Archive for Mathematical Logic manuscript No.
(will be inserted by the editor)

Implicational (semilinear) logics II: additional connectives and characterizations of semilinearity

Petr Cintula · Carles Noguera

Received: date / Accepted: date

Abstract This is the continuation of the paper [4]. We continue the abstract study of non-classical logics based on the kind of generalized implication connectives they possess and we focus on semilinear logics, i.e. those that are complete with respect to the class of models where the implication defines a linear order. We obtain general characterizations of semilinearity in terms of the intersection-prime extension property, the syntactical semilinearity metarule and the class of finitely subdirectly irreducible models. Moreover, we consider extensions of the language with lattice connectives and generalized disjunctions, study their interplay with implication and obtain axiomatizations and further descriptions of semilinear logics in terms of disjunctions and the proof by cases property.

Keywords Abstract algebraic logic · implicational logics · disjunctional logics · semilinear logics · non-classical logics · transfer theorems.

Mathematics Subject Classification (2000) 03B22 · 03B47 · 03B52 · 03G99

1 Introduction

The paper [4] started a new approach to Abstract Algebraic Logic (AAL) in which, instead of the usual equivalence-based classification of logical systems leading to the well-known Leibniz hierarchy of protoalgebraic logics (see [8]), we presented an alternative setting based on implication connectives. By studying the properties

Petr Cintula
Institute of Computer Science, Czech Academy of Sciences
Pod Vodárenskou věží 2, 182 07 Prague, Czech Republic
URL: http://www2.cs.cas.cz/ cintula/
E-mail: cintula@cs.cas.cz

Carles Noguera
Institute of Information Theory and Automation, Czech Academy of Sciences
Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic
URL: http://www.carlesnoguera.cat/
E-mail: noguera@utia.cas.cz

of implications (understood as generalized connectives defined by sets of formulae in two variables and, possibly, with parameters), we obtained a refinement of the Leibniz hierarchy with several new classes of *weakly p-implicational logics*. Moreover, it was shown that implications define an order relation in the semantical counterpart of these logics, i.e. in their reduced matrix models. This yielded a natural definition of *semilinear implications* (and the corresponding classes of *weakly p-implicational semilinear logics*) as those that endow the logic with a complete semantics of *linearly ordered* matrix models. The paper also discussed the usefulness of this notion to provide an AAL framework for prominent systems of many-valued logics studied in the literature as *fuzzy logics* (see e.g. [2]).

Semilinearity was characterized in [4] (only for *finitary* logics) in terms of a purely syntactical property: the metarule called *Semilinearity Property* (SLP):

$$\frac{\Gamma, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}.$$

An important question was whether, given a logic L with a semilinear implication $\Rightarrow$, the class of linearly ordered models $\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$ is intrinsically determined or it depends on the chosen implication. We showed that (assuming finitarity of L again and even finiteness of the set of formulae defining $\Rightarrow$) such class is indeed intrinsically defined and it coincides with a well understood subclass of models of the logic, that is, the *relatively finitely subdirectly irreducible* reduced models. In symbols: we proved that for every finite semilinear implication $\Rightarrow$ in a finitary logic L, $\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L}) = \mathbf{MOD}^{*}(\mathrm{L})_{\mathrm{RFSI}}$. Moreover, in such context, this gave another characterization of semilinearity, now of a semantical nature.

The existence of prominent examples of *infinitary* semilinear logics in the literature (such as the well-known infinitely-valued Łukasiewicz logic [14]) demonstrates the limitations of these characterization theorems.

On the other hand, the paper [6] carried out an investigation of another logical connective, disjunction, from a similar point of view. In a completely general context of Tarskian structural consequence relations (not limited to protoalgebraic logics), we studied the properties of generalized disjunctions (following [8], defined by arbitrary sets of formulae). We argued that their essential feature is the *Proof by Cases Property* (PCP). Again, some characterization theorems seemed to require the assumption of finitary in previous works, but we identified a suitable generalization of finitarity that allowed to prove all the desired results. It is a well-known fact that, in a finitary logic, theories always can be decomposed as intersection of finitely meet-irreducible theories (also called $\cap$-*prime theories*); equivalently: if L is a finitary logic, T is a theory, and $T \nvdash_{\mathrm{L}} \varphi$, then there is an $\cap$-prime theory $T' \supseteq T$ such that $T' \nvdash_{\mathrm{L}} \varphi$. This property is called in [6] *Intersection-Prime Extension Property* (IPEP). It yields a class of logics (properly extending the class of finitary logics),[1] in which many interesting results about disjunction connectives could be proved. Therefore, there is a strong intuition that the IPEP might also be the missing condition in a more general characterization theorem for semilinear implications.

As a last precedent, we have to mention the book chapter [5] where we presented a rather simplified AAL-based framework for fuzzy logics. In particular, it was always assumed that implication was either a primitive connective or given by

[1] It can also be shown, by elaborating examples in [6], that not all logics have the IPEP.

a formula in two variables, and most results where proved under the assumption of finitarity (thus leaving out, as mentioned above, several important fuzzy logics), due to the limitations of the characterization theorems in [4]. Moreover, the chapter used the notions of disjunction studied in [6] to provide axiomatizations and alternative characterizations of fuzzy logics in terms of PCP and other properties involving disjunction.

The present paper is intended as a necessary second part of [4]. Its main goal is to contribute to AAL by continuing the general study of implicational (semilinear) logics. More precisely, we want to pursue two aims: (a) we want to achieve a better understanding of semilinearity itself, by obtaining fully general characterization theorems in terms of semantical and syntatical properties of the implication, (b) we want to describe the rôle of additional connectives (disjunction, conjunction, constants for truth and falsity) and their interplay with implication, in particular obtaining further equivalent descriptions of semilinearity.

To fullfil the first goal we will need to bridge the gap between syntax and semantics, because of the different nature of the expected characterizations of semilinearity. We do it by proving a transfer theorem[2] for the SLP, i.e. showing that if the metarule holds in the logic, then it also holds in the semantics. This is achieved by essentially describing, inside the logic, its semantical models by making use of double implications (giving equality in these models) and a sufficient amount of propositional variables to name all the elements in the algebraic domain. Therefore, we might need to extend the sets of variables without changing the properties of our logics. We do it by resorting to *natural extensions* as studied in [8].

In three particular steps of the proof of this transfer theorem (once in Lemma 2, twice in Lemma 3), we need to assume a reasonable assumption on the cardinality of the logic.[3] For this reason, in this paper we will always assume that logics are structural consequences relations with the additional assumption established in the following convention.

Convention 1 *Let $\mathcal{L}$ be a propositional language with a set of variables Var. We say that a structural consequence relation* L *in $\mathcal{L}$ is a* logic *if for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ we have: if $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is $\Gamma_0 \subseteq \Gamma$ with $|\Gamma_0| \leq |Var|$ such that $\Gamma_0 \vdash_{\mathrm{L}} \varphi$.*

The vast majority of logics considered in the literature do satisfy this restriction, in particular:

- when L is a finitary logic, because we assume that Var is always infinite,
- when $|Var| = |Fm_{\mathcal{L}}|$ (e.g. if the set of logical connectives is countable), because then we can take $\Gamma_0 = \Gamma$

[2] *Transfer theorems* are typical results in AAL, in which one shows that a (lattice theoretic or closure system) property of the theories of the logic also holds for the set of filters over any algebra of the logic.

[3] One of the reasons is the very construction of a natural extension that we take from [8] which, as we have shown in [3], only works under this cardinality restriction. We are aware of alternative unpublished constructions of natural extensions (personal communications by Ramon Jansana and Adam Přenosil) that work without our cardinality restriction and would even allow us to prove Lemma 3 in full generality. However, for Lemma 2 the restriction is still needed (a counterexample can be given), so we prefer to stick to this presentation of natural extensions based on already published results.

