From fuzzy sets to mathematical fuzzy logic

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Outline

Logic is the science that studies the relation of logical consequence:

when a proposition is a logical consequence of (entailed by, follows from, derived from) a set of propositions.

Logic studies correct reasoning.

The notion of correct reasoning is not unique, so there is a multiplicity of logics.

In Mathematics we typically assume the Bivalence Principle.

Bivalence Principle

Every proposition is either true or false.

The usual logic for mathematical reasoning is classical logic.

In classical logic every predicate yields a perfect division between those objects it applies to, and those it does not. We call them *crisp*.

Examples: prime number, even number, monotonic function, continuous function, divisible group, ... (any mathematical predicate)

Formulae of classical logic are evaluated at $\mathcal{B}_2$ (the two-element Boolean algebra).

Correct reasoning in classical logic

Definition

Given $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ we say that φ is a logical consequence of Γ, denoted $\Gamma \models_{\mathcal{B}_2} \varphi$, iff for every $\mathcal{B}_2$-evaluation e:
if $e(\gamma) = 1$ for every $\gamma \in \Gamma$, then $e(\varphi) = 1$.

Remark

$$\frac{\begin{array}{c}\psi_1 \\ \psi_2 \\ \vdots \\ \psi_n\end{array}}{\varphi}$$

is a correct reasoning iff there is no interpretation making the premises true and the conclusion false.

Example: *modus ponens*

$$\frac{\begin{array}{l}\varphi \\ \varphi \to \psi\end{array}}{\psi}$$

Sorites paradox [Eubulides of Miletus, IV century BC]

A man who has no money is poor. If a poor man earns one euro, he remains poor. Therefore, a man who has one million euros is poor.

Formalization:

p_n: *A man who has exactly* n *euros is poor*

$$\frac{\begin{array}{l} p_0 \\ p_0 \to p_1 \\ p_1 \to p_2 \\ p_2 \to p_3 \\ \vdots \\ p_{999999} \to p_{1000000} \end{array}}{p_{1000000}}$$

- There is no doubt that the premise p_0 is true.
- There is no doubt that the conclusion $p_{1000000}$ is false.
- For each i, the premise $p_i \rightarrow p_{i+1}$ seems to be true.
- The reasoning is logically correct (application of *modus ponens* one million times).
- We have a paradox!

The predicates that generate this kind of paradoxes are called *vague*.

Remark

A predicate is vague iff it has borderline cases, i.e. there are objects for which we cannot tell whether they fall under the scope of the predicate.

Example: Consider the predicate *tall*. Is a man measuring 1.78 meters tall?

- It is not a problem of ambiguity. Once we fix an unambiguous context, the problem remains.
- It is not a problem of uncertainty. Uncertainty typically appears when some relevant information is not known. Even if we assume that all relevant information is known, the problem remains.
- It cannot be solved by establishing a crisp definition of the predicate. The problem is: with the meaning that the predicate *tall* has in the natural language, whatever it might be, is a man measuring 1.78 meters tall?

Many-valued logics to deal with vagueness

1949 Sören Halldén in *The Logic of Nonsense* proposes a three-valued logic to model vague predicates.

1955 Stephan Körner proposes an alternative three-valued treatment.

1965 Lotfi Zadeh proposes Fuzzy Set Theory (FST) as a mathematical treatment of vagueness and imprecision. FST becomes an extremely popular paradigm for engineering applications, known also as *Fuzzy Logic*.

1969 Goguen shows how to combine Zadeh's fuzzy sets and Łukasiewicz logic to solve the sorites paradox.

Fuzzy sets as a model for vague predicates

Formally a fuzzy set is a pair $\langle X, \mu \rangle$ where X is a classical set and $\mu : X \to [0, 1]$ is a function (called *membership function*) that maps every object $x \in X$ to its membership degree $\mu(x) \in [0, 1]$.

Example: For the predicate *tall* take $X := [0.3, 2.4]$ (containing all possible heights) and

$$\mu(x) = \begin{cases} 0 & \text{if } x \leq 1.2, \\ \frac{5}{3}x - 2 & \text{if } 1.2 \leq x \leq 1.8, \\ 1 & \text{if } x \geq 1.8. \end{cases}$$

If fuzzy sets interpret atomic vague propositions, their set-theoretic operations correspond to logical connectives for vague propositions.

