2

$(\overline{1})$ $\overline{1}$

(R') $\overline{1} \to (\varphi \to \varphi)$

(Push) $\varphi \to (\overline{1} \to \varphi)$

(MP) $\varphi, \varphi \to \psi \vdash \psi$

(As) $\varphi \vdash (\varphi \to \psi) \to \psi$

(Sf) $\varphi \to \psi \vdash (\psi \to \chi) \to (\varphi \to \chi)$

(Pf) $\psi \to \chi \vdash (\varphi \to \psi) \to (\varphi \to \chi)$

$(\mathrm{E}_{\rightsquigarrow 1})$ $\varphi \to (\psi \to \chi) \vdash \psi \to (\varphi \rightsquigarrow \chi)$

(Symm_1) $\varphi \rightsquigarrow \psi \vdash \varphi \to \psi$

(Adj) $\varphi, \psi \vdash \varphi \wedge \psi$

(Res_1) $\psi \to (\varphi \to \chi) \vdash \varphi \,\&\, \psi \to \chi$

Basic substructural logic

SL: Galatos-Ono logic expanded with $\bot$ ($\bot \to \varphi$; $\top = \bot \to \bot$)

Bounded non-associative full Lambek logic

A big family of substructural logics

Convention

A logic is substructural if it is a finitary extension of $\mathbf{SL}$.

Prominent axiomatic extensions – 1

Definition

Let us consider the following consecutions:

a_1	$\varphi \mathbin{\&} (\psi \mathbin{\&} \chi) \to (\varphi \mathbin{\&} \psi) \mathbin{\&} \chi$	*re-associate to the left*
a_2	$(\varphi \mathbin{\&} \psi) \mathbin{\&} \chi \to \varphi \mathbin{\&} (\psi \mathbin{\&} \chi)$	*re-associate to the right*
e	$\varphi \to (\psi \to \chi) \vdash \psi \to (\varphi \to \chi)$	*exchange*
c	$\varphi \to (\varphi \to \psi) \vdash \varphi \to \psi$	*contraction*
i	$\vdash \psi \to (\varphi \to \psi)$	*left weakening*
o	$\overline{0} \to \varphi$	*right weakening*

Given any $X \subseteq \{a_1, a_2, e, c, i, o\}$ by SL_X we denote the expansion by X. If both a_1 and a_2 are in X we replace them by the symbol a. Analogously if both i and o are in X we replace them by the symbol w.

SL_a is the bounded version of FL.

Prominent axiomatic extensions – 2

Each one of the following axiomatic extensions of SL is axiomatized by any one of the corresponding indicated rules:

$\mathrm{SL_{a_1}}$

1. $\vdash (\varphi \,\&\, \psi \to \chi) \to (\psi \to (\varphi \to \chi))$
2. $\vdash (\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$
3. $\vdash (\varphi \to (\psi \rightsquigarrow \chi)) \to (\psi \rightsquigarrow (\varphi \to \chi))$

$\mathrm{SL_{a_2}}$

1. $\vdash (\psi \to (\varphi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)$
2. $\vdash (\psi \rightsquigarrow (\varphi \to \chi)) \to (\varphi \to (\psi \rightsquigarrow \chi))$

$\mathrm{SL_e}$

1. $\vdash \varphi \,\&\, \psi \to \psi \,\&\, \varphi$
2. $\vdash (\varphi \rightsquigarrow \psi) \to (\varphi \to \chi)$
3. $\vdash (\varphi \to \psi) \to (\varphi \rightsquigarrow \chi)$

$\mathrm{SL_c}$

1. $\vdash \varphi \to \varphi \,\&\, \varphi$
2. $\vdash \varphi \wedge \psi \to \varphi \,\&\, \psi$

$\mathrm{SL_i}$

1. $\vdash \varphi \,\&\, \psi \to \psi$
2. $\psi \vdash \varphi \to \psi$
3. $\vdash \varphi \to \overline{1}$
4. $\vdash \varphi \,\&\, \psi \to \varphi$
5. $\vdash \varphi \,\&\, \psi \to \varphi \wedge \psi$