The paper is organized as follows. After this introduction, Section 2 proves the transfer of SLP, introduces logics with the IPEP, and proves a characterization of semilinearity in terms of IPEP, SLP, and relatively subdirectly irreducible models. Then Section 3 enriches the expressivity of weakly p-implicational logics by considering lattice protoconjuntions and protodisjunctions and truth-constants for top, bottom, and the unit. All these connectives are syntactically described by consecutions of the logic and semantically by showing their behaviour on matrix models (lattice) ordered by the implication. Moreover, we consider lattice conjunctions and disjunctions by strengthening the former connectives with, respectively, the properties of adjunction and proof by cases. Finally, Section 4, studies the interplay of generalized disjunctions with implications, introduce syntatical matching properties, and obtains other useful characterizations and axiomatization of semilinear logics in terms of disjunctions and their properties.

We have tried to keep the paper reasonably self-contained, although we need to assume some familiarity with the notation and the definitions from [4]. For a background knowledge on AAL and its methods we refer to [8–10].

2 Characterization of semilinear implications

As mentioned in the introduction, the main goal of this section is to improve the main characterization of semilinear implications [4, Theorem 16]. Let us now recall this theorem[4] along with some crucial notions from [4] that we need in this paper.

Theorem 1 (Characterization of semilinear implications [4]) *Let* L *be a logic in a language* $\mathcal{L}$ *with a weak p-implication* $\Rightarrow$*. Then the following are equivalent:*

1. $\Rightarrow$ *is* semilinear *in* L*, i.e.,* $\mathrm{L} = \models_{\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})}$.
2. L *has the* Linear Extension Property*,* LEP*, w.r.t.* $\Rightarrow$*, i.e.* $\Rightarrow$*-linear theories form a basis of* Th(L)*.*

If furthermore L *is finitary, we can add two more equivalent conditions:*

3. L *has the* Semilinearity Property*,* SLP*, w.r.t.* $\Rightarrow$*, i.e. for each set of formulae* $\Gamma \cup \{\varphi, \psi, \chi\}$ *we have* $\Gamma \vdash_{\mathrm{L}} \chi$ *whenever* $\Gamma, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \chi$ *and* $\Gamma, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} \chi$*.*
4. L *has the* transferred LEP *w.r.t.* $\Rightarrow$*, i.e. for each* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *the* $\Rightarrow$*-linear filters form a basis of* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$*.*
5. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}} \subseteq \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$*.*

If furthermore $\Rightarrow$ *is finite, we can add one more equivalent condition:*

6. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}} \subseteq \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$*.*

In order to improve the theorem, we first need a simple yet useful lemma.

Lemma 1 *Let* L *be a logic with a weak p-implication* $\Rightarrow$*. Let* $\mathbf{A}, \mathbf{B} \in \mathbf{MOD}(\mathrm{L})$ *and* $h\colon \mathbf{A} \to \mathbf{B}$ *be a homomorphism.*

- *If* h *is strict, then* $h(a) \leq^{\Rightarrow}_{\mathbf{B}} h(b)$ *implies* $a \leq^{\Rightarrow}_{\mathbf{A}} b$*.*
- *If* h *is surjective or* $\Rightarrow$ *is parameter-free, then* $a \leq^{\Rightarrow}_{\mathbf{A}} b$ *implies* $h(a) \leq^{\Rightarrow}_{\mathbf{B}} h(b)$*.*
- *If* h *is strict and* $\mathbf{B}$ *is* $\Rightarrow$*-linear, then so is* $\mathbf{A}$*.*

[4] Condition 4 was not explicitly a part of [4, Theorem 16], but we can add it by virtue of [4, Theorem 15].

Proof Let us take $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ and $\mathbf{B} = \langle \boldsymbol{B}, G\rangle$ and consider the following chain of implication/equivalencies:

$$\begin{aligned} a \leq^{\Rightarrow}_{\mathbf{A}} b &\Longleftrightarrow a \Rightarrow^{\boldsymbol{A}} b \subseteq F \\ &\Longrightarrow h[a \Rightarrow^{\boldsymbol{A}} b] \subseteq h[F] \subseteq G \\ &\Longleftarrow h(a) \Rightarrow^{\boldsymbol{B}} h(b) \subseteq G \\ &\Longleftrightarrow h(a) \leq^{\Rightarrow}_{\mathbf{B}} h(b) \end{aligned}$$

Both equivalencies hold by definition. The first implication is trivial because h is a homomorphism and so is second one because $h[a \Rightarrow^{\boldsymbol{A}} b] \subseteq h(a) \Rightarrow^{\boldsymbol{B}} h(b)$.

To complete the proof of the first two claims just observe that the strictness of h allows to replace the first implication by an equivalence; analogously for the surjectivity or the lack of parameters and the second implication.

The last claim is an easy consequence of the first one. □

The crucial result we need to achieve our goal is the fully general transfer theorem for the SLP, which in [5, Theorem 3.1.6] was proved only for weakly implicative logics with countable sets of formulae. We start by showing that, given a logic with that property, we can increase the set of variables without losing the SLP and essentially keeping the same logic. To this end, we need to introduce two useful notions: the cardinality of a structural consequence relation and the natural extension of a structural consequence relation to another one on a bigger set of variables.

Definition 1 The *cardinality* of a structural consequence relation L, card(L) in symbols, is the smallest cardinal κ such that for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ we have: if $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is $\Gamma_0 \subseteq \Gamma$ with $|\Gamma_0| < \kappa$ such that $\Gamma_0 \vdash_{\mathrm{L}} \varphi$.

Observe that a structural consequence relation L is *finitary* iff $\mathrm{card}(\mathrm{L}) \leq \omega$. Also note that a structural consequence relation is a logic in the new restricted sense of this paper (see Convention 1) iff $\mathrm{card}(\mathrm{L}) \leq |Var|^+$ (note that this restriction will be necessary for the proof of the next lemma).

Definition 2 Let L be a structural consequence relation in a language $\mathcal{L}$ with variables Var. Consider a structural consequence relation L′ in the language which has the same connectives as $\mathcal{L}$ and variables $Var' \supseteq Var$. We say that L′ is a *natural extension* of L if it extends L conservatively, i.e. for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ we have: $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \vdash_{\mathrm{L}'} \varphi$, and $\mathrm{card}(\mathrm{L}') = \mathrm{card}(\mathrm{L})$.

Lemma 2 *Let* L *be a logic in a language* $\mathcal{L}$ *(with variables* Var*) with a weak p-implication* $\Rightarrow$ *and let* L′ *be a natural extension in the language in the language which has the same connectives as* $\mathcal{L}$ *and variables* $Var' \supseteq Var$*. Then:*

– $\Rightarrow$ *is a weak p-implication in* L′
– $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A}) = \mathcal{F}i_{\mathrm{L}'}(\boldsymbol{A})$ *for each* $\mathcal{L}$*-algebra* $\boldsymbol{A}$

Proof The first claim is proved analogously to the proof of [4, Proposition 3]. As for the second claim, it is clear that $\mathcal{F}i_{\mathrm{L}'}(\boldsymbol{A}) \subseteq \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$; let us prove the converse inclusion. Take any $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ and a set $\Gamma \cup \{\varphi\}$ of formulae in the expanded language such that $\Gamma \vdash_{\mathrm{L}'} \varphi$. Since $\mathrm{card}(\mathrm{L}') \leq |Var|^+$, we can assume that

$|\Gamma| \leq |Var|$. Then there has to be $\sigma\colon Var' \to Var$ whose restriction to variables from $\Gamma \cup \{\varphi\}$ is one-one. Thus we know that $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma\varphi$ (from conservativity). Clearly, for each evaluation $e\colon Var' \to A$ there is an evaluation $e'\colon Var \to A$ such that $e'(\sigma p) = e(p)$ for each p appearing in $\Gamma \cup \{\varphi\}$. Thus if $e[\Gamma] \subseteq F$, then $e'[\sigma[\Gamma]] \subseteq F$ and so $e(\varphi) = e'(\sigma\varphi) \in F$.

