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A HENKIN-STYLE PROOF OF COMPLETENESS FOR FIRST-ORDER ALGEBRAIZABLE LOGICS

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Abstract. This paper considers Henkin's proof of completeness of classical first-order logic and extends its scope to the realm of algebraizable logics in the sense of Blok and Pigozzi. Given a propositional logic L (for which we only need to assume that it has an algebraic semantics $\mathbb{L}$ and a suitable disjunction) we axiomatize two natural first-order extensions $L\forall^m$ and $L\forall$ and prove that the former is complete with respect to all models over algebras from $\mathbb{L}$, while the latter is complete with respect to all models over relatively finitely subdirectly irreducible algebras. While the first completeness result is relatively straightforward, the second requires non-trivial modifications of Henkin's proof by making use of the disjunction connective. As a byproduct, we also obtain a form of Skolemization provided that the algebraic semantics admits regular completions. The relatively modest assumptions on the propositional side allow for a wide generalization of previous approaches by Rasiowa, Sikorski, Hájek, Horn, and others and help to illuminate the 'essentially first-order' steps in the classical Henkin's proof.

§1. Introduction. The problem of completeness of classical first-order predicate logic was formulated, for the first time in precise mathematical terms, in 1928 by Hilbert and Ackermann in [17] and solved positively by Gödel in his Ph.D. thesis one year later [8, 9]. In 1947 Henkin [15, 16] presented an alternative simpler proof that has become standard in logic textbooks. The main advantage of Henkin's proof is that it shows how to construct a term-model to invalidate a derivation in the calculus. Roughly speaking, the proof has two stages: first he constructs an appropriate theory (i.e. a maximal consistent Henkin theory) which, secondly, is used to define the desired term model.

In 1950 Rasiowa and Sikorski [25] gave an alternative proof in which, by avoiding the first step on Henkin's construction, they obtained a term-model valued on a general (not necessarily two-valued) Boolean algebra. By means of the famous Rasiowa–Sikorski Lemma, they managed to factorize such model to the desired two-valued term-model. In [26] they generalized this modified Henkin proof to

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intuitionistic first-order predicate logic. Following the ideas of Mostowski [21], they used the order relation on Heyting algebras to interpret the existential (resp. universal) quantification of a formula as the supremum (resp. infimum) of the values of its instances. As in the classical case, they construct a counterexample model over an arbitrary Heyting algebra, which is later embedded into a complete one to ensure that the interpretation of quantifiers is always defined.

After that, we can distinguish two different lines of research in the investigation of non-classical first-order logics. The first one arises from another, less well-known, work of Gödel. Indeed, in [10] he considered linearly ordered models for (propositional) intuitionistic logic, which inspired Dummett to introduce in 1959 a superintuitionistic propositional calculus, obtained by adding the axiom of *prelinearity* $(\varphi \to \psi) \vee (\psi \to \varphi)$, and proved its completeness with respect to linearly ordered Heyting algebras [5]. This system has been called *Gödel–Dummett logic* and served as the propositional basis for a new non-classical first-order logic introduced by Horn [18] in 1969 as the extension of intuitionistic first-order logic obtained by adding the axioms of prelinearity and *constant domains* $(\forall x)(\varphi \vee \chi) \to (\forall x)\varphi \vee \chi$ (for x not free in χ). His proof of the completeness theorem for this logic can be seen as analogous to Rasiowa–Sikorski's approach to the completeness of classical first-order logic: one obtains a term-model valued on a Heyting algebra that is later factorized and turned into a model valued on a linearly ordered Heyting algebra (and then embedded into a particular complete algebra over the naturally ordered real unit interval $[0, 1]$). Let us stress that linearly ordered Heyting algebras play in the variety of Heyting algebras the same rôle as the two-element Boolean algebra in the variety of Boolean algebras: besides being (obviously) exactly those which are linearly ordered (disregarding the trivial algebra) they share a deeper universal-algebraic property: they are exactly the finitely subdirectly irreducible ones (again disregarding the trivial algebra). Thus both proofs (by Horn and Rasiowa–Sikorski) can be seen as instances of a general pattern.

The second line of research was started by Rasiowa in her monograph [24] published in 1974 where she generalized her approach from intuitionism to a rather wide class of propositional logics which she called *implicative logics*.[1] She axiomatized first-order logics based on these systems, that have an algebraic semantics ordered by means of an implication connective which, as in the case of intuitionism, does not always ensure the existence of suprema and infima for the interpretation of quantifiers (which enjoy the shifts typical from intuitionism, but not the axiom of constant domains). For this reason Rasiowa had to deal with the possibility of leaving some formulae with an undefined truth-value in some particular models, but nevertheless obtaining a completeness theorem with respect to those where all formulae can be interpreted. The completeness theorem of all these logics is obtained in a uniform way by direct generalization of the first part of the proof for intuitionistic logic: one obtains a term-model which is valued over an arbitrary (not necessarily finitely subdirectly irreducible) algebra. In fact, if applied to Gödel–Dummett propositional logic her approach would result

[1] Rasiowa's implicative logics are characterized by the presence of an implication connective satisfying identity, transitivity, *modus ponens*, congruence w.r.t. all other connectives, and weakening.

in a first-order logic different from the one considered by Horn (these two systems are separated e.g. by the axiom of constant domains as shown in [18, page 404]).[2]

Classical and Gödel–Dummett logics are not the only first-order logics for which (relatively) finitely subdirectly irreducible algebras are a preferred semantics. Indeed, in [11] Hájek started studying the class of the so-called *fuzzy logics* (which has Gödel–Dummett as a prominent example) whose finitely subdirectly irreducible algebras are exactly those whose underlying order forms a chain and are seen as an intended semantics for both propositional and first-order formalisms. Since Hájek's fuzzy logics are, in particular, implicative logics in the sense of Rasiowa, he could apply Rasiowa's methodology. Therefore, he also had to handle first-order structures not rich enough to interpret the values of all quantified formulae (he called *safe models* those with all suprema and infima that are necessary to interpret formulae).[3] As in the case of Gödel–Dummett logic, Rasiowa's axiomatization would not provide completeness w.r.t. the intended semantics (i.e. w.r.t. safe models based on chains), and so Hájek, following Horn, was forced to add the axiom of constant domains. Interestingly enough, his proof (unlike the previously mentioned ones) is close in spirit to the original Henkin approach: he constructs an appropriate theory such that the term-model built using this theory has the desired properties. Later, in cooperation with Cintula, he generalized his approach to wider classes of fuzzy logics and to languages of arbitrary cardinality[4] by identifying the crucial rôle of disjunction (not only in the axiom of constant domains) and distinguishing two forms of *Henkin theories* in the process [12]. Roughly speaking, one of the forms requires the existence of witnesses to validity of existential statements whereas the other one requires witnesses to non-validity of universal statements.

The logics we have mentioned so far are only the tip of the iceberg. Currently a plethora of non-classical logics, introduced and motivated from diverse points of view, are being studied and they usually require first-order predicate formalisms to guarantee a sufficient expressive power. It is natural to wonder whether these

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[2]This is perhaps a surprising observation because in finitary *propositional* logics (either classical or non-classical) we have an algebraic completeness theorem which can always be restricted to (finitely) subdirectly irreducible algebras (for logics whose semantics is not closed under quotients, one has to consider *relatively* finitely sudirectly irreducible algebras).

[3]The reader might wonder why Rasiowa and Hájek (and us in this paper) do not restrict to semantics over completely ordered algebras (such as $[0, 1]$-valued algebras), where the truth-values for quantified formulae would always be defined without having to resort to the rather cumbersome partial semantics. The reason is that this would lead to non-axiomatizable logics as was first shown for first-order Łukasiewicz logic by Scarpellini who proved that the tautologies given by models over the standard $[0, 1]$-valued algebra are not recursively enumerable [28]; it was shown later that this set is actually Π_2-complete [23] (it is worth noting that Hay presented in [14] an axiomatization of the predicate $[0, 1]$-valued Łukasiewicz logic, but at the price of adding an infinitary deduction rule). In some cases one can even obtain non-arithmetical sets of tautologies over the semantics given by completely ordered algebras; see e.g. [13, 19, 20]. However, it is possible to have the best of both worlds, axiomatization and complete order in the semantics, provided that the corresponding class of algebras *admits regular completions*, i.e. if it is possible to embed any algebra in a completely ordered one while respecting infinite suprema and infima; see e.g. [22].

[4]All works mentioned above (except for the original Henkin approach) were restricted to countable first-order languages.

logics can be given, as classical logic, a semantics based on some kind of algebraic structures. More precisely, the question is what is the scope of the completeness theorem in the context of non-classical first-order logics.[5]

Fortunately, the question has been already given quite satisfactory answers for *propositional* logics. Indeed, *abstract algebraic logic* has concentrated on the relation between logics and algebraic semantics by taking as paradigmatic example the strong link between classical propositional logic and Boolean algebras. Research in this area has offered a deep understanding of wide classes of logics where this link holds in analogous ways. The most celebrated and developed setting is probably that introduced by Blok and Pigozzi when they defined the class of *algebraizable logics* [1]. Such class is wide enough to contain the majority of well-known particular propositional logics (including all Rasiowa's implicative logics and much more) and powerful enough to preserve a strong connection between logics and their corresponding algebraic semantics. Thus, they provide a suitable starting point for the investigation on the scope of the classical completeness theorem for first-order logics.[6] Indeed the Rasiowa–Sikorski definition of semantics for first-order languages can be preserved without changes, because in all algebraizable logics one can still find a suitable implication connective $\Rightarrow$, although maybe not given by a single symbol, but definable by a finite set of formulae. This generalized implication still induces an order relation in the algebras that allows to interpret quantifiers as suprema and infima.

To sum it up, the present paper is based on three distinct lines of research: (1) Blok–Pigozzi abstract study of propositional logics, (2) Rasiowa–Sikorski approach to first-order logics complete with respect to a general algebraic semantics, and (3) Horn–Hájek approach to first-order logics complete with respect to a semantics of (relatively) finitely subdirectly irreducible algebras. Our goal is to show that (2) assumed unnecessary conditions on propositional logics and can be extended so as to accommodate extensions of all propositional logics in (1). Furthermore the same can be done in (3) provided that the logic in question possesses a reasonable (generalized) disjunction connective. In this way, for each well-behaved propositional logic we axiomatize two natural first-order extensions: one will be complete w.r.t. models on all algebras, the other w.r.t. (sometimes more desirable) models over (relatively) finitely subdirectly irreducible algebras.