Lattice-ordered residuated unital groupoid or $\mathbf{SL}$-algebra is an algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 0, 1, \bot, \top \rangle$ such that $\langle A, \wedge, \vee, 0, 1, \bot, \top \rangle$ is a doubly pointed bounded lattice satisfying $x = 1 \cdot x = x \cdot 1$ and for all $a, b, c \in A$ we have

$$a \cdot b \leq c \quad \text{iff} \quad b \leq a \backslash c \quad \text{iff} \quad a \leq c/b\,.$$

Algebraic semantics – 2

Definition

Let L be any substructural logic obtained by adding a set of axioms AX and a set of rules R to SL. $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 0, 1, \bot, \top \rangle$ is an L-algebra if it is an SL-algebra such that:

- for every $\varphi \in AX$ and every $\boldsymbol{A}$-evaluation e, $e(\varphi) \geq 1$,
- for every $\Gamma \vdash \varphi \in R$ and every $\boldsymbol{A}$-evaluation e, if $e(\psi) \geq 1$ for every $\psi \in \Gamma$, then $e(\varphi) \geq 1$.

Variety of all SL-algebras: $\mathbb{SL}$.

Given a class $\mathbb{K} \subseteq \mathbb{SL}$, a set of formulae Γ and a formula φ, $\Gamma \models_{\mathbb{K}} \varphi$ if for every $\boldsymbol{A} \in \mathbb{K}$ and every $\boldsymbol{A}$-evaluation e, if $e(\psi) \geq 1$ for every $\psi \in \Gamma$, then $e(\varphi) \geq 1$.

Algebraic semantics – 3

Theorem

Let L *be a substructural logic and let* $\mathbb{L}$ *be the quasivariety of* L*-algebras. For every set of formulae* Γ *and every formula* φ *we have:* $\Gamma \vdash_{\mathrm{L}} \varphi$ *if, and only if,* $\Gamma \models_{\mathbb{L}} \varphi$.

Technically speaking, SL is an algebraizable logic and $\mathbb{SL}$ is its equivalent algebraic semantics with translations $E(p,q) = \{p \to q, q \to p\}$ and $\mathcal{E}(p) = \{p \wedge \overline{1} \approx \overline{1}\}$. The same holds for every finitary extension of SL and its corresponding quasivariety.

Filters and congruences – 1

Definition

Given a substructural logic L and an $\mathcal{L}_{\mathrm{SL}}$-algebra $\boldsymbol{A}$, a set $F \subseteq A$ is an L-filter if for every set of formulae Γ and every formula φ such that $\Gamma \vdash_{\mathrm{L}} \varphi$ and every $\boldsymbol{A}$-evaluation e it holds: if $e[\Gamma] \subseteq F$, then $e(\varphi) \in F$. By $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ we denote the set of all L-filters over $\boldsymbol{A}$.

Given $X \subseteq A$, the L-filter generated by X, denoted as $\mathrm{Fi}(X)$ is the least L-filter containing X.

Filters and congruences – 2

Let $\boldsymbol{Con}_{\mathbb{L}}(\boldsymbol{A})$ denote the lattice of congruences of $\boldsymbol{A}$ relative to $\mathbb{L}$, i.e. giving a quotient in $\mathbb{L}$. If $\mathbb{L}$ is a variety, then $\boldsymbol{Con}_{\mathbb{L}}(\boldsymbol{A})$ contains all congruences of $\boldsymbol{A}$. The Leibniz operator $\Omega_{\boldsymbol{A}}$ is defined, for any $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, as $\Omega_{\boldsymbol{A}}(F) = \{\langle a, b\rangle \in A^2 \mid a\backslash b, b\backslash a \in F\}$.

Proposition

Let L be a substructural logic and let $\boldsymbol{A}$ be an L-algebra. The Leibniz operator $\Omega_{\boldsymbol{A}}$ is a lattice isomorphism from $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ to $\boldsymbol{Con}_{\mathbb{L}}(\boldsymbol{A})$. Its inverse is the function that maps any $\theta \in \boldsymbol{Con}_{\mathbb{L}}(\boldsymbol{A})$ to the filter $\{a \in A \mid \langle a, a \wedge \overline{1}\rangle \in \theta\}$.