Lemma 3 *Let* L *be a logic in a language $\mathcal{L}$ with a weak p-implication $\Rightarrow$ satisfying the* SLP*, $Var' \supseteq Var$ a set of arbitrary cardinality, and $\mathcal{L}'$ the language with the same connectives as $\mathcal{L}$ and atoms from Var'. Then there is a natural extension* L′ *in $\mathcal{L}'$ which still enjoys the* SLP *w.r.t. $\Rightarrow$.*

Proof According to [8, Exercise 0.3.3] a natural extension can be syntactically defined by setting for any $\Gamma' \cup \{\varphi'\} \subseteq Fm_{\mathcal{L}'}$:

$$\Gamma' \vdash_{\mathrm{L}'} \varphi' \quad \text{iff} \quad \text{there are } \Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}} \text{ and a mapping } h\colon Var \to Fm_{\mathcal{L}'} \text{ such that } h(\varphi) = \varphi',\ h[\Gamma] \subseteq \Gamma', \text{ and } \Gamma \vdash_{\mathrm{L}} \varphi.$$

However, as shown in [3], such relation can only be guaranteed to be a consequence relation under the assumption that $\mathrm{card}(\mathrm{L}) \leq |Var|^+$, which is exactly the restriction we put on *logics* in this paper (Convention 1).

Thanks to the previous lemma, we know that $\Rightarrow$ is a weak p-implication in L′. Let us prove that $\Rightarrow$ satisfies the SLP. Take arbitrary formulae $\Gamma \cup \{\varphi, \psi, \chi\} \subseteq Fm_{\mathcal{L}'}$ and assume $\Gamma, \varphi \Rightarrow^{\mathcal{L}'} \psi \vdash_{\mathrm{L}'} \chi$ and $\Gamma, \psi \Rightarrow^{\mathcal{L}'} \varphi \vdash_{\mathrm{L}'} \chi$. Since $\mathrm{card}(\mathrm{L}') \leq |Var|^+$, there are sets $\Gamma' \subseteq \Gamma$, $\Delta_1 \subseteq \varphi \Rightarrow^{\mathcal{L}'} \psi$, and $\Delta_2 \subseteq \psi \Rightarrow^{\mathcal{L}'} \varphi$ such that $|\Gamma'| + |\Delta_1| + |\Delta_2| \leq |Var|$, $\Gamma', \Delta_1 \vdash_{\mathrm{L}'} \chi$, and $\Gamma', \Delta_2 \vdash_{\mathrm{L}'} \chi$. Thus $|V(\Gamma' \cup \Delta_1 \cup \Delta_2 \cup \{\chi\})| \leq |Var|$ (given any set of formulae Σ, by $V(\Sigma)$ we denote the set of variables occurring in formulae from Σ). By an easy set-theoretic reasoning (see e.g. [3, Lemma 2.5]) there have to be mappings $g, \bar{g}\colon Var' \to Var'$ such that $g[V(\Gamma' \cup \Delta_1 \cup \Delta_2 \cup \{\chi\})] \subseteq Var$ and $(\bar{g} \circ g)(\delta) = \delta$ for each $\delta \in \Gamma' \cup \Delta_1 \cup \Delta_2 \cup \{\chi\}$.

Thus, by structurality and conservativity, we get that $g[\Gamma'], g[\Delta_1] \vdash_{\mathrm{L}} g(\chi)$ and $g[\Gamma'], g[\Delta_2] \vdash_{\mathrm{L}} g(\chi)$. Since clearly $g[\Delta_1] \subseteq g(\varphi) \Rightarrow^{\mathcal{L}} g(\psi)$ and $g[\Delta_2] \subseteq g(\psi) \Rightarrow^{\mathcal{L}} g(\varphi)$, we obtain $g[\Gamma'], g(\varphi) \Rightarrow^{\mathcal{L}} g(\psi) \vdash_{\mathrm{L}} g(\chi)$ and $g[\Gamma'], g(\psi) \Rightarrow^{\mathcal{L}} g(\varphi) \vdash_{\mathrm{L}} g(\chi)$ and so, from the SLP of L, we obtain $g[\Gamma'] \vdash_{\mathrm{L}} g(\chi)$. Thus also $g[\Gamma'] \vdash_{\mathrm{L}'} g(\chi)$ and then, by structurality for the substitution $\bar{g}$, we have $\bar{g}[g[\Gamma']] \vdash_{\mathrm{L}'} \bar{g}(g(\chi))$ and so, finally, $\Gamma \vdash_{\mathrm{L}'} \chi$. □

Theorem 2 (Transfer of SLP**)** *Let* L *be a logic with a weak p-implication $\Rightarrow$ enjoying the* SLP*. Then* L *enjoys the* transferred SLP *w.r.t. $\Rightarrow$, i.e., for each $\mathcal{L}$-algebra $\boldsymbol{A}$ and each set $X \cup \{a, b\} \subseteq A$ it holds:*

$$\mathrm{Fi}(X \cup (a \Rightarrow^{\boldsymbol{A}} b)) \cap \mathrm{Fi}(X \cup (b \Rightarrow^{\boldsymbol{A}} a)) = \mathrm{Fi}(X).$$

Proof To prove the non-trivial direction we show that for each $t \notin \mathrm{Fi}(X)$ we have $t \notin \mathrm{Fi}(X, a \Rightarrow b)$ or $t \notin \mathrm{Fi}(X, b \Rightarrow a)$. Let us take, by virtue of the previous lemma, a natural extension L′ with the SLP for $Var' = Var \cup \{v_z \mid z \in A\}$.[5] Recall that, due to Lemma 2, $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A}) = \mathcal{F}i_{\mathrm{L}'}(\boldsymbol{A})$, therefore we will not distinguish between the operators $\mathrm{Fi}_{\mathrm{L}'}$ and Fi_{L}. Consider the following set of formulae:

$$\Gamma = \{v_z \mid z \in X\} \cup \bigcup_{\langle c,n\rangle \in \mathcal{L}, z_i \in A} c(v_{z_1}, \ldots v_{z_n}) \Leftrightarrow^{\mathcal{L}'} v_{c^{\boldsymbol{A}}(z_1,\ldots z_n)}.$$

[5] Note that if $\boldsymbol{A}$ is countable, we can assume that $\{v_z \mid z \in A\} \subseteq Var$ and take just $\mathrm{L}' = \mathrm{L}$.

Clearly, $\Gamma \nvdash_{\mathrm{L}'} v_t$ (because $\langle \boldsymbol{A}, \mathrm{Fi}(X)\rangle \in \mathbf{MOD}(\mathrm{L}')$ and for the $\boldsymbol{A}$-evaluation $e(v_z) = z$ we obtain $e[\Gamma] \subseteq \mathrm{Fi}(X)$ and $e(v_t) \notin \mathrm{Fi}(X)$ we know that $e(v_t) \notin \mathrm{Fi}(X)$). Thus by the SLP of L' we have $\Gamma, v_a \Rightarrow^{\mathcal{L}'} v_b \nvdash_{\mathrm{L}'} v_t$ or $\Gamma, v_b \Rightarrow^{\mathcal{L}'} v_a \nvdash_{\mathrm{L}'} v_t$. Assume (without loss of generality) the former case and define $T' = \mathrm{Th}_{\mathrm{L}'}(\Gamma, v_a \Rightarrow^{\mathcal{L}'} v_b)$. We show that the mapping $h\colon A \to Fm_{\mathcal{L}'}/\Omega T'$ defined as $h(z) = [v_z]_{T'}$ is a homomorphism by a simple chain of equalities:

$$\begin{aligned} h(c^{\boldsymbol{A}}(z_1,\dots,z_n)) &= [v_{c^{\boldsymbol{A}}(z_1,\dots,z_n)}]_{T'} \\ &= [c(v_{z_1},\dots,v_{z_n})]_{T'} \\ &= c^{\boldsymbol{Fm}_{\mathcal{L}}/\Omega T'}([v_{z_1}]_{T'},\dots,[v_{z_n}]_{T'}) \\ &= c^{\boldsymbol{Fm}_{\mathcal{L}}/\Omega T'}(h(z_1),\dots,h(z_n)). \end{aligned}$$