The paper is organized as follows. After this introduction, Section 2 gives the necessary technical notions to work in this framework. In Section 3, for each propositional logic L we present the axiomatizations for its corresponding minimal first-order logic $\mathrm{L}\forall^{\mathrm{m}}$ and (assuming the presence of disjunction connective) its extension $\mathrm{L}\forall$ and prove that the former is complete w.r.t. to all models (hence the name 'minimal' first-order logic), while the latter is complete w.r.t. models over relatively finitely subdirectly irreducible algebras. As a byproduct we obtain the proof of (a form of) Skolemization for logics whose algebraic semantics admits regular completions.

[5]The chapter [2] can be seen as a contribution towards this goal in the setting of fuzzy logics; indeed, like in the work of Hájek, its scope is restricted to the study of completeness w.r.t. linearly ordered algebras.

[6]To simplify the presentation, we will also assume the presence of a constant $\overline{1}$ in the language.

Our result can be also seen as an analysis of Henkin's proof. Indeed, first the relatively modest assumptions on the propositional side helps to illuminate the 'essentially first-order' steps in the classical Henkin's proof and, second, like Hájek and Cintula, we distinguish two kinds of Henkin theories and show that the 'universal' kind is crucial of the completeness proof, whereas the 'existential' one is closely linked to Skolemization.

§2. Setting the framework. This section presents the basic definitions and notational conventions for the paper (for further information on abstract algebraic logic notions see [6, 4]). We assume the usual notions of a propositional language $\mathcal{L}$, the absolutely free term algebra $\boldsymbol{Fm}_{\mathcal{L}}$ over a denumerable set of generators (propositional variables), and a finitary Hilbert-style proof system $\mathcal{AS}$ and its induced provability relation $\vdash_{\mathcal{AS}}$.

Our approach to propositional logics is based on the notion of *algebraizable logics* introduced by Blok and Pigozzi in [1]. They arguably provide the best paradigm, in abstract algebraic logic, for a class of logics with a strong link with an algebraic semantics, resembling as much as possible, the connection between classical propositional calculus and Boolean algebras. In their original presentation they are given in terms of a certain generalized notion of equivalence connective. However, we need to tailor them to our approach to first-order systems, in which the order relation induced by the implication plays an essential rôle, and so we equivalently formulate them in terms of a generalized notion of implication connective instead. For this we first need some useful notational conventions. Given a set $\Rightarrow(p, q)$ of formulae in two variables p and q, a set $\mathcal{T}(p)$ of equations in one variable p, formulae φ and ψ, and two sets T and S of formulae, we establish the following:

$\varphi \Rightarrow \psi$ denotes the set $\Rightarrow(\varphi, \psi)$
$\varphi \Leftrightarrow \psi$ denotes the set $(\varphi \Rightarrow \psi) \cup (\psi \Rightarrow \varphi)$
$T \vdash S$ means that $T \vdash \varphi$ for each $\varphi \in S$
$T \dashv\vdash S$ means that $T \vdash S$ and $S \vdash T$
$\Leftrightarrow[\mathcal{T}(\varphi)]$ denotes the set $\bigcup\{\alpha(\varphi) \Leftrightarrow \beta(\varphi) \mid \alpha \approx \beta \in \mathcal{T}\}$

CONVENTION 2.1. *Let $\mathcal{L}$ be a language with a truth constant $\overline{1}$, let $\Rightarrow(p, q)$ be a finite set of formulae in two variables, and let $\mathcal{T}(p)$ be a finite set of equations in one variable. In this paper a propositional logic* L *in $\mathcal{L}$ is identified with the provability relation $\vdash_{\mathcal{AS}}$ on $Fm_{\mathcal{L}}$ given by a finitary Hilbert-style system $\mathcal{AS}$ such that:*

$$\vdash_{\mathcal{AS}} \varphi \Rightarrow \varphi \qquad \varphi, \varphi \Rightarrow \psi \vdash_{\mathcal{AS}} \psi \qquad \varphi \Rightarrow \psi, \psi \Rightarrow \chi \vdash_{\mathcal{AS}} \varphi \Rightarrow \chi$$
$$\varphi \dashv\vdash_{\mathcal{AS}} \overline{1} \Rightarrow \varphi \qquad \varphi \dashv\vdash_{\mathcal{AS}} \Leftrightarrow[\mathcal{T}(\varphi)]$$
$$\varphi \Leftrightarrow \psi \vdash_{\mathcal{AS}} \circ(\chi_1, \ldots \chi_i, \varphi, \ldots, \chi_n) \Leftrightarrow \circ(\chi_1, \ldots \chi_i, \psi, \ldots, \chi_n)$$
for every n-ary $\circ \in \mathcal{L}$ and $i < n$.

We usually write simply $\vdash_{\mathrm{L}}$ instead of $\vdash_{\mathcal{AS}}$.

REMARK 2.2. *The required conditions describe the intended behavior of $\Rightarrow$ as a generalized implication connective, while $\Leftrightarrow$ is its corresponding equivalence obtained by symmetrization. In many well-known logics, $\Rightarrow$ is given by a primitive connective $\to$ (or definable just by one formula); this is the case of Rasiowa's implicative logics, which moreover are required to satisfy $\psi \vdash_{\mathrm{L}} \varphi \to \psi$. It is* *also important to remark that our convention does not cover all algebraizable logics because we assume the presence of the constant $\overline{1}$ with certain properties. Some algebraizable logics, such as certain fragments of relevance logic, are not included, but still the vast majority of algebraizable logics studied in the literature is covered including all Rasiowa's implicative logics (where $\overline{1}$ can be defined as $\varphi \to \varphi$) and all substructural logics in the sense of [7]. The restriction is important for the validity of the generalization rule (see Proposition 3.5 for its proof and [2, Example 4.1.16] for an example of an algebraizable logic where generalization fails). Of course, many results of the paper would also hold without the presence of $\overline{1}$.*

We recall now the basics of semantics. Let us fix from now on a logic L in a language $\mathcal{L}$. $\mathcal{L}$-algebras are algebras with signature $\mathcal{L}$; homomorphisms from $\boldsymbol{Fm}_{\mathcal{L}}$ to an $\mathcal{L}$-algebra $\boldsymbol{A}$ are called $\boldsymbol{A}$-evaluations.

Definition 2.3. *For any $\mathcal{L}$-algebra $\boldsymbol{A}$, we define the following set and binary relation:*

$$F_{\boldsymbol{A}} = \{a \mid \boldsymbol{A} \models \mathcal{T}^{\boldsymbol{A}}(a)\} \qquad\qquad a \leq_{\boldsymbol{A}} b \quad \textit{iff} \quad (a \Rightarrow^{\boldsymbol{A}} b) \subseteq F_{\boldsymbol{A}}.$$

$\boldsymbol{A}$ is an L*-algebra, in symbols: $\boldsymbol{A} \in \mathbb{L}$, if for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ and $x, y \in A$ hold:*

1. *$\Gamma \vdash_{\mathrm{L}} \varphi$ implies that for each $\boldsymbol{A}$-evaluation e we have $e(\varphi) \in F_{\boldsymbol{A}}$ whenever $e[\Gamma] \subseteq F_{\boldsymbol{A}}$,*
2. *$a \leq_{\boldsymbol{A}} b$ and $b \leq_{\boldsymbol{A}} a$ implies $a = b$.*[7]

Definition 2.4. *Given a class $\mathbb{K} \subseteq \mathbb{L}$, we define its induced semantical consequence relation for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ as:*

$$\Gamma \models_{\mathbb{K}} \varphi \quad \textit{iff} \quad \textit{for each}\, \boldsymbol{A} \in \mathbb{K} \textit{ and each}\, \boldsymbol{A}\textit{-evaluation}\, e \textit{ we have}$$
$$e(\varphi) \in F_{\boldsymbol{A}} \textit{ whenever}\, e[\Gamma] \subseteq F_{\boldsymbol{A}}.$$

It is not difficult to show that, thanks to the finitarity of $\vdash_{\mathrm{L}}$ and the finiteness of $\Rightarrow$, the consequence relation $\models_{\mathbb{L}}$ is also finitary. Therefore, $\mathbb{L}$ is in fact a quasivariety of algebras. Blok and Pigozzi call it the *equivalent algebraic semantics of* L. In particular, it gives a sound and complete semantics for the logic:

Theorem 2.5 ([1]). *Let* L *be a logic. Then $\vdash_{\mathrm{L}} = \models_{\mathbb{L}}$, i.e. $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbb{L}} \varphi$ for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$.*

A non-trivial L-algebra $\boldsymbol{A}$ is said to be *(finitely) subdirectly irreducible relative to* $\mathbb{L}$ if for every (finite non-empty) subdirect representation α of $\boldsymbol{A}$ with a family $\{\boldsymbol{A}_i \mid i \in I\} \subseteq \mathbb{L}$ there is $i \in I$ such that $\pi_i \circ \alpha$ is an isomorphism. $\mathbb{L}_{\mathrm{R(F)SI}}$ denotes the class of all (finitely) subdirectly irreducible algebras relative to $\mathbb{L}$. Of course $\mathbb{L}_{\mathrm{RSI}} \subseteq \mathbb{L}_{\mathrm{RFSI}}$. Both classes of algebras are also sound and complete semantics for the logic:

Theorem 2.6 ([4]). *Let* L *be a logic. Then*

$$\vdash_{\mathrm{L}} = \models_{\mathbb{L}_{\mathrm{RSI}}} = \models_{\mathbb{L}_{\mathrm{RFSI}}}.$$

[7]Note that in each L-algebra $\boldsymbol{A}$, $\leq_{\boldsymbol{A}}$ is an order, $F_{\boldsymbol{A}} = \{a \mid \overline{1}^{\boldsymbol{A}} \leq_{\boldsymbol{A}} a\}$, and it is an upper set w.r.t. $\leq_{\boldsymbol{A}}$.

Finally, for some results we need to assume that propositional logics are also endowed with a (generalized) disjunction connective. Following the notation introduced in [3], let $\nabla(p,q)$ be a set of formulae in two variables; for any pair of formulae φ, ψ, we define $\varphi \nabla \psi = \nabla(\varphi, \psi)$.