- Conjunction (intersection): t-norm.
- Disjunction (union): t-conorm.
- Negation (complement): negation function.
- Implication: residuum of the t-norm.

FST solution to sorites paradox

Motto: *Truth comes in degrees.*

- $X = \{0, 1, 2, \dots, 10^6\}$, $\mu : X \to [0, 1]$.
- The truth value of p_n will be $\mu(n)$.
- $\mu(0) = 1$ and $\mu(10^6) = 0$ (the first premise is completely true, the conclusion is completely false).
- Take $\varepsilon = 10^{-6}$ and $\mu(n) = 1 - n\varepsilon$.
- Compute the value of $p_n \to p_{n+1}$ by means of Łukasiewicz implication.
- $\mu(n) \to \mu(n+1) = 1 - \mu(n) + \mu(n+1) = 1 - (1 - n\varepsilon) + (1 - (n+1)\varepsilon) = 1 - \varepsilon < 1$ (all these premises have the same truth value: almost completely true).

Objections

1. Arbitrariness of the interpretation:
   - Why should we use Łukasiewicz functions?
   - Why should we use this evaluation of atomic propositions?
   - Why should we use $[0,1]$ as the set of truth-values?

   Where do the truth values (and their combinations) come from?
2. FST per se, strictly speaking, is not logic.

Requirements from Deductive (Formal, Mathematical) Logic

A logical system must have:

- a formal syntax
- a semantical consequence relation based on some notion of truth
- an (equivalent) notion of proof (completeness theorem)

Outline

The formal language (1st requirement)

- Denumerable set of variables X.
- Binary connective $\&$ for strong conjunction.
- Binary connective $\wedge$ for weak conjunction.
- Binary connective $\rightarrow$ for implication.
- Nullary connective $\overline{0}$ for falsity.

This gives a set of well-formed formulae $\mathrm{Fm}_{\mathcal{L}}$.

Defined connectives:

$$\varphi \vee \psi = ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$$

$$\neg\varphi = \varphi \to \overline{0}$$

$$\overline{1} = \neg\overline{0}$$

Semantics

- Set of truth-values: $[0, 1]$.
- $\&$ is interpreted by a left-continuous t-norm $*$.
- $\wedge$ is interpreted by the $\min$ function.
- $\rightarrow$ is interpreted by the residuum $\Rightarrow$ of $*$.
- $\overline{0}$ is interpreted by 0.
- $\vee$ is interpreted by the $\max$ function.
- $\neg$ is interpreted by the negation function $n(x) = x \Rightarrow 0$.
- $\overline{1}$ is interpreted by 1.

Why should the implication be interpreted as the residuum of a t-norm?

Requirements

- Truth-functionality: the connectives are interpreted by functions (a t-norm $*$ for strong conjunction, a function $\Rightarrow$ for implication).
- Normality: the connectives behave classically in $\{0, 1\}$.
- $\Rightarrow$ captures the order: $x \Rightarrow y = 1$ iff $x \leq y$.
- Truth-preserving *modus ponens*: the truer the premises are, the truer the conclusion is.

Why should the implication be interpreted as the residuum of a t-norm?

Modus ponens is a valid rule in this sense: if $v(\varphi) = 1$ and $v(\varphi) \Rightarrow v(\psi) = 1$, then $v(\psi) = 1$

But, we want more:

$v(\varphi) * [v(\varphi) \Rightarrow v(\psi)] \leq v(\psi)$.

Take the residuum:
$v(\varphi) \Rightarrow v(\psi) = \max\{c \in [0, 1] \mid v(\varphi) * c \leq v(\psi)\}$, i.e. the maximum value that fulfills the required inequality, and so making the *modus ponens* rule as powerful as possible.

- Each continuous t-norm is residuated.
- Actually, in the proof only left-continuity is used.

Proposition

Let $*$ be a t-norm. Then:
$*$ is residuated if, and only if, it is left-continuous.

Historical remarks: two converging lines

1930 Łukasiewicz and Tarski: $[0,1]$-valued logic that uses a particular continuous t-norm to interpret (strong) conjunction.

1942 Menger: Statistical metrics. Triangular norms (t-norms) as functions to deal with triangular inequality.

1957 Mostert, Shields: Representation of continuous t-norms as ordinal sums of three basic components.

1959 Dummett resumes Gödel's work from 1932 and proposes a $[0,1]$-valued logic that uses $\min$ function (a continuous t-norm) to interpret conjunction.