Thus $F = h^{-1}(T'/\Omega T') \in \mathcal{F}i_{\mathrm{L}'}(\boldsymbol{A}) = \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$. Clearly $X \subseteq F$ (for each $x \in X$ we have $v_x \in \Gamma \subseteq T'$ and so $h[X] \subseteq T'/\Omega T'$). Next we show (by a simple chain of (in)equations) that $a \Rightarrow^{\boldsymbol{A}} b \subseteq F$:

$$\begin{aligned} h[a \Rightarrow^{\boldsymbol{A}} b] &= \{[v_{\chi^{\boldsymbol{A}}(a,b,\vec{c})}]_{T'} \mid \chi \in \Rightarrow \text{ and } \vec{c} \in A^{<\omega}\} \\ &= \{[\chi(v_a, v_b, \vec{v_c})]_{T'} \mid \chi \in \Rightarrow \text{ and } \vec{c} \in A^{<\omega}\} \\ &\subseteq \{[\chi]_{T'} \mid \chi \in v_a \Rightarrow^{\mathcal{L}'} v_b\} \\ &\subseteq T'/\Omega T'. \end{aligned}$$

Thus $X \cup (a \Rightarrow^{\boldsymbol{A}} b) \subseteq F$ and, since clearly $t \notin F$ ($t \in F$ would imply $v_t \in T'$), we obtain $t \notin F \supseteq \mathrm{Fi}(X, a \Rightarrow^{\boldsymbol{A}} b)$. □

Next we show how we can use the SLP to identify $\Rightarrow$-linear filters, i.e., a family of filters parameterized by implication, with the so-called $\cap$-prime filters, a family of filters defined intrinsically for a given algebra $\boldsymbol{A}$ and logic L: a filter F is $\cap$-*prime* in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ if there is no pair of filters F_1, F_2 such that $F = F_1 \cap F_2$ and $F \subsetneq F_1, F_2$.[6]

Lemma 4 *Let* L *be a logic in a language* $\mathcal{L}$ *with a weak p-implication* $\Rightarrow$ *and* $\boldsymbol{A}$ *an* $\mathcal{L}$*-algebra. Then, any* $\Rightarrow$*-linear filter on* $\boldsymbol{A}$ *is* $\cap$*-prime. Moreover, if* $\Rightarrow$ *has the* SLP*, then every* $\cap$*-prime filter on* $\boldsymbol{A}$ *is* $\Rightarrow$*-linear.*

Proof First assume that F is not $\cap$-prime; i.e. $F = F_1 \cap F_2$ for some $F_i \supsetneq F$. Then there exist elements $a_1 \in F_1 \setminus F_2$ and $a_2 \in F_2 \setminus F_1$. If $a_1 \Rightarrow^{\boldsymbol{A}} a_2 \subseteq F \subseteq F_1$, then (by *modus ponens* and the fact that F_1 is a filter) also $a_2 \in F_1$, a contradiction. Analogously we arrive to a contradiction from the assumption that $a_2 \Rightarrow^{\boldsymbol{A}} a_1 \subseteq F$. Thus F is not $\Rightarrow$-linear.

We show the proof of the second claim for filters. Consider any $\mathcal{L}$-algebra $\boldsymbol{A}$ and any $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ and assume that F is not $\Rightarrow$-linear, i.e. there are $a_1, a_2 \in A$ such that $a_1 \Rightarrow^{\boldsymbol{A}} a_2 \not\subseteq F$ and $a_2 \Rightarrow^{\boldsymbol{A}} a_1 \not\subseteq F$. By the transferred SLP we know that $F = \mathrm{Fi}(F) = \mathrm{Fi}(F \cup (a_1 \Rightarrow^{\boldsymbol{A}} a_2)) \cap \mathrm{Fi}(F \cup (a_2 \Rightarrow^{\boldsymbol{A}} a_1))$, i.e. F is the intersection of two strictly bigger filters. □

---

[6] This property can be read as '*intersection-prime*' and is known in lattice theory as *finite meet-irreducibility*. In [8] these filters are called simply *prime*, but since in this paper we also define prime filters by using a suitable notion of disjunction, we need to separate both notions.

Recall the well-known fact (see e.g. [8, Proposition 1.3.4.]) that, for any matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$, we have $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ iff F is $\cap$-*prime*. Using this fact we can easily obtain the promised generalization of [4, Corollary 2] (indeed we know that every semilinear implication enjoys the SLP, which we know that transfers, and we can just apply the previous lemma).

Corollary 1 *In any protoalgebraic logic* L *it holds:* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}} = \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$ *for any weak p-implication* $\Rightarrow$ *enjoying the* SLP*.*

Now we are almost ready to formulate the promised characterization theorem. The final ingredient is the notion of $\cap$-prime extension property introduced in [6].

Definition 3 We say that L has the $\cap$-*prime extension property*, IPEP for short, if $\cap$-prime theories form a basis of Th(L).

Clearly, by virtue of Lemma 4, any logic with a weak p-implication $\Rightarrow$ enjoying the SLP enjoys the LEP w.r.t. $\Rightarrow$ iff it enjoys the IPEP. It is well known that all finitary logics enjoy the IPEP (see e.g. [8, Corollary 1.3.3]). In [6, Section 2.3] the authors give examples of (a) an infinitary logic enjoying the IPEP (due to the previous remark we know that any infinitary logic with an implication satisfying the LEP can be used as such example) and (b) a logic not enjoying the IPEP. Therefore, the class of IPEP logics is a non-trivial proper extension of that of finitary logics.

Theorem 3 (Characterization of semilinear logics) *Let* L *be a logic with a weak p-implication* $\Rightarrow$*. Then the following are equivalent:*

1. $\Rightarrow$ *is semilinear in* L*,*
2. L *has the* LEP *w.r.t.* $\Rightarrow$*.*
3. L *has the* IPEP *and any of the following conditions holds:*
   - *3a.* L *has the* SLP *w.r.t.* $\Rightarrow$*,*
   - *3b.* L *has the transferred* SLP *w.r.t.* $\Rightarrow$*,*
   - *3c.* $\Rightarrow$*-linear filters coincide with* $\cap$*-prime filters in each* $\mathcal{L}$*-algebra,*
   - *3d.* $\Rightarrow$*-linear theories coincide with* $\cap$*-prime theories,*
   - *3e.* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}} = \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$.

If furthermore L *is finitary, we can add two more equivalent conditions:*

4. L *has the transferred* LEP *w.r.t.* $\Rightarrow$*,*
5. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}} \subseteq \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$.

Proof Let us first prove that 1 implies 2.[7] Assume that $\Rightarrow$ is semilinear in L and take a theory T and a formula $\varphi \notin T$. Then there is a matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$ and an $\boldsymbol{A}$-evaluation e such that $e[T] \subseteq F$ and $e(\varphi) \notin F$. Consider the matrix $\mathbf{B} = \langle \boldsymbol{Fm}_{\mathcal{L}}, e^{-1}(F)\rangle$ and observe that $\mathbf{B} \in \mathbf{MOD}(\mathrm{L})$ and $e\colon \mathbf{B} \to \mathbf{A}$ is a strict matrix homomorphism. Clearly, $e^{-1}(F)$ is a theory such that $T \subseteq e^{-1}(F)$ and $\varphi \notin e^{-1}(F)$. By the third claim in Lemma 1, we know that $e^{-1}(F)$ is $\Rightarrow$-linear.

[7] This implication is already contained in Theorem 1; however its proof in [4] only refers to a more particular proof for weakly implicative logics from [1]. We present a full proof for the reader's convenience.