Definition 2.7. *A logic* L *is called* finitely disjunctional *if there is a finite set of formulae* $\nabla(p,q)$*, called a* disjunction*, such that* $\varphi \vdash_{\mathrm{L}} \varphi \nabla \psi$*,* $\psi \vdash_{\mathrm{L}} \varphi \nabla \psi$ *and it satisfies the* Proof by Cases Property *(*PCP *for short), i.e. for every set of formulae* Γ *and any formulae* φ, ψ, χ*:*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \chi \quad \textit{and} \quad \Gamma, \psi \vdash_{\mathrm{L}} \chi \quad \textit{imply} \quad \Gamma, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi.$$

As shown in [3], for any finitary logic the PCP can be equivalently written in the following (seemingly stronger) formulation, for any sets of formulae Γ, Φ, Ψ and any formula χ:[8]

$$\text{(sPCP)} \qquad \frac{\Gamma, \Phi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \Psi \vdash_{\mathrm{L}} \chi}{\Gamma, \Phi \nabla \Psi \vdash_{\mathrm{L}} \chi}.$$

The notion of disjunction is intrinsic for a given logic, i.e., for any pair ∇, ∇' of disjunctions in L we have $\varphi \nabla \psi \dashv\vdash_{\mathrm{L}} \varphi \nabla' \psi$. In many prominent cases, such as classical or intuitionistic logic, the lattice connective $\vee$ is itself a disjunction in the sense just defined. But this is not always the case (for example, in linear logic $\vee$ is not a disjunction, but the logic is still finitely disjunctional with the defined connective $(\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1})$). In the implicational fragment of Gödel–Dummett logic (as shown in [3]) it is not possible to define a disjunction with just one formula, but we can still obtain one by considering the set $\varphi \nabla \psi = \{(\varphi \to \psi) \to \psi, (\psi \to \varphi) \to \varphi\}$. However, some other prominent logics, like the full Lambek logic FL [7] or the implicational fragment of intuitionistic logic, are not finitely disjunctional [27, 3]; one would need a higher level of complexity and consider a disjunction defined by an infinite parameterized set of formulae (see [4, 3]).

We list a few properties of finitely disjunctional logics that will be needed in the upcoming text:

Proposition 2.8 ([4]). *Let* L *be a logic with a disjunction* ∇ *and* $\boldsymbol{A}$ *an* L*-algebra. Then:*

- $\varphi \nabla \psi \vdash_{\mathrm{L}} \psi \nabla \varphi$ $\quad (\mathrm{C}_\nabla)$
- $\varphi \nabla \varphi \vdash_{\mathrm{L}} \varphi$ $\quad (\mathrm{I}_\nabla)$
- $\varphi \nabla (\psi \nabla \chi) \dashv\vdash_{\mathrm{L}} (\varphi \nabla \psi) \nabla \chi$ $\quad (\mathrm{A}_\nabla)$
- $\Gamma \nabla \chi \vdash_{\mathrm{L}} \varphi \nabla \chi$ *whenever* $\Gamma \vdash_{\mathrm{L}} \varphi$
- $\boldsymbol{A} \in \mathbb{L}_{\mathrm{RFSI}}$ *iff for each* $a, b \in A$ *we have* $a \in F_{\boldsymbol{A}}$ *or* $b \in F_{\boldsymbol{A}}$ *whenever* $a \nabla^{\boldsymbol{A}} b \subseteq F_{\boldsymbol{A}}$.

§3. First-order logic.

3.1. Basic syntactic and semantic notions. Let us fix a logic L in a propositional language $\mathcal{L}$. As usual, a *predicate language* $\mathcal{P}$ is a triple $\langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$, where $\mathbf{P}$ is a non-empty set of predicate symbols, $\mathbf{F}$ is a set of function symbols, and $\mathbf{ar}$ is a function assigning to each symbol a natural number called the *arity* of the symbol; nullary function symbols are called *object constants*.

[8]Given sets $\Phi, \Psi \subseteq Fm_{\mathcal{L}}$, $\Phi \nabla \Psi$ denotes the set $\{\varphi \nabla \psi \mid \varphi \in \Phi, \psi \in \Psi\}$.

Let us further fix a predicate language $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$ and a denumerable set V whose elements are called *object variables*. The sets of $\mathcal{P}$*-terms*, *atomic* $\mathcal{P}$*-formulae*, and $\langle \mathcal{L}, \mathcal{P} \rangle$*-formulae* are defined as in classical logic. We omit the symbols for propositional or predicate languages when clear from the context (analogously any other notion parameterized by propositional or predicate languages). The notions of bound and free variables, closed terms, sentences, and substitutability are also defined in the standard way. Instead of $\xi_1, \ldots, \xi_n$ (where ξ_i's are terms or formulae and n is arbitrary or fixed by the context) we shall sometimes write just $\vec{\xi}$. Unless stated otherwise, by the notation $\varphi(\vec{z})$ we signify that *all* free variables of φ are among those in the vector of pairwise different object variables $\vec{z}$. If $\varphi(x_1, \ldots, x_n, \vec{z})$ is a formula and we replace all free occurrences of x_i's in φ by terms t_i, we denote the resulting formula in the context simply by $\varphi(t_1, \ldots, t_n, \vec{z})$. A theory T is a pair $\langle \mathcal{P}, \Gamma \rangle$, where $\mathcal{P}$ is a predicate language and Γ is a set of $\mathcal{P}$-formulae. For convenience we sometimes identify the theory T and its set of formulae Γ and say that T is a $\mathcal{P}$-theory to indicate that its language is $\mathcal{P}$.

Definition 3.1 (Structure). *A* $\mathcal{P}$-structure $\mathfrak{S}$ *is a pair* $\langle \boldsymbol{A}, \mathbf{S} \rangle$ *where* $\boldsymbol{A} \in \mathbb{L}$ *and* $\mathbf{S} = \langle S, \langle P_{\mathbf{S}} \rangle_{P \in \mathbf{P}}, \langle f_{\mathbf{S}} \rangle_{f \in \mathbf{F}} \rangle$, *where* S *is a non-empty domain;* $P_{\mathbf{S}}$ *is a function* $S^n \to A$, *for each* n*-ary predicate symbol* $P \in \mathbf{P}$*; and* $f_{\mathbf{S}}$ *is a function* $S^n \to S$ *for each* n*-ary function symbol* $f \in \mathbf{F}$.

An $\mathfrak{S}$-*evaluation* of the object variables is a mapping $\mathrm{v}\colon V \to S$; by $\mathrm{v}[x{\to}a]$ we denote the $\mathfrak{S}$-evaluation where $\mathrm{v}[x{\to}a](x) = a$ and $\mathrm{v}[x{\to}a](y) = \mathrm{v}(y)$ for each object variable $y \neq x$.

Definition 3.2 (Truth definition). *Let* $\mathfrak{S} = \langle \boldsymbol{A}, \mathbf{S} \rangle$ *be a* $\mathcal{P}$*-structure and* v *an* $\mathfrak{S}$*-evaluation. We define the* values *of the terms and the* truth values *of the formulae in* $\mathfrak{S}$ *for an evaluation* v *as:*

$$
\begin{array}{rcll}
\|x\|^{\mathfrak{S}}_{\mathrm{v}} & = & \mathrm{v}(x), & \\
\|f(t_1, \ldots, t_n)\|^{\mathfrak{S}}_{\mathrm{v}} & = & f_{\mathbf{S}}(\|t_1\|^{\mathfrak{S}}_{\mathrm{v}}, \ldots, \|t_n\|^{\mathfrak{S}}_{\mathrm{v}}), & \text{for } f \in \mathbf{F} \\
\|P(t_1, \ldots, t_n)\|^{\mathfrak{S}}_{\mathrm{v}} & = & P_{\mathbf{S}}(\|t_1\|^{\mathfrak{S}}_{\mathrm{v}}, \ldots, \|t_n\|^{\mathfrak{S}}_{\mathrm{v}}), & \text{for } P \in \mathbf{P} \\
\|\circ(\varphi_1, \ldots, \varphi_n)\|^{\mathfrak{S}}_{\mathrm{v}} & = & \circ^{\boldsymbol{A}}(\|\varphi_1\|^{\mathfrak{S}}_{\mathrm{v}}, \ldots, \|\varphi_n\|^{\mathfrak{S}}_{\mathrm{v}}), & \text{for } \circ \in \mathcal{L} \\
\|(\forall x)\varphi\|^{\mathfrak{S}}_{\mathrm{v}} & = & \inf_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathfrak{S}}_{\mathrm{v}[x\to a]} \mid a \in S\}, & \\
\|(\exists x)\varphi\|^{\mathfrak{S}}_{\mathrm{v}} & = & \sup_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathfrak{S}}_{\mathrm{v}[x\to a]} \mid a \in S\}. &
\end{array}
$$

If the infimum or supremum does not exist, we take the corresponding value as undefined. We say that $\mathfrak{S}$ *is* safe *iff* $\|\varphi\|^{\mathfrak{S}}_{\mathrm{v}}$ *is defined for each* $\mathcal{P}$*-formula* φ *and each* $\mathfrak{S}$*-evaluation* v. *Finally, we write* $\mathfrak{S} \models \varphi[\mathrm{v}]$ *if* $\|\varphi\|^{\mathfrak{S}}_{\mathrm{v}} \in F^{\boldsymbol{A}}$.

Definition 3.3 (Model). *Let* T *be a* $\mathcal{P}$*-theory and* $\mathbb{K} \subseteq \mathbb{L}$. *A* $\mathcal{P}$*-structure* $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M} \rangle$ *is called a* $\mathbb{K}$*-model of* T, *denoted as* $\mathfrak{M} \models T$, *if it is* safe, $\boldsymbol{A} \in \mathbb{K}$, *and* $\mathfrak{S} \models \varphi[\mathrm{v}]$ *for each* $\varphi \in T$ *and each* $\mathfrak{S}$*-evaluation* v.

We speak of '$\boldsymbol{A}$-model' instead of '$\{\boldsymbol{A}\}$-model' and we also use this term for safe structures over $\boldsymbol{A}$; we also write $\mathfrak{M} \models \varphi$ instead of $\mathfrak{M} \models \{\varphi\}$. Notice that, since each theory comes with a fixed predicate language, we need not to specify the language of $\mathfrak{M}$ when we say that it is a model of a theory T.