1960 Schweizer and Sklar: Development of statistical metric spaces.

- **1965** Zadeh: Fuzzy Set Theory (FST). He proposes $\min$ function for the intersection of fuzzy sets.
- **1969** Goguen: combined use of Zadeh's fuzzy sets and Łukasiewicz functions to solve the sorites paradox.
- **late 70s** Several researchers in FST propose the usage of t-norms to interpret intersection of fuzzy sets and conjunction in the logics. All known residuated t-norms are continuous.
- **1995** Fodor: Nilpotent Minimum, a left-continuous non-continuous t-norm.
- **1996** Esteva, Godo, Hájek: Product logic as a $[0, 1]$-valued logic that uses product t-norm to interpret (strong) conjunction.

1998 Hájek introduces the logic BL and conjectures that it is the logic of all continuous t-norms. He shows that Ł, G, and Π are axiomatic extensions of BL.

2000 Hájek, Cignoli, Esteva, Godo and Torrens: BL is the logic of all continuous t-norms.

2001 Esteva and Godo introduce the logic MTL and conjecture that it is the logic of all left-continuous t-norms. They show that the logic NM of Fodor's t-norm and BL are axiomatic extensions of MTL.

2002 Jenei and Montagna: MTL is the logic of all left-continuous t-norms.

Hilbert-style calculus for MTL

(MTL1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(MTL2)	$\varphi \& \psi \to \varphi$
(MTL3)	$\varphi \& \psi \to \psi \& \varphi$
(MTL4)	$\varphi \wedge \psi \to \varphi$
(MTL5)	$\varphi \wedge \psi \to \psi \wedge \varphi$
(MTL6)	$\varphi \& (\varphi \to \psi) \to \varphi \wedge \psi$
(MTL7a)	$(\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi)$
(MTL7b)	$(\varphi \& \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(MTL8)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(MTL9)	$\overline{0} \to \varphi$

Inference rule: *modus ponens*

$\Gamma \vdash_{\mathrm{MTL}} \varphi$ means that there is a proof of φ from Γ in this system.

The algebraic semantics

Definition

An MTL-algebra $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ is a residuated lattice satisfying the prelinearity equation:
$(x \rightarrow y) \vee (y \rightarrow x) \approx \overline{1}$.
If the lattice order is linear, we call it MTL-*chain*.

Let $*$ be a left-continuous t-norm and $\Rightarrow_*$ its residuum. Then $[0,1]_* = \langle [0,1], *, \Rightarrow_*, \min, \max, 0, 1 \rangle$ is an MTL-chain.

All MTL-chains over $[0,1]$ are of this form and are called *standard chains*.

Theorem

The class of all MTL*-algebras,* $\mathbb{MTL}$*, is a variety.*

Important subvarieties of $\mathbb{MTL}$:

1. $\mathbb{BL}$: $x\&(x \to y) \approx x \wedge y$ (divisibility equation).
2. $\mathbb{MV}$: divisibility + $\neg\neg x \approx x$ (involution equation).
3. $\mathbb{P}$: divisibility + $\neg x \vee ((x \to x\&y) \to y) \approx \overline{1}$ (cancellativity equation).
4. $\mathbb{G}$: $x \approx x\&x$ (contraction equation).
5. $\mathbb{BA}$: $x \vee \neg x \approx \overline{1}$ (excluded middle).
6. ...

Semantical notion of logical consequence (2nd requirement)

Definition

Given $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, we define the consequence relation $\Gamma \models_{\mathbb{MTL}} \varphi$ iff for every $\mathcal{A} \in \mathbb{MTL}$ and every $\mathcal{A}$-evaluation e: if $e[\Gamma] \subseteq \{\overline{1}^{\mathcal{A}}\}$, then $e(\varphi) = \overline{1}^{\mathcal{A}}$. (Preservation of complete truth)

Completeness (3rd requirement)

Theorem

For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{MTL}} \varphi$ *iff* $\Gamma \models_{\mathbb{MTL}} \varphi$.

$\mathbb{MTL}$ is the equivalent algebraic semantics of MTL.

Theorem

Every MTL*-algebra is representable as a subdirect product of* MTL*-chains.*

Corollary

For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{MTL}} \varphi$ *iff* $\Gamma \models_{\{\mathrm{MTL}-chains\}} \varphi$.

Remark

These properties remain valid for axiomatic extensions as well.