Next observe that the LEP implies the IPEP because $\Rightarrow$-linear filters are $\cap$-prime (Lemma 4). The fact that the LEP implies the SLP (3a) was established in [4, Proposition 11]. The next few implications are easy to prove: 3a implies 3b is Theorem 2; 3b implies 3c is Lemma 4; 3c implies 3d is trivial; and 3d implies 2 is straightforward due to the IPEP. Now we can attach 3e: 3a implies 3e was established in Corollary 1 and, since in any logic with the IPEP we have $\mathrm{L} = \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}}$ (a property called RFSI-*completeness* in [6], where this claim was proved in Lemma 2.6.), then 3e trivially implies 1.

Finally we add the last two conditions. Note that trivially 4 implies 2 and 3e implies 5. To complete the proof just recall that [8, Corollaries 1.3.3 and 1.3.6] show that finitarity implies both that $\cap$-prime filters form a basis of $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ for each $\mathcal{L}$-algebra $\boldsymbol{A}$ (a property called *transferred* IPEP) and the fact that $\mathrm{L} = \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}}$ (a property called RSI-*completeness* in [6]), and so 3c implies 4 and 5 implies 1. □

Remark 1 Observe that the finitary condition for the last two items of the previous theorem is unnecessarily strong. Clearly it would be sufficient to assume the transferred IPEP to add condition 4, and the RSI-completeness to add condition 5. Also it is easy to see that in condition 3, in all cases but 3d, the IPEP could be replaced by the weaker condition of RFSI-completeness[8] (we could have the circle: 2a→3a→3b→3c→3e→1). Nevertheless, these alternative side-conditions (transferred IPEP and R(F)SI-completeness) are rather obscure, and hence we choose to formulate our main characterization theorem in terms of finitary logics and logics with the IPEP.

3 Lattice connectives

In this section we review the consequences of the presence of lattice connectives in a weakly p-implicational logic, i.e. connectives that express the supremum and the infimum of two elements, and the least and the largest element of the domain of a matrix with respect to the order given by the implication. Furthermore, we also consider the connective representing the minimum element of the filter of a given matrix. The results we provide here are unsurprising and mostly (very) easy to prove yet they are interesting and consequential. We start by formally defining the connectives using consecutions from Table 1.

Definition 4 Let L be a logic with a weak p-implication $\Rightarrow$. We say that

- a (possibly definable) binary connective $\wedge$ is a *lattice protoconjunction* of $\Rightarrow$ in L if it satisfies the consecutions ($\wedge$1)–($\wedge$3),
- a (possibly definable) binary connective $\vee$ is a *lattice protodisjunction* of $\Rightarrow$ in L if it satisfies the consecutions ($\vee$1)–($\vee$3),
- a (possibly definable) truth constant $\bot$ is a *falsum* of $\Rightarrow$ in L if it satisfies the consecution ($\bot$),
- a (possibly definable) truth constant $\top$ is a *verum* of $\Rightarrow$ in L if it satisfies the consecution ($\top$).
- a (possibly definable) truth constant $\overline{1}$ is a *unit* of $\Rightarrow$ in L if it satisfies the consecution (Push) and (Pop).

[8] An example of an RFSI-complete logic that does not enjoy the IPEP has been given in he master thesis of Tomáš Lávička [13].

Table 1 Consecutions describing the lattice connectives

Consecution	symbol	Name
$\rhd \bot \Rightarrow \varphi$	$(\bot)$	*ex falso quodlibet*
$\rhd \varphi \Rightarrow \top$	$(\top)$	*verum ex quolibet*
$\varphi \rhd \overline{1} \Rightarrow \varphi$	(Push)	push
$\overline{1} \Rightarrow \varphi \rhd \varphi$	(Pop)	pop
$\rhd \varphi \Rightarrow \varphi \vee \psi$	$(\vee 1)$	upper bound
$\rhd \psi \Rightarrow \varphi \vee \psi$	$(\vee 2)$	upper bound
$\varphi \Rightarrow \chi, \psi \Rightarrow \chi \rhd \varphi \vee \psi \Rightarrow \chi$	$(\vee 3)$	supremality
$\rhd \varphi \wedge \psi \Rightarrow \varphi$	$(\wedge 1)$	lower bound
$\rhd \varphi \wedge \psi \Rightarrow \psi$	$(\wedge 2)$	lower bound
$\chi \Rightarrow \varphi, \chi \Rightarrow \psi \rhd \chi \Rightarrow \varphi \wedge \psi$	$(\wedge 3)$	infimality

Furthermore we say that a lattice protoconjunction $\wedge$ is a *lattice conjunction* if it is a conjunction,[9] i.e., satisfies additionally also the consecution

$$(\wedge 0) \qquad \varphi, \psi \rhd \varphi \wedge \psi$$

Finally we say that a lattice protodisjunction $\wedge$ is a *(strong) lattice disjunction* if it is a (strong) disjunction,[10] i.e., satisfies the (strong) Proof by Cases Property:[11]

PCP If $\mathit{\Gamma}, \varphi \vdash_{\mathrm{L}} \chi$ and $\mathit{\Gamma}, \psi \vdash_{\mathrm{L}} \chi$, then $\mathit{\Gamma}, \varphi \vee \psi \vdash_{\mathrm{L}} \chi$

sPCP If $\mathit{\Gamma}, \mathit{\Phi} \vdash_{\mathrm{L}} \chi$ and $\mathit{\Gamma}, \mathit{\Psi} \vdash_{\mathrm{L}} \chi$, then $\mathit{\Gamma}, \mathit{\Phi} \vee \mathit{\Psi} \vdash_{\mathrm{L}} \chi$.

The notions of lattice protodisjunctions and disjunctions are mutually incomparable. On one hand, while the connective $\vee$ of the logic $\mathrm{FL_e}$ (a prominent substructural logics, see e.g. [11]) is not a disjunction (cf. [6, Example 3.3.]) it is clearly a *lattice* protodisjunction for $\to$. On the other hand, the disjunction of this logic $p \vee' q = (p \wedge \overline{1}) \vee (q \wedge \overline{1})$ is clearly not a lattice disjunction ($(\vee 1)$ would entail $p \to \overline{1}$, a contradiction).

The analogous claim holds for lattice (proto)conjunctions. Indeed, consider a matrix $\mathbf{A}$ with domain $\{\bot, a, b\}$, filter $\{a, b\}$ and operations $\Rightarrow$ and $\wedge$ defined as:

$\Rightarrow$	$\bot$	a	b
$\bot$	a	a	b
a	$\bot$	a	$\bot$
b	$\bot$	$\bot$	b

$\wedge$	$\bot$	a	b
$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$
b	$\bot$	$\bot$	b

Clearly, $\wedge$ is a lattice protoconjunction of $\Rightarrow$ in $\models_{\mathbf{A}}$ which is not a conjunction. Conversely, consider e.g. the 4-valued Boolean algebra $\boldsymbol{B}_4$ expanded with a connective $\wedge'$ defined as: $x \wedge' y = \top$ if $x = y = \top$ and $\bot$ otherwise. In this expansion of classical logic $\wedge'$ is a conjunction but not a lattice protoconjunction.

9 In AAL literature (see e.g. [12]) a connective $\wedge$ is called a *conjunction*, and the logic *conjunctive*, if $\varphi, \psi \dashv\vdash \varphi \wedge \psi$. Note that any finitely weakly implicational logic with a conjunction is weakly implicative.

10 In general AAL literature (see e.g. [6]) a connective $\vee$ is called a disjunction, and the logic disjunctive, if $\mathit{\Gamma}, \varphi \vee \psi \vdash_{\mathrm{L}} \chi$ iff $\mathit{\Gamma}, \varphi \vdash_{\mathrm{L}} \chi$ and $\mathit{\Gamma}, \psi \vdash_{\mathrm{L}} \chi$. We will see an even more general approach towards disjunction in the next section.

11 Given sets of formulae $\mathit{\Phi}$ and $\mathit{\Psi}$, we define the set $\mathit{\Phi} \vee \mathit{\Psi}$ as $\{\varphi \vee \psi \mid \varphi \in \mathit{\Phi},\ \psi \in \mathit{\Psi}\}$.