Definition 3.4 (Consequence relation). *Let $\mathbb{K} \subseteq \mathbb{L}$. A $\mathcal{P}$-formula φ is a semantical (sentential) consequence of a $\mathcal{P}$-theory T w.r.t. the class $\mathbb{K}$, in symbols $T \models_{\mathbb{K}} \varphi$, if for each $\mathbb{K}$-model $\mathfrak{M}$ of T we have $\mathfrak{M} \models \varphi$.*

Note that both in the definition of model and semantical consequence, the language of the theory T plays a minor rôle; basically they could be formulated just for sets of formulae. Indeed we can prove that $\langle \mathcal{P}, \Gamma \rangle \models_{\mathbb{K}} \varphi$ iff $\langle \mathcal{P}', \Gamma \rangle \models_{\mathbb{K}} \varphi$ for all $\mathcal{P}' \supseteq \mathcal{P}$ iff $\langle \mathcal{P}', \Gamma \rangle \models_{\mathbb{K}} \varphi$ for some $\mathcal{P}' \supseteq \mathcal{P}$ (actually, due to the safeness restriction, this is not as trivial to prove as in classical predicate logic).

In the next proposition we show that the generalization rule is valid in every $\models_{\mathbb{K}}$ and a rule form of the constants domain axiom holds in $\models_{\mathbb{L}_{\mathrm{RFSI}}}$.

Proposition 3.5. *For any logic* L *we have $\varphi \models_{\mathbb{L}} (\forall x)\varphi$. If furthermore* L *is a finitely disjunctional logic, then we also have: $\varphi \mathbin{\nabla} \psi \models_{\mathbb{L}_{\mathrm{RFSI}}} ((\forall x)\varphi) \mathbin{\nabla} \psi$ whenever x is not free in ψ.*

Proof. The first claim is straightforward: Indeed, for any $\boldsymbol{A}$-model $\mathfrak{M}$ of φ and any $\mathfrak{M}$-evaluation v, we know that $\bar{1}^{\boldsymbol{A}} \leq_{\boldsymbol{A}} \|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x\to a]}$ for each $a \in M$ and thus also $\bar{1}^{\boldsymbol{A}} \leq_{\boldsymbol{A}} \inf^{\boldsymbol{A}}\{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x\to a]} \mid a \in M\}$.

To prove the second claim, consider an $\mathbb{L}_{\mathrm{RFSI}}$-model $\mathfrak{M}$ of $\varphi \mathbin{\nabla} \psi$ and an $\mathfrak{M}$-evaluation v. If $\mathfrak{M} \models \psi[\mathrm{v}]$ we are done. Assume that $\mathfrak{M} \not\models \psi[\mathrm{v}]$, then also $\mathfrak{M} \not\models \psi[\mathrm{v}[x\to a]]$ for each $a \in M$ (because x is not free in ψ). Using the characterization of $\mathbb{L}_{\mathrm{RFSI}}$ from Proposition 2.8 we know that $\mathfrak{M} \models \varphi[\mathrm{v}[x\to a]]$; the rest works as in the first case. ⊣

As it is well known, $\varphi \mathbin{\nabla} \psi \not\models_{\mathbb{HA}} ((\forall x)\varphi) \mathbin{\nabla} \psi$, where $\mathbb{HA}$ is the class of Heyting algebras. The same rule also fails for the class $\mathbb{G}$ of G-algebras, as shown by Horn in [18, page 404]. Therefore, unlike in the propositional case (Theorem 2.6), the consequence relations $\models_{\mathbb{L}_{\mathrm{RFSI}}}$ and $\models_{\mathbb{L}}$ need not coincide.

3.2. Axiomatic systems. The goal of this subsection is to propose axiomatizations for the two natural semantical consequence relations we have introduced and show their basic properties.

Definition 3.6. *Let* L *be a logic in $\mathcal{L}$ presented by an axiomatic system $\mathcal{AS}$. The* minimal predicate logic over L *(in a predicate language $\mathcal{P}$), denoted as* $\mathrm{L}\forall^{\mathrm{m}}$, *is given by the following axiomatic system:*[9]

- (P) *the axioms and rules resulting from those of $\mathcal{AS}$ by substituting variables by $\langle \mathcal{L}, \mathcal{P} \rangle$-formulae*
- (∀1) $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\varphi(x, \vec{z}) \Rightarrow \varphi(t, \vec{z})$, *where t is substitutable for x in φ*
- (∃1) $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi(t, \vec{z}) \Rightarrow (\exists x)\varphi(x, \vec{z})$, *where t is substitutable for x in φ*
- (∀2) $\chi \Rightarrow \varphi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \chi \Rightarrow (\forall x)\varphi$, *where x is not free in χ*
- (∃2) $\varphi \Rightarrow \chi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)\varphi \Rightarrow \chi$, *where x is not free in χ.*

[9] Note that we have omitted the propositional language $\mathcal{L}$ in the symbol $\mathrm{L}\forall^{\mathrm{m}}$ for it is always that of L. Omitting the symbol for the predicate language could be more confusing. Properly, we should first define the notion of proof relative to a predicate language $\mathcal{P}$, denoting it by means of $\vdash^{\mathcal{P}}$ and then prove that as in the semantical case the language of $\mathcal{P}$ plays a little rôle here, i.e., for each $\mathcal{P}$-theory $\Gamma \cup \{\varphi\}$ we have: $\Gamma \vdash^{\mathcal{P}} \varphi$ iff $\Gamma \vdash^{\mathcal{P}'} \varphi$ for all $\mathcal{P}' \supseteq \mathcal{P}$ iff $\Gamma \vdash^{\mathcal{P}'} \varphi$ for some $\mathcal{P}' \supseteq \mathcal{P}$. This can be proved either syntactically (as in the classical logic) or obtained as a consequence of the completeness theorem.

If L *is finitely disjunctional, we also define a stronger predicate logic over* L *(in a predicate language $\mathcal{P}$), denoted here as* L$\forall$,[10] *as the extension of* L$\forall^{\mathrm{m}}$ *by:*

$(\forall 2)^{\nabla}$ $(\chi \Rightarrow \varphi) \nabla \psi \vdash_{\mathrm{L}\forall} (\chi \Rightarrow (\forall x)\varphi) \nabla \psi$, *where x is not free in χ and ψ*

$(\exists 2)^{\nabla}$ $(\varphi \Rightarrow \chi) \nabla \psi \vdash_{\mathrm{L}\forall} ((\exists x)\varphi \Rightarrow \chi) \nabla \psi$, *where x is not free in χ and ψ.*

Let us list some (easy to prove) theorems and derivable rules to demonstrate that the quantification theory is not too different from the classical one (we assume that x is not free in χ, and x' does not occur in $\varphi(x, \vec{z})$):

$(\forall 0)$	$\varphi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\varphi$	$(\forall 0)^{\nabla}$	$\varphi \nabla \chi \vdash_{\mathrm{L}\forall} (\forall x)\varphi \nabla \chi$
$(T1)$	$\varphi \Rightarrow \psi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\varphi \Rightarrow (\forall x)\psi$	$(T2)$	$\varphi \Rightarrow \psi \vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)\varphi \Rightarrow (\exists x)\psi$
$(T3)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} \chi \Leftrightarrow (\forall x)\chi$	$(T4)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)\varphi \Leftrightarrow \chi$
$(T5)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\varphi(x, \vec{z}) \Leftrightarrow (\forall x')\varphi(x', \vec{z})$	$(T6)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)\varphi(x, \vec{z}) \Leftrightarrow (\exists x')\varphi(x', \vec{z})$
$(T7)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)(\forall y)\varphi \Leftrightarrow (\forall y)(\forall x)\varphi$	$(T8)$	$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)(\exists y)\varphi \Leftrightarrow (\exists y)(\exists x)\varphi.$

The following three theorems state crucial properties of our first-order logics. The first one is easily proved by induction using (P), $(T1)$, and $(T2)$. The second one is the usual Constants Theorem (proved almost as in classical logic) showing that free variables behave as constants naming arbitrary elements. The third one requires a more elaborated proof and shows the rôle of disjunction.

THEOREM 3.7 (Congruence Property). *Let φ, ψ, δ be sentences, χ a formula, and $\hat{\chi}$ a formula obtained from χ by replacing some occurrences φ by ψ. Then for $\vdash \in \{\vdash_{\mathrm{L}\forall^{\mathrm{m}}}, \vdash_{\mathrm{L}\forall}\}$:*

$$\vdash \varphi \Leftrightarrow \varphi \qquad \varphi \Leftrightarrow \psi \vdash \psi \Leftrightarrow \varphi \qquad \varphi \Leftrightarrow \delta, \delta \Leftrightarrow \psi \vdash \varphi \Leftrightarrow \psi. \qquad \varphi \Leftrightarrow \psi \vdash \chi \Leftrightarrow \hat{\chi}.$$

THEOREM 3.8 (Constants Theorem). *Let $\vdash \in \{\vdash_{\mathrm{L}\forall^{\mathrm{m}}}, \vdash_{\mathrm{L}\forall}\}$, $T \cup \{\varphi(x, \vec{z})\}$ be a theory, and c a constant not occurring there. Then $\Sigma \vdash \varphi(c, \vec{z})$ iff $\Sigma \vdash \varphi(x, \vec{z})$*

THEOREM 3.9 (Strong Proof by Cases Property). *For any $\mathcal{P}$-theories T, Φ, Ψ and any $\mathcal{P}$-formula χ:*

$$\text{(sPCP)} \qquad \frac{T, \Phi \vdash_{\mathrm{L}\forall} \chi \qquad T, \Psi \vdash_{\mathrm{L}\forall} \chi}{T, \Phi \nabla \Psi \vdash_{\mathrm{L}\forall} \chi}.$$

PROOF. First we prove that for each set of formulae $\Gamma \cup \{\varphi\}$ such that $\Gamma \vdash_{\mathrm{L}\forall} \varphi$ we have $\Gamma \nabla \psi \vdash_{\mathrm{L}\forall} \varphi \nabla \psi$ for any sentence ψ. We show $\Gamma \nabla \psi \vdash_{\mathrm{L}\forall} \delta \nabla \psi$ for each δ appearing in the proof of φ from Γ. If $\delta \in \Gamma$ or is an axiom, the proof is trivial. Now assume that $\Gamma' \vdash_{\mathrm{L}\forall} \delta$ is the rule used to obtain δ. By hypothesis $\Gamma \nabla \psi \vdash_{\mathrm{L}\forall} \Gamma' \nabla \psi$. Since $\Gamma' \nabla \psi \vdash_{\mathrm{L}\forall} \delta \nabla \psi$ (for (P) due to the second item of Proposition 2.8, for $(\forall 2)$ and $(\exists 2)$ it is due resp. to $(\forall 2)^{\nabla}$ and $(\exists 2)^{\nabla}$, and for the latter it is due to (A_{∇}) of Proposition 2.8), the proof of this claim is done.