Some axiomatic extensions of MTL

BL	=	MTL	+	$\varphi \wedge \psi \to \varphi \& (\varphi \to \psi)$	(Div)
Ł	=	BL	+	$\neg\neg\varphi \to \varphi$	(Inv)
G	=	BL	+	$\varphi \to \varphi \& \varphi$	(C)
Π	=	BL	+	$\neg\psi \vee ((\psi \to \varphi \& \psi) \to \varphi)$	(Can)
CPC	=	MTL	+	$\varphi \vee \neg\varphi$	(EM)

...

Strong standard completeness of MTL

Theorem

For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,
$\Gamma \vdash_{\mathrm{MTL}} \varphi$ *iff* $\Gamma \models_{\{[0,1]_* \mid * \text{ left-continuous t-norm}\}} \varphi$.

Solution to the objection of arbitrariness

The logical consequence does not depend on particular arbitrary choices, but on *all possible interpretations*.

Outline

Mathematical Fuzzy Logic is the study of systems of fuzzy logic with the methodology of Mathematical Logic (metamathematics of fuzzy logics).

Other systems of fuzzy logic

- Removing bottom: Logics based on hoops on the real interval (proponents: Esteva, Godo, Hájek, Montagna 2002)
- Removing commutativity: Logics based non-commutative t-norms (proponents: Hájek 2002)
- Adding expressive power: Logics based on expanded standard algebras (with Δ, $\sim$, truth-constants, etc.)
- Leaving the real interval behind: Logics complete w.r.t. rational, finite, hyperreal, and in general (expansions of) linearly ordered MTL-algebras.
- General classes of fuzzy logics: Core and Δ-core fuzzy logics (proponents: Hájek and Cintula 2006)
- Removing integrality: Uninorm logics (proponents: Metcalfe, Montagna, 2007)

Relation to other families of logics

$$(lin) \quad (\varphi \to \psi) \vee (\psi \to \varphi)$$

$$(lin_1) \quad ((\varphi \to \psi) \wedge \overline{1}) \vee ((\psi \to \varphi) \wedge \overline{1})$$

- G is the axiomatic extension of intuitionistic logic obtained by adding (lin)
- MTL is the axiomatic extension of FL_{ew} (monoidal logic) obtained by adding (lin)
- UL is the axiomatic extension of FL_e (linear logic) obtained by adding (lin_1)

Fuzzy logics are a subfamily of substructural logics

Taking intermediate truth degrees seriously

1. Truth degrees as syntactical objects: add a truth-constant $\overline{r}$ for each truth value r (Pavelka logic, Hájek RPL, Novák's evaluated syntax, algebraic approach by Esteva et al).
2. Logics preserving truth degrees: $\Gamma \models_{\mathcal{A}} \varphi$ iff for every $\mathcal{A}$-evaluation e: if $\min e[\Gamma] \leq e(\varphi)$. (Preservation of partial truth)

The essence of Mathematical Fuzzy Logic

1. Completeness w.r.t. a semantics based on chains (Hájek, Pavelka, Novák, Esteva, Godo, Montagna ...)
2. Fuzzy logics are the logics of chains (Běhounek-Cintula 2006)
3. Mathematical definition: semilinear logics (Cintula-Noguera 2010)

Agenda of Mathematical Fuzzy Logic

- Semantics: algebraic and relational semantics, algebraic study of logical properties, ...
- Model theory: study of first-order models
- Set theory: axiomatic set theory over fuzzy logics
- Proof theory: sequent systems, hypersequents systems, ...
- Recursion theory: computational complexity, arithmetical complexity

Agenda of Mathematical Fuzzy Logic

- Game theory: dialogical games, Rényi-Ulam games, evaluation games
- Philosophy of Logic: liar paradox, vagueness paradoxes
- Applied logical calculi: logic programming, possibilistic and probabilistic reasoning, description logics, similarity-based reasoning, ...

International working group (www.mathfuzzlog.org)

Handbook of Mathematical Fuzzy Logic, to appear in College Publications, 2010

Conference *Logic, Algebra and Truth Degrees*, 7 - 11 September 2010, Prague

Conclusions

- MFL is the study of fuzzy logics arising from FST.
- Fuzzy logics allow a mathematical treatment of *some aspects* of vagueness, those corresponding to gradual properties.
- Fuzzy logics are logics of graduality.
- MFL is a very active branch of Mathematical Logic with many challenging open problems.