It is very easy to check that, given a weak p-implication, all the connectives considered in this section are *intrinsic* in the sense of the next proposition.

Proposition 1 *Let* L *be a logic with a weak p-implication* $\Rightarrow$ *and* $*, *'$ *be two lattice protodisjunctions or two lattice protoconjunctions for* $\Rightarrow$ *in* L. *Then* $\vdash_{\mathrm{L}} \varphi * \psi \Leftrightarrow \varphi *' \psi$.

Furthermore let c *and* c' *be two units, two verums, or two falsums for* $\Rightarrow$ *in* L. *Then* $\vdash_{\mathrm{L}} c \Leftrightarrow c'$.

The next two straightforward propositions justify the names of our connectives by formalizing their expected relation with the order given on any reduced matrix by any chosen weak p-implication.

Proposition 2 *Let* L *be a logic with a weak p-implication* $\Rightarrow$ *and* $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$.

- *If* $\overline{1}$ *is a unit for* $\Rightarrow$ *in* L, *then* $\overline{1}^{\boldsymbol{A}} = \min_{\leq_{\mathbf{A}}} F$.
- *If* $\top$ *is a verum for* $\Rightarrow$ *in* L, *then* $\top^{\boldsymbol{A}} = \max_{\leq_{\mathbf{A}}} A$ *and* $\top^{\boldsymbol{A}} \in F$.
- *If* $\bot$ *is a falsum for* $\Rightarrow$ *in* L, *then* $\bot^{\boldsymbol{A}} = \min_{\leq_{\mathbf{A}}} A$, *and* $\bot^{\boldsymbol{A}} \in F$ *iff* $A = F$.
- *If there is a unit for* $\Rightarrow$ *in* L, *then any lattice protoconjunction is a lattice conjunction.*
- *If* $\Rightarrow$ *is a Rasiowa implication, then* L *has a conservative expansion* L$'$ *with a constant which is both unit and verum of* $\Rightarrow$ *in* L$'$. *Also, if* $\Rightarrow$ *is a Rasiowa implication, any theorem (e.g. any formula from* $p \Rightarrow p$*) satisfies the properties of unit and verum.*[12]

Proposition 3 *Let* L *be a logic with a weak p-implication* $\Rightarrow$, $\wedge$ *and* $\vee$ *binary connectives, and* $\Rightarrow'(p, q, \overrightarrow{r})$ *a parameterized set of formulae. Then the following three conditions are equivalent:*

1. $\Rightarrow'$ *is a weak p-implication in* L *and* $\wedge$ *is a lattice protoconjunction w.r.t.* $\Rightarrow'$.
2. *For every* $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$, *the reduct* $\langle A, \wedge^{\boldsymbol{A}}\rangle$ *is a meet semilattice,* $x \wedge^{\boldsymbol{A}} y = \inf_{\leq_{\mathbf{A}}^{\Rightarrow'}}\{x, y\}$, *and* $x \in F$ *whenever* $x \wedge^{\boldsymbol{A}} y \in F$.
3. *The following (sets of) consecutions are derivable in* L:
   - (e$\wedge$1) $\quad \varphi \wedge \psi \rhd \varphi$
   - (iC$_\wedge$) $\quad \rhd \varphi \wedge \psi \Leftrightarrow \psi \wedge \varphi$
   - (iI$_\wedge$) $\quad \rhd \varphi \wedge \varphi \Leftrightarrow \varphi$
   - (iA$_\wedge$) $\quad \rhd \varphi \wedge (\psi \wedge \chi) \Leftrightarrow (\varphi \wedge \psi) \wedge \chi$
   - (Def$_\wedge$) $\quad \varphi \Rightarrow' \psi \rhd \varphi \Leftrightarrow \varphi \wedge \psi \qquad \varphi \Leftrightarrow \varphi \wedge \psi \rhd \varphi \Rightarrow' \psi$.

Also, the following three conditions are equivalent:

1. $\Rightarrow'$ *is a weak p-implication in* L *and* $\vee$ *is a lattice protodisjunction w.r.t.* $\Rightarrow'$.
2. *For every* $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$, *the reduct* $\langle A, \vee^{\boldsymbol{A}}\rangle$ *is a join semilattice,* $x \vee^{\boldsymbol{A}} y = \sup_{\leq_{\mathbf{A}}^{\Rightarrow'}}\{x, y\}$, *and* $x \vee^{\boldsymbol{A}} y \in F$ *whenever* $x \in F$.
3. *The following (sets of) consecutions are derivable in* L:
   - (e$\vee$1) $\quad \varphi \rhd \varphi \vee \psi$
   - (iC$_\vee$) $\quad \rhd \varphi \vee \psi \Leftrightarrow \psi \vee \varphi$
   - (iI$_\vee$) $\quad \rhd \varphi \vee \varphi \Leftrightarrow \varphi$
   - (iA$_\vee$) $\quad \rhd \varphi \vee (\psi \vee \chi) \Leftrightarrow (\varphi \vee \psi) \vee \chi$
   - (Def$_\vee$) $\quad \varphi \Rightarrow' \psi \rhd \psi \Leftrightarrow \varphi \vee \psi \qquad \psi \Leftrightarrow \varphi \vee \psi \rhd \varphi \Rightarrow' \psi$

[12] This example shows that, as implications can be given by sets of formulae with more than two variables, also a truth constant can sometimes be defined by a formula with variables.

Proof We prove only the first claim (the second one is analogous). The implications '1 implies 2' and '2 implies 3' are easy to prove. To prove the remaining implication '3 implies 1' we need to show validity of all defining properties of $\Rightarrow'$ (reflexivity, transitivity, *modus ponens*, and symmetrized congruence) and of $\wedge$ (($\wedge$1)–($\wedge$3)). We show the case of transitivity; the remaining properties can be analogously proved. Each of the following formulae is provable from $\{\varphi \Rightarrow' \psi, \psi \Rightarrow' \chi\}$:

1.	$\varphi \Leftrightarrow \varphi \wedge \psi$	assumption $\varphi \Rightarrow' \psi$ and (Def$_\wedge$)
2.	$\varphi \wedge \chi \Leftrightarrow (\varphi \wedge \psi) \wedge \chi$	1. and congruence of $\Leftrightarrow$
3.	$\psi \Leftrightarrow \psi \wedge \chi$	assumption $\psi \Rightarrow' \chi$ and (Def$_\wedge$)
4.	$\varphi \wedge \psi \Leftrightarrow \varphi \wedge (\psi \wedge \chi)$	3. and congruence of $\Leftrightarrow$
5.	$\varphi \wedge \psi \Leftrightarrow (\varphi \wedge \psi) \wedge \chi$	4., (iA$_\wedge$), and transitivity of $\Leftrightarrow$
6.	$\varphi \Leftrightarrow (\varphi \wedge \psi) \wedge \chi$	1., 5., and transitivity of $\Leftrightarrow$
7.	$\varphi \Leftrightarrow \varphi \wedge \chi$	2., 6., and transitivity of $\Leftrightarrow$
8.	$\varphi \Rightarrow' \chi$	7. and (Def$_\wedge$) □

As a corollary of this proposition (esp. the consecutions (Def$_\vee$) and (Def$_\wedge$)) and the interderivability of all symmetrized weak p-implications we obtain:

Corollary 2 *Let* L *be a logic with weak p-implications* $\Rightarrow$ *and* $\Rightarrow'$ *and a connective which is a lattice protoconjunction (or a lattice protodisjunction) for both* $\Rightarrow$ *and* $\Rightarrow'$. *Then* $\varphi \Rightarrow \psi \vdash_{\mathrm{L}} \varphi \Rightarrow' \psi$.

We also obtain a corollary showing that the usual way of defining implication from equivalence and conjunction (or from equivalence and disjunction) works in general.

Corollary 3 *Let* L *be a logic with a weak p-implication* $\Rightarrow$.