Now, from $T, \Phi \vdash_{\mathrm{L}\forall} \chi$ and $T, \Psi \vdash_{\mathrm{L}\forall} \chi$, using the claim we have just proved, we obtain $T \nabla \chi, \Psi \nabla \chi \vdash_{\mathrm{L}\forall} \chi \nabla \chi$ and $T \nabla \psi, \Phi \nabla \psi \vdash_{\mathrm{L}\forall} \chi \nabla \psi$ for every $\psi \in \Psi$, and hence $T \nabla \Psi, \Phi \nabla \Psi \vdash_{\mathrm{L}\forall} \chi \nabla \Psi$. Using (C_{∇}) and (I_{∇}) of Proposition 2.8 we can complete the proof. $\dashv$

[10] As no other variants of first-order logics are considered in the paper, we have decided to use the simple notation L$\forall$; otherwise some superscript (e.g. $^{\mathrm{FSI}}$) would be needed. Observe that there is no need to mention the used disjunction in the symbol for L$\forall$ (as all disjunctions are interderivable). It can also be easily shown that axioms $(\forall 2)$ and $(\exists 2)$ are redundant in the axiomatization of L$\forall$.

3.3. Completeness theorem. In this subsection we show that the axiomatic systems $\mathrm{L}\forall^{\mathrm{m}}$ and $\mathrm{L}\forall$ are respectively presentations of the semantically defined first-order logics $\models_{\mathbb{L}}$ and $\models_{\mathbb{L}_{\mathrm{RFSI}}}$. The proofs of soundness (i.e., $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} \subseteq \models_{\mathbb{L}}$ and $\vdash_{\mathrm{L}\forall} \subseteq \models_{\mathbb{L}_{\mathrm{RFSI}}}$) are easy. To prove the reverse inclusions we need the notions of prime and $\forall$-Henkin theory; unless said otherwise $\vdash$ stands for either $\vdash_{\mathrm{L}\forall^{\mathrm{m}}}$ or $\vdash_{\mathrm{L}\forall}$.

Definition 3.10 (Prime and $\forall$-Henkin theories). *Let $\mathcal{P}$ be a predicate language. A $\mathcal{P}$-theory T is*

- Prime *(in $\vdash_{\mathrm{L}\forall}$) if for each pair of $\mathcal{P}$-sentences φ, ψ we have $T \vdash_{\mathrm{L}\forall} \varphi$ or $T \vdash_{\mathrm{L}\forall} \psi$ whenever $T \vdash_{\mathrm{L}\forall} \varphi \nabla \psi$.*
- $\forall$-Henkin *(in $\vdash$) if for each $\mathcal{P}$-formula ψ such that $T \nvdash (\forall x)\psi(x)$ there is a constant c in $\mathcal{P}$ such that $T \nvdash \psi(c)$.*

The next definition is sound thanks to the congruence property of $\Leftrightarrow$ stated in Theorem 3.7.

Definition 3.11 (Lindenbaum–Tarski algebra). *Let φ be a $\mathcal{P}$-sentence and T a $\mathcal{P}$-theory. We define*

$$[\varphi]_T^{\vdash} = \{\psi \mid \psi \text{ a } \mathcal{P}\text{-sentence and } T \vdash \varphi \Leftrightarrow \psi\}.$$

The Lindenbaum–Tarski algebra of T (in $\vdash$), denoted by $\mathbf{LindT}_T^{\vdash}$, has the domain $L_T^{\vdash} = \{[\varphi]_T^{\vdash} \mid \varphi \text{ a } \mathcal{P}\text{-sentence}\}$, and operations (for each n-ary connective c of L and each $\mathcal{P}$-sentences $\varphi_1, \dots, \varphi_n$):

$$\circ^{\mathbf{LindT}_T^{\vdash}}([\varphi_1]_T^{\vdash}, \dots, [\varphi_n]_T^{\vdash}) = [\circ(\varphi_1, \dots, \varphi_n)]_T^{\vdash}.$$

We omit the superscript $\vdash$ whenever it is irrelevant or clear from the context.

Proposition 3.12. *Let T be a $\mathcal{P}$-theory. Then*

1. *$[\varphi]_T \in F_{\mathbf{LindT}_T}$ iff $T \vdash \varphi$. Thus in particular $[\varphi]_T \leq_{\mathbf{LindT}_T} [\psi]_T$ iff $T \vdash \varphi \Rightarrow \psi$.*
2. *$\mathbf{LindT}_T \in \mathbb{L}$.*
3. *$\mathbf{LindT}_T^{\vdash_{\mathrm{L}\forall}} \in \mathbb{L}_{\mathrm{RFSI}}$ iff T is prime.*

Proof. The first claim is established by a chain of simple observations: $[\varphi]_T \in F_{\mathbf{LindT}_T}$ iff $\mathbf{LindT}_T \models \mathcal{T}^{\mathbf{LindT}_T}([\varphi]_T)$ iff for each $\alpha \approx \beta \in \mathcal{T}$, $\alpha^{\mathbf{LindT}_T}([\varphi]_T) = \beta^{\mathbf{LindT}_T}([\varphi]_T)$ iff for each $\alpha \approx \beta \in \mathcal{T}$, $[\alpha(\varphi)]_T = [\beta(\varphi)]_T$ iff for each $\alpha \approx \beta \in \mathcal{T}$, $T \vdash \alpha(\varphi) \Leftrightarrow \beta(\psi)$ iff $T \vdash {\Leftrightarrow}[\mathcal{T}(\varphi)]$ iff $T \vdash \varphi$.

To prove the second claim we show the two conditions from Definition 2.3. The second contition is a simple consequence of the first claim. In order to show the first condition let us assume that $\Gamma \vdash_{\mathrm{L}} \psi$ and fix a $\mathbf{LindT}_T$-evaluation e such that $e[\Gamma] \subseteq F_{\mathbf{LindT}_T}$. Let us inductively define a mapping σ from propositional formulae to $\langle \mathcal{L}, \mathcal{P}\rangle$-sentences: $\sigma(v) \in e(v)$ (arbitrarily for each propositional variable v) and $\sigma(\circ(\varphi_1, \dots, \varphi_n)) = \circ(\sigma\varphi_1, \dots, \sigma\varphi_n)$ for each n-ary connective $\circ$. Now we show by induction that for each propositional formula φ, $[\sigma\varphi]_T = e(\varphi)$. For variables it is clear; if $\circ$ is a connective, we have $[\sigma\circ(\varphi_1, \dots, \varphi_n)]_T = [\circ(\sigma\varphi_1, \dots, \sigma\varphi_n)]_T = \circ^{\mathbf{LindT}_T}([\sigma\varphi_1]_T, \dots, [\sigma\varphi_n]_T) = \circ^{\mathbf{LindT}_T}(e(\varphi_1), \dots, e(\varphi_n)) = e(\circ(\varphi_1, \dots, \varphi_n))$.
Since $e[\Gamma] \subseteq F_{\mathbf{LindT}_T}$, we have $T \vdash \sigma[\Gamma]$. From $\Gamma \vdash_{\mathrm{L}} \psi$ we obtain $\sigma[\Gamma] \vdash \sigma\psi$ (due to (P)). Taken together, we have $T \vdash \sigma\psi$ and so $e(\psi) = [\sigma(\psi)]_T \in F_{\mathbf{LindT}_T}$.

The last claim easily follows from the first one, the definition of prime theory, and Proposition 2.8. ⊣

LEMMA 3.13. *Let T be a $\forall$-Henkin $\mathcal{P}$-theory and $\mathbf{C}$ the set of all closed $\mathcal{P}$-terms. Then for any $\mathcal{P}$-formula φ with only one free variable x holds:*

$$[(\forall x)\varphi]_T = \inf_{\leq_{\mathbf{LindT}_T}} \{[\varphi(c)]_T \mid c \in \mathbf{C}\} \qquad [(\exists x)\varphi]_T = \sup_{\leq_{\mathbf{LindT}_T}} \{[\varphi(c)]_T \mid c \in \mathbf{C}\}.$$

PROOF. We prove only the first claim (the second one is completely analogous). Recall that $[\varphi]_T \leq_{\mathbf{LindT}_T} [\psi]_T$ iff $T \vdash \varphi \Rightarrow \psi$. From this and ($\forall$1) we obtain that $[(\forall x)\varphi]_T$ is a lower bound of $\{[\varphi(c)]_T \mid c \in \mathbf{C}\}$.