Distributivity and generalized disjunctions

If L is an axiomatic extension of SL), then $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ forms a distributive lattice.

$\nabla(p, q, \overrightarrow{r})$ is a (p-)disjunction if it satisfies

(PD) $\varphi \vdash_{\mathrm{L}} \varphi \nabla \psi$ and $\psi \vdash_{\mathrm{L}} \varphi \nabla \psi$.

and the Proof by Cases Property (PCP):

$$\frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, \varphi \nabla \psi \vdash \chi}.$$

Theorem (Czelakowski)

Every finitary protoalgebraic distributive logic has a p-disjunction.

Almost MP-based logics: definition – 1

$\star$-formula: built using variables $Var \cup \{\star\}$.
Let φ be a $\star$-formula, δ be a $\star$-formula, and σ a $\star$-substitution defined as $\sigma(\star) = \varphi$ and $\sigma p = p$ for $p \in Var$. By $\delta(\varphi)$ we denote the $\star$-formula $\sigma\delta$.

Definition

Given a set of $\star$-formulae Γ, we define the set Γ^* of $\star$-formulae as the smallest set such that

- $\star \in \Gamma^*$ and
- $\delta(\gamma) \in \Gamma^*$ for each $\delta \in \Gamma$ and each $\gamma \in \Gamma^*$.

Definition

Let bDT be a set of $\star$-formulae. A substructural logic L is almost (MP)-based w.r.t. the set of basic deduction terms bDT if:

- the set bDT is closed under all $\star$-substitutions σ such that $\sigma(\star) = \star$,
- L has a presentation where the only deduction rules are *modus ponens* and those from $\{\varphi \vdash \gamma(\varphi) \mid \varphi \in Fm_{\mathcal{L}_{\mathrm{SL}}}, \gamma \in \mathrm{bDT}\}$, and
- for each $\beta \in \mathrm{bDT}$ and each formulae φ, ψ, there exist $\beta_1, \beta_2 \in \mathrm{bDT}^*$ such that:

$$\vdash_{\mathrm{L}} \beta_1(\varphi \to \psi) \to (\beta_2(\varphi) \to \beta(\psi)).$$

L is called (MP)-based if it admits the empty set as a set of basic deduction terms.

Almost MP-based logics: usefulness and open problem

In A General Framework for Mathematical Fuzzy Logic, chapter II of *Handbook of Mathematical Fuzzy Logic*, College Publications, 2011, we introduced the notion of almost MP-based logics, shown it for FL (and its axiomatic extensions) and used it to obtain Local Deduction Theorem, p-disjunction, and axiomatization of semilinear extensions.

The question was left open for SL and other non-associative logics.

Consider the following Hilbert system $\mathcal{AS}$:

$$
\begin{array}{rl}
(\mathrm{Adj}_{\&}) & \varphi \to (\psi \to \psi \mathbin{\&} \varphi) \\
(\mathrm{Adj}_{\&\rightsquigarrow}) & \varphi \to (\psi \rightsquigarrow \varphi \mathbin{\&} \psi) \\
(\&\wedge) & (\varphi \wedge \overline{1}) \mathbin{\&} (\psi \wedge \overline{1}) \to \varphi \wedge \psi \\
(\wedge 1) & \varphi \wedge \psi \to \varphi \\
(\wedge 2) & \varphi \wedge \psi \to \psi \\
(\wedge 3) & (\chi \to \varphi) \wedge (\chi \to \psi) \to (\chi \to \varphi \wedge \psi) \\
(\vee 1) & \varphi \to \varphi \vee \psi \\
(\vee 2) & \psi \to \varphi \vee \psi \\
(\vee 3) & (\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi) \\
(\mathrm{Push}) & \varphi \to (\overline{1} \to \varphi) \\
(\mathrm{Pop}) & (\overline{1} \to \varphi) \to \varphi
\end{array}
$$