- *Assume that there is a connective* $\wedge$ *satisfying the conditions* (e$\wedge$1), (iC$_\wedge$), (iI$_\wedge$), *and* (iA$_\wedge$). *Then the connective* $\Rightarrow'$ *defined as* $\varphi \Leftrightarrow \varphi \wedge \psi$ *is a weak p-implication and* $\wedge$ *is a lattice protoconjunction for* $\Rightarrow'$.
- *Assume that there is a connective* $\vee$ *satisfying the conditions* (e$\vee$1), (iC$_\vee$), (iI$_\vee$), *and* (iA$_\vee$). *Then the connective* $\Rightarrow'$ *defined as* $\psi \Leftrightarrow \varphi \vee \psi$ *is a weak p-implication and* $\vee$ *is a lattice protodisjunction for* $\Rightarrow'$.

Next we look at the relation between the presence of lattice connectives and algebraizability of the logic. We prove an unsurprising generalization of known facts.

Proposition 4 *Let* L *be a logic with a weak p-implication* $\Rightarrow$ *such that the following two conditions are met:*

- *there is a formula* χ *such that* $\vdash_{\mathrm{L}} \chi$ *and* $p \vdash_{\mathrm{L}} \chi \Rightarrow p$ *(e.g.* $\overline{1}$ *if there is a unit for* $\Rightarrow$ *in* L *or* $p \to p$ *if* $p \vdash_{\mathrm{L}} (p \to p) \to p$*)*
- *there is a lattice protoconjunction or a protodisjunction of* $\Rightarrow$.

Then, L *is an algebraically p-implicational logic.*[13]

[13] If only the first condition is met, then we obtain that L is *order-algebraizable* in the sense of Raftery [15].

Proof We show that, based on the second assumption, $\langle p \wedge \chi, \chi \rangle$ or $\langle p \vee \chi, p \rangle$ is an algebraizing pair. In the first case we need to show that $p \dashv\vdash_{\mathrm{L}} p \wedge \chi \Leftrightarrow \chi$. One direction is easy: due to ($\wedge$2) we have $p \vdash_{\mathrm{L}} p \wedge \chi \Rightarrow \chi$ and also $p \vdash_{\mathrm{L}} \chi \Rightarrow p \wedge \chi$ (due to $p \vdash_{\mathrm{L}} \chi \Rightarrow p$, $\vdash_{\mathrm{L}} \chi \Rightarrow \chi$, and ($\wedge$3)). The second direction: clearly $p \wedge \chi \Leftrightarrow \chi \vdash_{\mathrm{L}} p \wedge \chi$ (due to $\vdash_{\mathrm{L}} \chi$) and so $p \wedge \chi \Leftrightarrow \chi \vdash_{\mathrm{L}} p$ (due to ($\wedge$1)).

In the second case we need to show that $p \dashv\vdash_{\mathrm{L}} p \vee \chi \Leftrightarrow p$. One direction is easy: due to ($\vee$1) we have $p \vdash_{\mathrm{L}} p \Rightarrow p \vee \chi$ and also $p \vdash_{\mathrm{L}} p \vee \chi \Rightarrow p$ (due to $p \vdash_{\mathrm{L}} \chi \Rightarrow p$, $\vdash_{\mathrm{L}} p \Rightarrow p$, and ($\vee$3)). The second direction: clearly $p \vee \chi \Leftrightarrow p \vdash_{\mathrm{L}} \chi \Rightarrow p$ (due to ($\vee$2) and transitivity) and so $p \vee \chi \Leftrightarrow p \vdash_{\mathrm{L}} p$ (due to $\vdash_{\mathrm{L}} \chi$). □

We conclude this section by studying the behavior of lattice connectives in the setting of semilinear logics.

Proposition 5 *Let* L *be a logic with a weak* semilinear *p-implication* $\Rightarrow$. *Then:*

1. *Any lattice protoconjunction of* $\Rightarrow$ *in* L *is a lattice conjunction.*
2. *Any lattice protodisjunction of* $\Rightarrow$ *in* L *is a strong lattice disjunction.*
3. L *has a conservative expansion* L$'$ *(where* $\Rightarrow$ *is still a semilinear weak p-implication) with both a lattice conjunction and a strong disjunction of* $\Rightarrow$ *in* L$'$.
4. *If there are a lattice protodisjunction* $\vee$ *and a lattice protoconjunction* $\wedge$ *for* $\Rightarrow$ *in* L, *then the reduct* $\langle A, \wedge, \vee \rangle$ *of any* $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})$ *is a distributive lattice.*

Proof 1. Clearly $\varphi \Rightarrow \psi \vdash_{\mathrm{L}} \varphi \Rightarrow \varphi \wedge \psi$ and so $\varphi, \psi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \varphi \wedge \psi$. Analogously $\varphi, \psi, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} \varphi \wedge \psi$ and so, by the SLP, $\varphi, \psi \vdash_{\mathrm{L}} \varphi \wedge \psi$.

2. Assume that $\Gamma, \Phi \vdash_{\mathrm{L}} \varphi$ and $\Gamma, \Psi \vdash_{\mathrm{L}} \varphi$ and we will show that $\Gamma, \Phi \vee \Psi \vdash_{\mathrm{L}} \varphi$. Let us consider any $\mathbf{A} = \langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$ and an $\mathbf{A}$-evaluation e such that $e[\Gamma \cup (\Phi \vee \Psi)] \subseteq F$ and we show that $e(\varphi) \in F$. If $e[\Gamma \cup \Phi] \subseteq F$, the claim is trivial (since $\Gamma, \Phi \vdash_{\mathrm{L}} \varphi$). If $e(\chi) \notin F$ for some $\chi \in \Phi$, then $e[\Psi] \subseteq F$ (otherwise there would be $\psi \in \Psi$ such that $e(\chi \vee \psi) = \max\{e(\chi), e(\psi)\} \notin F$, a contradiction with $e[\Phi \vee \Psi] \subseteq F$) and so $e(\varphi) \in F$ (since $\Gamma, \Psi \vdash_{\mathrm{L}} \varphi$).

The last two claims are straightforward. □

4 Interplay of disjunction and implication

In the previous section, protodisjunctions and disjunctions (as well as protoconjunctions and conjunctions) were either primitive binary connectives or defined by a formula with two variables. However, some works in AAL (see e.g. [8]) have considered generalized disjunctions given by parameterized sets of formulae, in the same fashion as generalized equivalence are commonly studied in AAL. These generalized disjunctions have been extensively studied in [6] where they are shown to be useful connectives with important consequences in a wide class of logics. They have been also considered in [5] in the framework of weakly implicative logics. However, their interplay with weak p-implications has not yet been systematically described. This is the goal of the present section.

Let us start recalling some basic notions from [6]. The convention used for parameterized disjunctions is analogous to that used for implications (and equivalencies). Indeed if $\nabla(p, q, \overrightarrow{r})$ is a set of formulae in two variables p, q and possible parameters $\overrightarrow{r}$, we define $\varphi \nabla \psi$ as $\bigcup\{\nabla(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in Fm_{\mathcal{L}}^{\leq\omega}\}$. Given sets $\Phi, \Psi \subseteq Fm_{\mathcal{L}}$, $\Phi \nabla \Psi$ denotes the set $\bigcup\{\varphi \nabla \psi \mid \varphi \in \Phi, \psi \in \Psi\}$. We start with a useful convention:

Convention 2 *A set* $\nabla(p, q, \overrightarrow{r})$ *of formulae is a* p-protodisjunction *(or just* protodisjunction *if there are no parameters* $\overrightarrow{r}$*) in* L *whenever*

(PD) $\quad \varphi \vdash_{\mathrm{L}} \varphi \nabla \psi \quad$ *and* $\quad \psi \vdash_{\mathrm{L}} \varphi \nabla \psi.$

Throughout all the paper the sets ∇ *are assumed to be p-protodisjunctions.*

Definition 5 We say that a (p-)protodisjunction ∇ is a *(p-)disjunction* if it enjoys the *Proof by Cases Property*[14] in L if for any set $\Gamma, \varphi, \psi, \chi$ of formulae we have:

PCP $\quad$ If $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{L}} \chi$, then $\Gamma, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi$.