Assume that $[\chi]_T \not\leq_{\mathbf{LindT}_T} [(\forall x)\varphi]_T$. Without loss of generality we assume that x is not free in χ (because we know that $[(\forall x)\varphi]_T = [(\forall y)\varphi]_T$ if y does not occur in $\varphi(x)$). Thus $T \nvdash \chi \Rightarrow (\forall x)\varphi$ and so $T \nvdash \chi \Rightarrow \varphi(x)$ (by rule ($\forall$2)) and $T \nvdash (\forall x)(\chi \Rightarrow \varphi(x))$ (by derived rule ($\forall$0)). By the $\forall$-Henkin property of T we obtain a constant $d \in \mathbf{C}$ such that $T \nvdash \chi \Rightarrow \varphi(d)$. Thus finally $[\chi]_T \not\leq_{\mathbf{LindT}_T} [\varphi(d)]_T$, i.e. $[\chi]_T$ is not a lower bound of $\{[\varphi(c)]_T \mid c \in \mathbf{C}\}$. ⊣

DEFINITION 3.14 (Canonical model). *Given a $\forall$-Henkin $\mathcal{P}$-theory T, its* canonical model *(in $\vdash$) $\mathfrak{CM}_T^{\vdash}$ is defined as the $\mathcal{P}$-structure $\langle \mathbf{LindT}_T^{\vdash}, \mathbf{S}\rangle$ where the domain of $\mathbf{S}$ consists of the closed $\mathcal{P}$-terms,*

- *$f_{\mathbf{S}}(t_1, \dots, t_n) = f(t_1, \dots, t_n)$ for each n-ary function symbol $f \in \mathcal{P}$, and*
- *$P_{\mathbf{S}}(t_1, \dots, t_n) = [P(t_1, \dots, t_n)]_T^{\vdash}$ for each n-ary predicate symbol $P \in \mathcal{P}$.*

Now we can easily prove the following proposition which shows that $\mathfrak{CM}_T$ is indeed a $\mathcal{P}$-model of T:

PROPOSITION 3.15. *Let T be a $\forall$-Henkin $\mathcal{P}$-theory. Then for each $\mathcal{P}$-sentence φ we have $\|\varphi\|^{\mathfrak{CM}_T^{\vdash}} = [\varphi]_T$ and so $\mathfrak{CM}_T^{\vdash} \models \varphi$ if, and only if, $T \vdash \varphi$.*

The following two results, actually first-order versions of Lindenbaum lemma, give the final ingredients to obtain completeness.

THEOREM 3.16. *Let $\mathcal{P}$ be a predicate language and $T \cup \{\varphi\}$ a $\mathcal{P}$-theory such that $T \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi$. Then there is a predicate language $\mathcal{P}' \supseteq \mathcal{P}$ and a $\mathcal{P}'$-theory $T' \supseteq T$ such that T' is $\forall$-Henkin in $\mathrm{L}\forall^{\mathrm{m}}$ and $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi$.*

PROOF. Let $\mathcal{P}'$ be an expansion of $\mathcal{P}$ by countably many new object constants, and take $T' = \langle \mathcal{P}', T\rangle$. Take any $\mathcal{P}'$-formula $\psi(x)$, such that $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\psi(x)$. Thus $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \psi(x)$ and so $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \psi(c)$ for some c not occurring in $T' \cup \{\psi\}$ (since T' contains just $\mathcal{P}$-formulae and ψ is a finite object, there is always such $c \in \mathcal{P}'$ and so we can use Constants Theorem). ⊣

The analogous result for $\mathrm{L}\forall$ is more involved and we will obtain it as consequence of the upcoming Theorem 3.25; for now we only formulate it:

THEOREM 3.17. *Let $\mathcal{P}$ be a predicate language and $T \cup \{\varphi\}$ a $\mathcal{P}$-theory such that $T \nvdash_{\mathrm{L}\forall} \varphi$. Then there is a predicate language $\mathcal{P}' \supseteq \mathcal{P}$ and a $\mathcal{P}'$-theory $T' \supseteq T$ such that T' is prime, $\forall$-Henkin in $\mathrm{L}\forall$, and $T' \nvdash_{\mathrm{L}\forall} \varphi$.*

THEOREM 3.18 (Completeness theorem for $\mathrm{L}\forall^{\mathrm{m}}$ and $\mathrm{L}\forall$). *Let* L *be a logic and $T \cup \{\varphi\}$ a $\mathcal{P}$-theory. Then*

$$T \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi \quad \textit{iff} \quad T \models_{\mathbb{L}} \varphi \qquad \textit{and} \qquad T \vdash_{\mathrm{L}\forall} \varphi \quad \textit{iff} \quad T \models_{\mathbb{L}_{\mathrm{RFSI}}} \varphi.$$

A natural question is when these completeness results can be refined to the corresponding classes of models over *completely ordered* algebras, thus avoiding the *safeness* issue. A sufficient condition to achieve this is to assume that the class $\mathbb{L}$ or $\mathbb{L}_{\mathrm{RFSI}}$ admits regular completions, in the sense of the following definition:

DEFINITION 3.19. *Let* L *be a logic. We say that* $\mathbb{K} \subseteq \mathbb{L}$ admits regular completions *if for every* $\boldsymbol{A} \in \mathbb{K}$*, there exists* $\boldsymbol{B} \in \mathbb{K}$ *such that* $\leq_{\boldsymbol{B}}$ *is a complete order and there is an embedding from* $\boldsymbol{A}$ *to* $\boldsymbol{B}$ *preserving all existing suprema and infima.*

In this case, given an $\boldsymbol{A}$-model $\mathfrak{M}$ we can construct a model $\mathfrak{M}'$ over a completely ordered algebra $\boldsymbol{A}^c$ such that $\boldsymbol{A}^c$ is a regular completion of $\boldsymbol{A}$, $\mathfrak{M}$ and $\mathfrak{M}'$ have the same domain, and $\mathfrak{M}'$ interprets the function and predicate symbols in an obvious way. Then for any formula φ and any $\mathfrak{M}$-evaluation e we have $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}} = \|\varphi\|^{\mathfrak{M}'}_{\mathrm{v}}$. Thus, in a way, for classes admitting regular completions we can assume that all models are build over completely ordered algebras.

3.4. $\exists$-Henkin theories, Skolemization and proof of Theorem 3.17. In this subsection we work only in L$\forall$ and so $\vdash$ always stands for $\vdash_{\mathrm{L}\forall}$. We show that some logics L$\forall$ admit a form of Skolemization (the one which allows to erase existential quantifiers in a formula by conservatively introducing new functional symbols) restricted to a certain class Σ of formulae which are *term-closed* (i.e. for each $\varphi(x, \vec{y}) \in \Sigma$, each language $\mathcal{P}$, and each sequence of closed $\mathcal{P}$-terms $\vec{t}$, we have $\varphi(x, \vec{t}) \in \Sigma$) and the logic L$\forall$ enjoys "Skolemization for constants", formally defined as:

DEFINITION 3.20. *We say that* L$\forall$ *is* Σ*-preSkolem if* $T \cup \{\varphi(c)\}$ *is a conservative expansion of* $T \cup \{(\exists x)\varphi(x)\}$ *for each language* $\mathcal{P}$*, each* $\mathcal{P}$*-theory* T*, each* $\mathcal{P}$*-formula* $\varphi(x) \in \Sigma$ *and any constant* $c \notin \mathcal{P}$*.*

For example, every logic is trivially $\emptyset$-preSkolem; intuitionistic and most substructural logics are Σ-preSkolem for Σ being the class of all formulae; some other logics (e.g. fuzzy logics expanded by the Monteiro–Baaz Δ connective) are Σ-preSkolem for Σ being the class of provably classical formulae.

Now we are almost ready to prove the fundamental lemma, but first we observe why it needs to be formulated in such complex fashion. In the process of extending a theory T into a $\forall$-Henkin extension T' we obtain a formula φ unprovable in T we want to keep unprovable in T'. In classical logic we just add $\neg\varphi$ to T and proceed from there. In our non-classical setting the situation is not that simple and so we need to "store" the formulae we want to keep unprovable in a special set Ψ. We will construct those sets with the help of the disjunction ∇. Since ∇ is in general not given by a single formula, we need to work with sets of theories instead of just sets of formulae.

DEFINITION 3.21. *A set of* $\mathcal{P}$*-theories* Ψ *is* deductively directed *if for each* $T, S \in \Psi$ *there is* $R \in \Psi$ *such that* $T \vdash R$ *and* $S \vdash R$*; we call* R *an* upper bound *of* T *and* S *in* Ψ*.*

CONVENTION 3.22. *Let* Ψ *be a set of* $\mathcal{P}$*-theories and* T *a* $\mathcal{P}$*-theory. We write* $T \nVdash \Psi$ *whenever* $T \nvdash S$ *for each* $S \in \Psi$*.*

Definition 3.23. *Let $\mathcal{P} \subseteq \mathcal{P}'$ be predicate languages. We say that a $\mathcal{P}'$-theory T is:*

- $\mathcal{P}$-$\forall$-Henkin *if for each $\mathcal{P}$-formula $\varphi(x)$ such that $T \nvdash (\forall x)\varphi(x)$ there is a constant $c \in \mathcal{P}'$ such that $T \nvdash \varphi(c)$.*[11]
- Σ-$\mathcal{P}$-$\exists$-Henkin *if for each $\mathcal{P}$-formula $\varphi(x) \in \Sigma$ such that $T \vdash (\exists x)\varphi(x)$ there is a constant $c \in \mathcal{P}'$ such that $T \vdash \varphi(c)$.*
- Σ-Henkin *if it is $\mathcal{P}'$-$\forall$-Henkin and Σ-$\mathcal{P}'$-$\exists$-Henkin.*

Lemma 3.24 (Fundamental Lemma). *Let T be a $\mathcal{P}$-theory and Ψ a deductively directed set of finite closed $\mathcal{P}$-theories such that $T \not\Vdash \Psi$. Then the following hold:*

1. *There exist $\mathcal{P}' \supseteq \mathcal{P}$, a $\mathcal{P}'$-theory $T' \supseteq T$, and a deductively directed set of finite closed $\mathcal{P}'$-theories $\Psi' \supseteq \Psi$, such that $T' \not\Vdash \Psi'$ and each theory $S \supseteq T'$ in arbitrary language is $\mathcal{P}$-$\forall$-Henkin whenever $S \not\Vdash \Psi'$.*
2. *If* L$\forall$ *is Σ-preSkolem, then there exist $\mathcal{P}' \supseteq \mathcal{P}$ and a $\mathcal{P}'$-theory $T' \supseteq T$ such that $T' \not\Vdash \Psi$ and each theory $S \supseteq T'$ in arbitrary language is Σ-$\mathcal{P}$-$\exists$-Henkin whenever $S \not\Vdash \Psi$.*
3. *There is a prime $\mathcal{P}$-theory $T' \supseteq T$ such that $T' \not\Vdash \Psi$.*

Proof. 1. We construct the extensions by transfinite recursion. Let $\mathcal{P}'$ be the expansion of $\mathcal{P}$ by new constants $\{c_\nu \mid \nu < ||\mathcal{P}||\}$ (by $||\mathcal{P}||$ we denote the cardinality of the set of $\mathcal{P}$-formulae). We enumerate all $\mathcal{P}$-formulae with one free variable by ordinals as χ_μ for $\mu < ||\mathcal{P}||$. Now we will construct $\mathcal{P}'$-theories T_μ and sets of finite closed $\mathcal{P}'$-theories Ψ_μ such that $T_\mu \subseteq T_\nu$ and $\Psi_\mu \subseteq \Psi_\nu$ for each $\mu \leq \nu$, $T_\mu \not\Vdash \Psi_\mu$, and Ψ_μ is deductively directed. For each $\mu \leq ||\mathcal{P}||$ we define: $T_{<\mu} = \bigcup_{\nu<\mu} T_\nu$ and $\Psi_{<\mu} = \bigcup_{\nu<\mu} \Psi_\nu$ and notice that, by the induction assumption, we have that $\Psi_{<\mu}$ is deductively directed and $T_{<\mu} \not\Vdash \Psi_{<\mu}$ (otherwise there would be a $\nu < \mu$ and $R \in \Psi_\nu$ such that $T_{<\mu} \vdash R$; due to finitarity and finiteness of R there would be a $\nu' < \mu$ and a finite set $T_0 \subseteq T_{\nu'} \subseteq T_{<\mu}$ such that $T_0 \vdash R$; a contradiction with $T_{\max\{\nu,\nu'\}} \not\Vdash \Psi_{\max\{\nu,\nu'\}}$).