New Hilbert-system – 2

$$
\begin{array}{rl}
(\mathrm{Bot}) & \bot \to \varphi \\
(\mathrm{Res}') & \psi \,\&\, (\varphi \,\&\, (\varphi \to (\psi \to \chi))) \to \chi \\
(\mathrm{Res}'_{\rightsquigarrow}) & (\varphi \,\&\, (\varphi \to (\psi \rightsquigarrow \chi))) \,\&\, \psi \to \chi \\
(\mathrm{T}') & (\varphi \to (\varphi \,\&\, (\varphi \to \psi)) \,\&\, (\psi \to \chi)) \to (\varphi \to \chi) \\
(\mathrm{T}'_{\rightsquigarrow}) & (\varphi \rightsquigarrow ((\varphi \rightsquigarrow \psi) \,\&\, \varphi) \,\&\, (\psi \to \chi)) \to (\varphi \rightsquigarrow \chi) \\
(\mathrm{MP}) & \varphi, \varphi \to \psi \vdash \psi \\
(\mathrm{Adj_u}) & \varphi \vdash \varphi \wedge \overline{1} \\
(\alpha) & \varphi \vdash \delta \,\&\, \varepsilon \to \delta \,\&\, (\varepsilon \,\&\, \varphi) \\
(\alpha') & \varphi \vdash \delta \,\&\, \varepsilon \to (\delta \,\&\, \varphi) \,\&\, \varepsilon \\
(\beta) & \varphi \vdash \delta \to (\varepsilon \to (\varepsilon \,\&\, \delta) \,\&\, \varphi) \\
(\beta') & \varphi \vdash \delta \to (\varepsilon \rightsquigarrow (\delta \,\&\, \varepsilon) \,\&\, \varphi)
\end{array}
$$

Theorem

$\mathcal{AS}$ is an axiomatic system for SL*.*

Definition

Given arbitrary formulae δ, ε, we define the following $\star$-formulae:

$$\alpha_{\delta,\varepsilon} = (\delta \,\&\, \varepsilon \to \delta \,\&\, (\varepsilon \,\&\, \star))$$

$$\alpha'_{\delta,\varepsilon} = (\delta \,\&\, \varepsilon \to (\delta \,\&\, \star) \,\&\, \varepsilon)$$

$$\beta_{\delta,\varepsilon} = (\delta \to (\varepsilon \to (\varepsilon \,\&\, \delta) \,\&\, \star)$$

$$\beta'_{\delta,\varepsilon} = (\delta \to (\varepsilon \rightsquigarrow (\delta \,\&\, \varepsilon) \,\&\, \star)$$

SL is almost (MP)-based – 2

Lemma

For every $\star$*-formula* $\gamma \in \{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon} \mid \delta, \varepsilon$ *formulae*$\}$ *and every pair of formulae* φ, ψ*, we have:* $\varphi \to \psi \vdash_{\mathrm{SL}} \gamma(\varphi) \to \gamma(\psi)$.

Lemma

For each $\gamma \in \mathrm{bDT}_{\mathrm{SL}}$ *and each formulae* φ, ψ *there is* $\gamma' \in \mathrm{bDT}^*_{\mathrm{SL}}$ *such that*

$$\vdash \gamma'(\varphi \to \psi) \to (\gamma(\varphi) \to \gamma(\psi)).$$

Theorem

SL *is almost* (MP)*-based with respect to the set*

$$\mathrm{bDT}_{\mathrm{SL}} = \{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon} \mid \delta, \varepsilon \text{ formulae}\}.$$

Simplifications in extensions

Recall the conjugates in FL: $\lambda_\varepsilon = \varepsilon \to \star \,\&\, \varepsilon$ and $\rho_\varepsilon = \varepsilon \rightsquigarrow \varepsilon \,\&\, \star$.

