Abstract Algebraic Logic: Theory and Applications. Petr Cintula and Carles Noguera

3. Description
3.1. Contents of the course (maximum one page)

Abstract Algebraic Logic is a relatively new subfield of Mathematical Logic. It is a natural evolution of Algebraic Logic, which is the branch of Mathematical Logic that studies logical systems by giving them a semantics based on some particular kind of algebraic structures. It can be traced back to George Boole and his study of classical propositional logic by means of a two-element algebra that became its canonical semantics. Linbenbaum-Tarski method was introduced to show completeness of classical logic with respect to the semantics given by Boolean algebras. This method identifies pairs of formulae whose equivalence can be proved from a given theory and shows that this defines a congruence in the algebra of formulae which actually gives a Boolean algebra in the quotient. Analogous proofs were later used to show the completeness of non-classical logics with respect to their corresponding algebraic semantics (e.g. intuitionistic logic w.r.t. Heyting algebras). The fact that it could be analogously repeated in many propositional logics led to more general studies where it was used to show completeness theorems for broad classes of logics such as Rasiowa's implicative logics (studied in her monograph [10]). Abstract Algebraic Logic (AAL) was born as the natural next step to be taken in this evolution: the abstract study of logical systems through the generalization of the Lindenbaum-Tarski process to arbitrary logics. The last two decades have seen the florescence of this subfield of Algebraic Logic resulting in a deep theory of the correspondence between logics and classes of algebras (or logical matrices defined over the algebras) which has very recently obtained its own code in the Mathematics Subject Classification of AMS: 03G27. A very strong link between propositional logics and algebraic semantics was identified and systematically studied first by Blok and Pigozzi, when they introduced the notion of algebraizable logic in [1]. In particular, they introduced the crucial technical notion: the Leibniz operator which maps any theory of a logic to the congruence relation of the formulae which are provably equivalent in the presence of such theory. Then, other classes of logics have been obtained by relaxing the link with their algebraic semantics in terms of properties of the Leibniz operator, which gave rise to the Leibniz hierarchy. This classification has become the core theory of AAL because its classes have been usefully characterized and used to obtain the so-called bridge theorems, i.e. results connecting logical properties to equivalent algebraic properties in the semantics (see [5,7]). An extension of the theory encompassing logics outside the hierarchy has been developed in [6].

The aim of this course is to present an up-to-date and self-contained introduction to AAL. We want to present the field as a collection of useful notions and results for any researcher in non-classical logics, for it provides a uniform approach to propositional systems and a number of deep theorems that allow to understand their properties in terms of equivalent algebraic properties of their semantics. In the first two lessons, we start from very basic syntactical and semantical notions in algebraic logic in a very elementary theory needed to obtain three increasingly strong algebraic completeness theorems. For the sake of simplicity we restrict these first results to the class of weakly implicative logics, which provide a quite general framework (containing most well-known propositional logics: classical, intuitionistic, linear, relevant, fuzzy, and substructural logics in the sense of [8]) but still very simple. In the next two sessions we present the core theory of AAL (following [5,7,9]) now at its full generality by introducing the notion of Leibniz operator for arbitrary logic and presenting its associated hierarchy and characterizations. In this framework we survey several bridge theorems and illustrate on particular examples their usefulness in the algebraic study of non-classical logics.

In the same level of generality we perform an abstract study of disjunction connectives and their associated properties and applications (see [4]). Finally, to conclude the course as it started, in a very down-to-earth fashion, we restrict to another particular well-known class of non-classical logics: fuzzy logics. Following our works [2,3] we mathematically define them as semilinear logics and use the AAL tools introduced in the previous lessons to provide a powerful uniform approach to these logical systems.

3.2. Tentative outline

Session 1: Basic notions of algebraic logic: formulae, proofs, logical matrices, filters, closure operators, closure systems, Schmidt Theorem, abstract Lindenbaum Lemma. Completeness theorem w.r.t. the class of all models. (Weakly) implicative logics and examples.

Session 2: Lindenbaum-Tarski method for weakly implicative logics: Leibniz congruence, reduced matrices, and completeness theorem w.r.t. the class of reduced models. Operators on classes of matrices. Relatively (finitely) subdirectly irreducible matrices (RFSI). Completeness theorem w.r.t. RFSI reduced models. Algebraizability and order algebraizability. Examples.

Session 3: Leibniz operator on arbitrary logics. Leibniz hierarchy: protoalgebraic, equivalential and (weakly) algebraizable logics. Regularity and finiteness conditions. Alternative characterizations of the classes in the hierarchy.

Session 4: Bridge theorems (deduction theorems, Craig interpolation, Beth definability). Generalized disjunctions and proof by cases properties (PCP) and their role in AAL.

Session 5: Applications of AAL to Mathematical Fuzzy Logic: the theory of semilinear logics.

3.3. References

[1] W.J. Blok and D. Pigozzi. *Algebraizable logics.* Memoirs of the American Mathematical Society 396, vol. 77, 1989.
[2] Petr Cintula, Carles Noguera. Implicational (Semilinear) Logics I: A New Hierarchy. *Archive for Mathematical* Logic 49 (2010) 417-446.
[3] Petr Cintula, Carles Noguera. A General Framework for Mathematical Fuzzy Logic. In *Handbook of Mathematical Fuzzy Logic – Volume 1*. Studies in Logic, Mathematical Logic and Foundations, vol. 37, London, College Publications, pp. 103-207, 2011.
[4] Petr Cintula, Carles Noguera. The proof by cases property and its variants in structural consequence relations. To appear in *Studia Logica*, 2011.
[5] Janusz Czelakowski. *Protoalgebraic Logics*. Trends in Logic, vol 10, Dordercht, Kluwer, 2001.
[6] Josep Maria Font and Ramon Jansana. *A General Algebraic Semantics for Sentential Logics*. Springer-Verlag, 1996.
[7] Josep Maria Font, Ramon Jansana, and Don Pigozzi. A survey of Abstract Algebraic Logic. *Studia Logica*, 74(1--2, Special Issue on Abstract Algebraic Logic II):13--97, 2003.
[8] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, Studies in Logic and the Foundations of Mathematics, vol. 151, Amsterdam, Elsevier, 2007.
[9] James Raftery. Order algebraizable logics. Submitted, 2009.
[10] Helena Rasiowa. *An Algebraic Approach to Non-Classical Logics*. North-Holland, Amsterdam, 1974.

3.4. Prerequisites

This course is intended to be highly self-contained. Students are only assumed to have a basic knowledge on classical propositional logic, elementary Set Theory, and some rudiments of Universal Algebra. Knowledge of particular non-classical logics is not necessary but would be helpful in understanding the examples used to illustrate the theory and its applications.