We start by taking $T_0 = T$ and $\Psi_0 = \Psi$, which fulfil our conditions. For the induction step we distinguish two possibilities:

- **(H1):** If $T_{<\mu} \vdash R \mathbin{\nabla} (\forall x)\chi_\mu(x)$ for some $R \in \Psi_{<\mu}$, then we define $T_\mu = T_{<\mu} \cup \{(\forall x)\chi_\mu(x)\}$ and $\Psi_\mu = \Psi_{<\mu}$.
- **(H2):** Otherwise we define $T_\mu = T_{<\mu}$ and $\Psi_\mu = \Psi_{<\mu} \cup \{R \mathbin{\nabla} \chi_\mu(c_\mu) \mid R \in \Psi_{<\mu}\}$.

The structural conditions on T_μ and Ψ_μ are clearly met (note that elements of Ψ_μ remain finite due to our restriction to finite disjunctions). Next we show that, no matter which possibility (H1) or (H2) occurred, $T_\mu \not\Vdash \Psi_\mu$ and Ψ_μ is deductively directed.

- **(H1):** Ψ_μ is obviously deductively directed. Assume, for a contradiction, that $T_\mu = T_{<\mu} \cup \{(\forall x)\chi_\mu(x)\} \vdash R'$ for some $R' \in \Psi_\mu$. Take an upper bound $\hat{R}$ of R and R' and notice that $T_{<\mu}, (\forall x)\chi_\mu(x) \vdash \hat{R}$ and $T_{<\mu}, R \vdash \hat{R}$. Thus by Theorem 3.9 we obtain $T_{<\mu}, R \mathbin{\nabla} (\forall x)\chi_\mu(x) \vdash \hat{R}$ and so $T_{<\mu} \vdash \hat{R}$. Since $\hat{R} \in \Psi_{<\mu}$ we have a contradiction with $T_{<\mu} \not\Vdash \Psi_{<\mu}$.

[11] Notice that when $\mathcal{P}' = \mathcal{P}$ we obtain the already defined (without the prefix $\mathcal{P}$) notion of $\forall$-Henkin theory.

(H2): Assume that $T_\mu = T_{<\mu} \vdash R$ for some $R \in \Psi_\mu$. From the induction assumption we know that $T_{<\mu} \nvdash R$ for each $R \in \Psi_{<\mu}$ and so R has to be of the form $R' \mathbin{\nabla} \chi_\mu(c_\mu)$ for some $R' \in \Psi_{<\mu}$. Since c_μ does not appear in $T_{<\mu} \cup \Psi_{<\mu}$, we can use Theorem 3.8 to obtain $T_\mu \vdash R' \mathbin{\nabla} \chi_\mu(x)$, and, by $(\forall 0)^\nabla$, $T_\mu \vdash R' \mathbin{\nabla} (\forall x)\chi_\mu(x)$, a contradiction with the fact that we are in the case (H2). To show that Ψ_μ is deductively directed we distinguish four cases: first if both $R, R' \in \Psi_{<\mu}$ then they have an upper bound already in $\Psi_{<\mu}$. Second assume that $R \in \Psi_{<\mu}$ and $R' = S \nabla \chi_\mu(c_\mu)$ for some $S \in \Psi_{<\mu}$. Let $\hat{R} \in \Psi_{<\mu}$ be an upper bound of R and S. Thus $\hat{R} \mathbin{\nabla} \chi_\mu(c_\mu) \in \Psi_\mu$ is an upper bound of R (trivially) and R' (by the sPCP and the trivial fact that $\chi_\mu(c_\mu) \vdash \hat{R} \mathbin{\nabla} \chi_\mu(c_\mu)$). The final two cases are analogous.

Now take $T' = T_{<||\mathcal{P}||}$ and $\Psi' = \Psi_{<||\mathcal{P}||}$. Thus by the induction assumption $T' \nVdash \Psi'$. Let now S be any theory such that $T' \subseteq S$ and $S \nVdash \Psi'$. We show that S is $\mathcal{P}$-$\forall$-Henkin. Clearly for each $\mu < ||\mathcal{P}||$ if $S \nvdash (\forall x)\chi_\mu(x)$, then we must have used case (H2) (otherwise $T_\mu \vdash (\forall x)\chi_\mu(x)$ and so $S \vdash (\forall x)\chi_\mu(x)$). If $S \vdash \chi_\mu(c_\mu)$, then $S \vdash R \mathbin{\nabla} \chi_\mu(c_\mu)$ for any $R \in \Psi_{<\mu}$. Since we have used case (H2), we know that $R \mathbin{\nabla} \chi_\mu(c_\mu) \in \Psi_\mu$—a contradiction with $S \nVdash \Psi'$.

2. We proceed by transfinite recursion as in 1. Let $\bar{\Sigma}$ be the set of all $\mathcal{P}$-formulae of the form $\varphi(x) \in \Sigma$ and let $\mathcal{P}'$ be the expansion of $\mathcal{P}$ by new constants $\{c_\nu \mid \nu < ||\bar{\Sigma}||\}$. We enumerate all formulae from $\bar{\Sigma}$ by ordinals as $\chi_\mu(x)$. Now we will construct $\mathcal{P}'$-theories T_μ such that $T_\mu \subseteq T_\nu$ and $T_\mu \nVdash \Psi$. For each $\mu \leq ||\mathcal{P}||$ we define $T_{<\mu} = \bigcup_{\nu<\mu} T_\nu$ and notice that, by the induction assumption, we have $T_{<\mu} \nVdash \Psi$ (for reasons similar to the previous case). We start by taking $T_0 = T$, which fulfils our conditions. For the induction step we distinguish two possibilities:

(W1): If $T_{<\mu} \cup \{(\exists x)\chi_\mu(x)\} \nVdash \Psi$, we define $T_\mu = T_{<\mu} \cup \{\chi_\mu(c_\mu)\}$.

(W2): Otherwise we define $T_\mu = T_{<\mu}$.

In the case (W1) we use the fact that $T_{<\mu} \cup \{\chi_\mu(c_\mu)\}$ is a conservative expansion of $T_{<\mu} \cup \{(\exists x)\chi_\mu(x)\}$ (because L$\forall$ is Σ-preSkolem) to obtain $T_\mu \nVdash \Psi$. In the case (W2) we obtain it trivially.

Take $T' = T_{<||\bar{\Sigma}||}$ and observe that $T' \nVdash \Psi$. Let S be an arbitrary theory such that $T' \subseteq S$ and $S \nVdash \Psi$. We show that S is Σ-$\mathcal{P}$-$\exists$-Henkin. If $S \vdash (\exists x)\chi_\mu(x)$ then we used case (W1) (from $T_{<\mu} \cup \{(\exists x)\chi_\mu(x)\} \vdash R$ for some $R \in \Psi$ we would obtain $S \vdash R$, a contradiction). Thus $T_\mu \vdash \chi_\mu(c_\mu)$ and so $S \vdash \chi_\mu(c_\mu)$.

3. We say that T is maximally consistent w.r.t. Ψ if $T \nVdash \Psi$ and for each $\varphi \notin T$ there is $R \in \Psi$ such that $T, \varphi \vdash R$. By Zorn's Lemma be obtain a theory $T' \supseteq T$ which is maximally consistent w.r.t. Ψ. Let us check that T' is prime. Assume that $\varphi \notin T'$ and $\psi \notin T'$. Thus there are $R, S \in \Psi$ such that $T', \varphi \vdash R$ and $T', \psi \vdash S$; take an upper bound $\hat{R}$ of R and S and using the sPCP we obtain that $T', \varphi \mathbin{\nabla} \psi \vdash \hat{R}$ and so $T' \nvdash \varphi \mathbin{\nabla} \psi$. $\dashv$

Besides proving the Skolemization, the next theorem serves another purpose: as any logic is $\emptyset$-preSkolem and $\emptyset$-Henkin theories are just $\forall$-Henkin, it yields the promised proof of Theorem 3.17.