Proposition

- $\vdash_{\mathrm{SL}} \gamma_{\bar{1},\bar{1}}(\varphi) \leftrightarrow \varphi$ for each $\gamma \in \{\alpha, \alpha', \beta, \beta'\}$
- $\vdash_{\mathrm{SL_e}} \alpha_{\delta,\varepsilon}(\varphi) \leftrightarrow \alpha'_{\varepsilon,\delta}(\varphi)$
- $\vdash_{\mathrm{SL_e}} \beta_{\delta,\varepsilon}(\varphi) \leftrightarrow \beta'_{\delta,\varepsilon}(\varphi)$
- $\vdash_{\mathrm{SL_a}} \varphi \to \gamma_{\delta,\varepsilon}(\varphi)$ for each $\gamma \in \{\alpha, \beta\}$
- $\vdash_{\mathrm{SL_a}} \lambda_\varepsilon(\varphi) \to \alpha'_{\delta,\varepsilon}(\varphi)$ and $\vdash_{\mathrm{SL_a}} \rho_\varepsilon(\varphi) \to \beta'_{\delta,\varepsilon}(\varphi)$
- $\vdash_{\mathrm{SL_a}} \lambda_\varepsilon(\varphi) \leftrightarrow \alpha'_{\bar{1},\varepsilon}(\varphi)$ and $\vdash_{\mathrm{SL_a}} \rho_\varepsilon(\varphi) \leftrightarrow \beta'_{\bar{1},\varepsilon}(\varphi)$
- $\vdash_{\mathrm{SL_{ae}}} \varphi \to \lambda_\varepsilon(\varphi)$ and $\vdash_{\mathrm{SL_{ae}}} \varphi \to \rho_\varepsilon(\varphi)$

Basic deduction terms in extensions

Logic L	$\mathrm{bDT_L}$
SL	$\{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon}, \star \wedge \overline{1} \mid \delta, \varepsilon$ formulae$\}$
$\mathrm{SL_w}$	$\{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon} \mid \delta, \varepsilon$ formulae$\}$
$\mathrm{SL_e}$	$\{\alpha_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \star \wedge \overline{1} \mid \delta, \varepsilon$ formulae$\}$
$\mathrm{SL_{ew}}$	$\{\alpha_{\delta,\varepsilon}, \beta_{\delta,\varepsilon} \mid \delta, \varepsilon$ formulae$\}$
$\mathrm{SL_a}$	$\{\lambda_\varepsilon, \rho_\varepsilon, \star \wedge \overline{1} \mid \varepsilon$ a formula$\}$
$\mathrm{SL_{ae}}$	$\{\star \wedge \overline{1}\}$
$\mathrm{SL_{aew}}$	$\emptyset$

Closure under conjunctions

Definition

Given a set of $\star$-formulae Γ, an SL-algebra $\boldsymbol{A}$, and a set $X \subseteq A$, we define

- $\Pi(\Gamma)$ as the smallest set of $\star$-formulae containing $\Gamma \cup \{\overline{1}\}$ and closed under $\&$.
- $\Gamma^{\boldsymbol{A}}$ as the set of unary polynomials built using terms from Γ with coefficients from A and variable $\star$, i.e., $\{\delta(\star, a_1, \ldots, a_n) \mid \delta(\star, p_1, \ldots, p_n) \in \Gamma$ and $a_1, \ldots, a_n \in A\}$.
- $\Gamma^{\boldsymbol{A}}(X)$ as the set $\{\delta^{\boldsymbol{A}}(x) \mid \delta(\star) \in \Gamma^{\boldsymbol{A}}$ and $x \in X\}$.

Local Deduction Theorem

Theorem

Let L *be an almost* (MP)*-based substructural logic with a set of basic deduction terms* bDT*. Let* $\boldsymbol{A}$ *be an* $\mathcal{L}_{\mathrm{SL}}$*-algebra and* $X \cup \{x\} \subseteq A$*. Then* $y \in \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X, x)$ *iff* $\gamma^{\boldsymbol{A}}(x) \backslash y \in \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X)$ *for some* $\gamma \in (\Pi(\mathrm{bDT}^{*}))^{\boldsymbol{A}}$.

Corollary

Let L *be an almost* (MP)*-based substructural logic with a set of basic deduction terms* bDT*. Then for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae the following holds:*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathrm{L}} \gamma(\varphi) \to \psi \text{ for some } \gamma \in \Pi(\mathrm{bDT}^{*}).$$

Filter generation

Corollary

Let L *be an almost* (MP)*-based substructural logic with a set of basic deduction terms* bDT*. Let* $\boldsymbol{A}$ *be an* L*-algebra and* $X \subseteq A$*. Then* $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X) = \{a \in A \mid a \geq x$ *for some* $x \in (\Pi(\mathrm{bDT}^*))^{\boldsymbol{A}}(X)\}$*.*