THEOREM 3.25. *Let Σ be a term-closed class of formulae. Then the following are equivalent:*

1. L$\forall$ *is Σ-preSkolem.*
2. *For each $\mathcal{P}$-theory $T \cup \{\varphi\}$ such that $T \nvdash \varphi$ there is $\mathcal{P}' \supseteq \mathcal{P}$ and a prime Σ-Henkin $\mathcal{P}'$-theory $T' \supseteq T$ such that $T' \nvdash \varphi$.*

Moreover, if $\mathbb{L}_{\mathrm{RFSI}}$ admits regular completions, we can add:

3. *$T \cup \{(\forall \vec{y})\varphi(f_\varphi(\vec{y}), \vec{y})\}$ is a conservative expansion of $T \cup \{(\forall \vec{y})(\exists x)\varphi(x, \vec{y})\}$ for each language $\mathcal{P}$, each $\mathcal{P}$-theory T, each $\mathcal{P}$-formula $\varphi(x, \vec{y}) \in \Sigma$ and any functional symbol $f_\varphi \notin \mathcal{P}$ of the proper arity.*

PROOF. We show first that 1 implies 2. Assume that $T \nvdash \varphi$ for some $\mathcal{P}$-formulae $T \cup \{\varphi\}$. We proceed by induction over the set of natural numbers N. Take $T_0 = T$ and $\Psi_0 = \{\{\varphi\}\}$, $\mathcal{P}_0 = \mathcal{P}$. We construct predicate languages $\mathcal{P}_i$, $\mathcal{P}_i$-theories T_i, and deductively directed sets Ψ_i of finite closed $\mathcal{P}_i$-theories such that $T_i \nVdash \Psi_i$ and $\mathcal{P}_i \subseteq \mathcal{P}_j$, $T_i \subseteq T_j$, and $\Psi_i \subseteq \Psi_j$ for $i \leq j$. Observe that the theory T_0, the set Ψ_0 and the language $\mathcal{P}_0$ fulfil these conditions. The induction step is defined according to the following two cases:

- If i is odd: use part 1 of Lemma 3.24 for $\mathcal{P}_i$, T_i, and Ψ_i; define their successors as $\mathcal{P}'_i$, T'_i, and Ψ'_i.
- If i is even: use part 2 of Lemma 3.24 for $\mathcal{P}_i$, T_i, and Ψ_i; define their successors as $\mathcal{P}'_i$, T'_i, and Ψ_i.

Finally, we use the third part of Lemma 3.24 for $\mathcal{P}' = \bigcup\{\mathcal{P}_i \mid i \in \mathsf{N}\}$, $\hat{T} = \bigcup\{T_i \mid i \in \mathsf{N}\}$, and $\Psi' = \bigcup\{\Psi_i \mid i \in \mathsf{N}\}$ and define T' as $\hat{T}'$.

Obviously T' is prime, $T_i \subseteq T'$, and $T' \nVdash \Psi_i$ for each i. Thus from parts 1 and 2 of Lemma 3.24 and the definition of $\mathcal{P}'$ we obtain that T' is Σ-Henkin.

Next we prove that 2 implies 3. We denote $T \cup \{(\forall \vec{y})\varphi(f_\varphi(\vec{y}), \vec{y})\}$ as T_1 and $T \cup \{(\forall \vec{y})(\exists x)\varphi(x, \vec{y})\}$ as T_2. We want to show that $T_2 \nvdash \chi$ implies $T_1 \nvdash \chi$ for each formula χ. We know that there is $\mathcal{P}' \supseteq \mathcal{P}$ and a prime Σ-Henkin $\mathcal{P}'$-theory $T' \supseteq T_2$ such that $T' \nvdash \chi$, and hence $\mathfrak{CM}_{T'} \not\models \chi$. For each sequence $\vec{t}$ of closed $\mathcal{P}'$-terms $T' \vdash (\exists x)\varphi(x, \vec{t})$ (by ($\forall$1)) and hence there is a $\mathcal{P}'$-constant $c_{\vec{t}}$ such that $T' \vdash \varphi(c_{\vec{t}}, \vec{t})$ (we know that $\varphi(x, \vec{t}) \in \Sigma$ because Σ is term-closed). Since $c_{\vec{t}}$ is an element of the domain of $\mathfrak{CM}_{T'}$, we can expand $\mathfrak{CM}_{T'}$ into a model $\mathfrak{M}$ with one additional functional symbol defined as: $(f_\varphi)_{\mathfrak{M}}(\vec{t}) = c_{\vec{t}}$. Recall that thanks to our assumption that $\mathbb{L}_{\mathrm{RFSI}}$ admits regular completions, we can assume that $\mathfrak{M}$ is safe (as it is defined over completely ordered algebra). Since, for each $\mathcal{P}'$-formula, obviously, $\mathfrak{M} \models \psi$ iff $\mathfrak{CM}_{T'} \models \psi$, we obtain: $\mathfrak{M}$ is a model of T and $\mathfrak{M} \not\models \chi$. Also clearly $\mathfrak{M} \models (\forall y)\varphi(f_\varphi(\vec{y}), \vec{y})$, and thus the proof is done.

The proof of 2 implies 1 is analogous: we can read it as introduction of a nullary function symbol $(f_\varphi)_{\mathfrak{M}} = c_\emptyset$. We only show that the additional requirement is not needed. Assume that $\mathfrak{M}$ is not safe, i.e., there is a formula ψ in the language $\mathcal{P}' \cup \{f_\varphi\}$ such that the set $\{\|\psi\|^{\mathfrak{M}}_{\mathrm{v}[x\to a]} \mid a \in M\}$ has no supremum or infimum. Consider the formula ψ' resulting from ψ by replacing the constant f_φ by $c_\emptyset$. Observe that $\{\|\psi\|^{\mathfrak{M}}_{\mathrm{v}[x\to a]} \mid a \in M\} = \{\|\psi'\|^{\mathfrak{CM}_{T'}}_{\mathrm{v}[x\to a]} \mid a \in \mathfrak{CM}_{T'}\}$. Thus $\mathfrak{CM}_{T'}$ is not safe. a contradiction.

The final implication is trivial. $\dashv$

REFERENCES

[1] Willem J. Blok and Don L. Pigozzi, ***Algebraizable logics***, Memoirs of the American Mathematical Society, vol. 396, American Mathematical Society, Providence, RI, 1989, Freely downloadable from `http://orion.math.iastate.edu/dpigozzi/`.

[2] Petr Cintula and Carles Noguera, *A general framework for mathematical fuzzy logic*, ***Handbook of mathematical fuzzy logic - volume 1*** (Petr Cintula, Petr Hájek, and Carles Noguera, editors), Studies in Logic, Mathematical Logic and Foundations, vol. 37, College Publications, London, 2011, pp. 103–207.

[3] ———, *The proof by cases property and its variants in structural consequence relations*, ***Studia Logica***, vol. 101 (2013), no. 4, pp. 713–747.

[4] Janusz Czelakowski, ***Protoalgebraic logics***, Trends in Logic, vol. 10, Kluwer, Dordrecht, 2001.

[5] Michael Dummett, *A propositional calculus with denumerable matrix*, this Journal, vol. 24 (1959), no. 2, pp. 97–106.

[6] Josep Maria Font, Ramon Jansana, and Don L. Pigozzi, *A survey of Abstract Algebraic Logic*, ***Studia Logica***, vol. 74 (2003), no. 1–2, Special Issue on Abstract Algebraic Logic II, pp. 13–97.

[7] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono, ***Residuated lattices: An algebraic glimpse at substructural logics***, Studies in Logic and the Foundations of Mathematics, vol. 151, Elsevier, Amsterdam, 2007.

[8] Kurt Gödel, *Über die Vollständigkeit des Logikkalküls*, ***Ph.D. thesis***, University Of Vienna, 1929.

[9] ———, *Die Vollständigkeit der Axiome des logischen Funktionenkalküls*, ***Monatshefte für Mathematik und Physik***, vol. 37 (1930), pp. 349–360.

[10] ———, *Zur intuitionistischen Arithmetik und Zahlentheorie*, ***Ergebnisse eines mathematischen Kolloquiums***, vol. 4 (1933), pp. 34–38.

[11] Petr Hájek, ***Metamathematics of fuzzy logic***, Trends in Logic, vol. 4, Kluwer, Dordrecht, 1998.

[12] Petr Hájek and Petr Cintula, *On theories and models in fuzzy predicate logics*, this Journal, vol. 71 (2006), no. 3, pp. 863–880.

[13] Petr Hájek and Franco Montagna, *A note on the first-order logic of complete BL-chains*, ***Mathematical Logic Quarterly***, vol. 54 (2008), no. 4, pp. 435–446.

[14] Louise Schmir Hay, *Axiomatization of the infinite-valued predicate calculus*, this Journal, vol. 28 (1963), no. 1, pp. 77–86.

[15] Leon Henkin, *The completeness of formal systems*, ***Ph.D. thesis***, University Of Princeton, 1947.

[16] ———, *The completeness of the first-order functional calculus*, this Journal, vol. 14 (1949), no. 3, pp. 159–166.

[17] David Hilbert and Wilhelm Ackermann, ***Grundzüge der theoretischen Logik (principles of theoretical logic)***, Springer-Verlag, Berlin, 1928.

[18] Alfred Horn, *Logic with truth values in a linearly ordered Heyting algebras*, this Journal, vol. 34 (1969), no. 3, pp. 395–408.

[19] Franco Montagna, *Three complexity problems in quantified fuzzy logic*, ***Studia Logica***, vol. 68 (2001), no. 1, pp. 143–152.

[20] ———, *On the predicate logics of continuous t-norm BL-algebras*, ***Archive for Mathematical Logic***, vol. 44 (2005), no. 1, pp. 97–114.

[21] Andrzej Mostowski, *Axiomatizability of some many valued predicate calculi*, ***Fundamenta Mathematicae***, vol. 50 (1961), no. 2, pp. 165–190.

[22] Hiroakira Ono, *Completions of algebras and completeness of modal and substructural logics*, ***Advances in modal logic*** (Philippe Balbiani, Nobu-Yuki Suzuki, Frank Wolter, and Michael Zakharyaschev, editors), vol. 4, Kings College Publications, 2003, pp. 335–353.

[23] Matthias Emil Ragaz, *Arithmetische Klassifikation von Formelmengen der unendlichwertigen Logik*, ***Ph.D. thesis***, Swiss Federal Institute of Technology, Zürich, 1981.

[24] Helena Rasiowa, ***An algebraic approach to non-classical logics***, North-Holland, Amsterdam, 1974.

[25] Helena Rasiowa and Roman Sikorski, *A proof of the completeness theorem of Gödel*, ***Fundamenta Mathematicae***, vol. 37 (1950), pp. 193–200.

[26] ———, ***The mathematics of metamathematics***, Panstwowe Wydawnictwo Naukowe, Warsaw, 1963.

[27] Kentaro Sato, *Proper semantics for substructural logics, from a stalker theoretic point of view*, ***Studia Logica***, vol. 88 (2008), no. 2, pp. 295–324.

[28] Bruno Scarpellini, *Die Nichtaxiomatisierbarkeit des unendlichwertigen Prädikatenkalküls von Łukasiewicz*, this Journal, vol. 27 (1962), no. 2, pp. 159–170.

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