Describing p-disjunctions

Theorem

Let L *be an almost* (MP)*-based substructural logic with a set of basic deduction terms* bDT*. Then the set* $\nabla_{\mathrm{L}} = \{\gamma_1(p) \vee \gamma_2(q) \mid \gamma_1, \gamma_2 \in (\mathrm{bDT} \cup \{\star \wedge \overline{1}\})^*\}$ *is p-disjunction in* L*.*

L	$\mathrm{bDT}_{\mathrm{L}}$	(p-)disjunction in L
SL	$\{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon}, \star \wedge \overline{1} \mid \delta, \varepsilon\}$	$\{\gamma_1(p) \vee \gamma_2(q) \mid \gamma_1, \gamma_2 \in \mathrm{bDT}_{\mathrm{SL}}\}$
SL_{a}	$\{\lambda_\varepsilon, \rho_\varepsilon, \star \wedge \overline{1} \mid \varepsilon\}$	$\{\gamma_1(p) \vee \gamma_2(q) \mid \gamma_1, \gamma_2 \in \mathrm{bDT}_{\mathrm{SL}_{\mathrm{a}}}\}$
$\mathrm{SL}_{\mathrm{ae}}$	$\{\star \wedge \overline{1}\}$	$\{(\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1})\}$
$\mathrm{SL}_{\mathrm{aew}}$	$\emptyset$	$\{\varphi \vee \psi\}$

Intersection of filters

Theorem

For each SL*-algebra* $\boldsymbol{A}$ *and each* $X, Y \subseteq A$ *we have*
$\mathrm{Fi}(X) \cap \mathrm{Fi}(Y) = \mathrm{Fi}(X \nabla^{\boldsymbol{A}}_{\mathrm{SL}} Y)$.

Axiomatization of positive universal classes

Theorem

Let L be a substructural logic with a p-disjunction ∇, and let $\mathcal{C}$ be a set of positive equational clauses. Then:

$$\models_{\{\boldsymbol{A}\in\mathbb{L} \mid \boldsymbol{A}\models\mathcal{C}\}} = \mathrm{L} + \bigcup\{\nabla_{i\in\mathcal{I}_C}(\delta_i \leftrightarrow \varepsilon_i) \mid C \in \mathcal{C}\}.$$

Intersection of axiomatic extensions

Theorem

Let L be a substructural logic with a p-disjunction ∇, and let L_1, L_2 be axiomatic extensions of L by sets of axioms AX_1 and AX_2, respectively. Without loss of generality we can assume that AX_1 and AX_2 are written in disjoint sets of variables. Then:

$$\mathrm{L}_1 \cap \mathrm{L}_2 = \mathrm{L} + \bigcup\{\varphi \nabla \psi \mid \varphi \in AX_1 \text{ and } \psi \in AX_2\}.$$

Equational bases for generated varieties

Theorem

Let $\mathcal{C}$ be a set of positive equational clauses. Then an equational base for the variety of SL-algebras generated by those satisfying $\mathcal{C}$ can be obtained by adding the following:

$$\overline{1} \approx \overline{1} \wedge [\nabla_{i \in \mathcal{I}_C} (\delta_i \leftrightarrow \varepsilon_i)] \text{ for each } C \in \mathcal{C}.$$

Joints of subvarieties

Theorem

Let $\mathbb{L}$ be a quasivariety of SL-algebras, ∇ a p-disjunction for the corresponding logic, and let $\mathbb{L}_1$, $\mathbb{L}_2$ be relative subvarieties of $\mathbb{L}$ given by sets of equations $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Without loss of generality we can assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ are written in disjoint sets of variables. Then:

$$\mathbb{L}_1 \vee \mathbb{L}_2 = \mathbb{L} + \bigcup \{((\delta_1 \leftrightarrow \varepsilon_1) \nabla (\delta_2 \leftrightarrow \varepsilon_2)) \wedge \overline{1} \approx \overline{1} \mid \delta_1 \approx \varepsilon_1 \in \mathcal{E}_1, \delta_2 \approx \varepsilon_2 \in \mathcal{E}_2\}.$$

Thank you for your attention!