We say that a (p-)protodisjunction ∇ is a *strong (p-)disjunction* if it enjoys the *strong Proof by Cases Property* in L if for every sets Γ, Φ, Ψ of formulae and every formula χ we have:

sPCP $\quad$ If $\Gamma, \Phi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \Psi \vdash_{\mathrm{L}} \chi$, then $\Gamma, \Phi \nabla \Psi \vdash_{\mathrm{L}} \chi$.

The notion of p-disjunction is intrinsic for a given logic, i.e., for any pair ∇, ∇' of p-disjunctions in L we have $\varphi \nabla \psi \dashv\vdash_{\mathrm{L}} \varphi \nabla' \psi$. Moreover, the existence of disjunction connectives allows to classify logics:

Definition 6 We say that a logic L is *(strongly) (p-)disjunctional* if it has a (strong) (p-)disjunction. We use the term *disjunctive* instead of disjunctional if the disjunction is given by a single parameter-free formula.[15]

In the mentioned previous paper [6] we have provided separating examples for all these classes of logics (and some more classes defined therein). We continue by introducing two natural syntactical conditions that will play an important rôle in the interplay of disjunctions and implications: a version of *modus ponens* with disjunction ($\mathrm{DMP}^{\nabla}_{\Rightarrow}$) and a natural generalization of the prelinearity axiom used in fuzzy logics ($\mathrm{P}^{\nabla}_{\Rightarrow}$). ($\mathrm{DMP}^{\nabla}_{\Rightarrow}$) simply follows from the PCP and (MP), but, more importantly, we show that the presence of this simple consecution together with a suitable property of *implication* entails the PCP (analogously for ($\mathrm{P}^{\nabla}_{\Rightarrow}$) and the SLP). Thus, these consecutions are indeed natural binding conditions for implication and disjunction.

Proposition 6 *Let* L *be a logic with a weak p-implication* $\Rightarrow$ *and p-protodisjunction* ∇*. If* ∇ *is a p-disjunction or a lattice protodisjunction of* $\Rightarrow$*, then it holds:*

($\mathrm{DMP}^{\nabla}_{\Rightarrow}$) $\quad \varphi \Rightarrow \psi, \varphi \nabla \psi \vdash_{\mathrm{L}} \psi \quad$ *and* $\quad \varphi \Rightarrow \psi, \psi \nabla \varphi \vdash_{\mathrm{L}} \psi.$

If $\Rightarrow$ *enjoys the* SLP*, then it holds:*

($\mathrm{P}^{\nabla}_{\Rightarrow}$) $\quad \vdash_{\mathrm{L}} (\varphi \Rightarrow \psi) \nabla (\psi \Rightarrow \varphi).$

---

[14] We could have introduced the PCP as a double direction meta-rule (as it was done and studied e.g. in [9] under the name *Property of Disjunction*). However, the reverse direction of this meta-rule could obviously be equivalently replaced by (PD) (one direction is obvious, for the other one observe that from $\varphi \nabla \psi \vdash_{\mathrm{L}} \varphi \nabla \psi$, we would obtain $\varphi \vdash_{\mathrm{L}} \varphi \nabla \psi$ and $\psi \vdash_{\mathrm{L}} \varphi \nabla \psi$). Thus, we prefer our definition because it keeps the interesting implication separated from the trivial one that we can always assume.

[15] Note that in [8] the term 'disjunctive' means 'p-disjunctional' in our sense.

Proof The first claim easily follows from ($\vee$3) if ∇ is a lattice protodisjunction, and from the obvious derivations $\varphi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \psi$ and $\psi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \psi$ if ∇ enjoys the PCP. The second one is proved using the SLP and the following instance of (PD): $\varphi \Rightarrow \psi \vdash_{\mathrm{L}} (\varphi \Rightarrow \psi) \nabla (\psi \Rightarrow \varphi)$ and $\psi \Rightarrow \varphi \vdash_{\mathrm{L}} (\varphi \Rightarrow \psi) \nabla (\psi \Rightarrow \varphi)$. □

So far we have seen two special kinds of filters: $\Rightarrow$-linear and $\cap$-prime filters; we will now add one more kind parameterized by a p-disjunction.

Definition 7 Let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra. A filter $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is called ∇-*prime* if for every $a, b \in A$, $a \nabla^{\boldsymbol{A}} b \subseteq F$ implies $a \in F$ or $b \in F$.

Analogously to Lemma 4 we could easily show that each ∇-prime filter is $\cap$-prime and, if ∇ has the PCP, then the converse implication also holds.[16] More interesting for this paper is the relation between ∇-prime and $\Rightarrow$-linear filters.

Lemma 5 *Let* L *be a logic with a weak p-implication* $\Rightarrow$ *and a p-protodisjunction* ∇*, and* $\boldsymbol{A}$ *an* $\mathcal{L}$*-algebra.*

- *If* L *satisfies* ($\mathrm{DMP}^{\nabla}_{\Rightarrow}$)*, then each* $\Rightarrow$*-linear filter in* $\boldsymbol{A}$ *is* ∇*-prime.*
- *If* L *satisfies* ($\mathrm{P}^{\nabla}_{\Rightarrow}$)*, then each* ∇*-prime filter in* $\boldsymbol{A}$ *is* $\Rightarrow$*-linear.*

Proof To prove the first claim assume that F is $\Rightarrow$-linear and $a \nabla^{\boldsymbol{A}} b \subseteq F$. We know that $a \Rightarrow^{\boldsymbol{A}} b \subseteq F$ or $b \Rightarrow^{\boldsymbol{A}} a \subseteq F$. Thus from ($\mathrm{DMP}^{\nabla}_{\Rightarrow}$) we obtain that $b \in F$ or $a \in F$.

To prove the second one assume that F is not $\Rightarrow$-linear, i.e. there are elements a, b, x, y such that $x \in a \Rightarrow^{\boldsymbol{A}} b$, $y \in b \Rightarrow^{\boldsymbol{A}} a$, and $x, y \notin F$. Since we clearly have $x \nabla^{\boldsymbol{A}} y \subseteq (a \Rightarrow^{\boldsymbol{A}} b) \nabla^{\boldsymbol{A}} (b \Rightarrow^{\boldsymbol{A}} a)$ and L satisfies ($\mathrm{P}^{\nabla}_{\Rightarrow}$), we obtain $x \nabla^{\boldsymbol{A}} y \subseteq F$, i.e. F is not ∇-prime. □

Our next goal is to formulate (and prove) an analog of Theorem 3 where the properties of the implication are replaced by analogous properties of the disjunction. To this end, we first recall the *(transferred)* ∇*-prime extension property*, *(transferred)* PEP for short, defined as expected: ∇-prime theories form a basis of Th(L) (∇-prime filters form a basis of $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ for each $\mathcal{L}$-algebra $\boldsymbol{A}$, respectively). Using this, we can formulate a proposition that links the properties of disjunctions and implication from a different perspective.

Proposition 7 *Let* L *be a logic with a weak p-implication* $\Rightarrow$ *and p-protodisjunction* ∇*.*

- *If* L *satisfies* ($\mathrm{DMP}^{\nabla}_{\Rightarrow}$)*, we have:*
  1. *if* $\Rightarrow$ *has the* LEP*, then* ∇ *has the* PEP *and so also the* sPCP
  2. *if* $\Rightarrow$ *has the* SLP*, then* ∇ *has the* PCP.
- *If* L *satisfies* ($\mathrm{P}^{\nabla}_{\Rightarrow}$)*, we have:*
  3. *if* ∇ *has the* PEP*, then* $\Rightarrow$ *has the* LEP
  4. *if* ∇ *has the* sPCP*, then* $\Rightarrow$ *has the* SLP.
- *If* L *satisfies* ($\mathrm{P}^{\nabla}_{\Rightarrow}$) *and* $\Rightarrow$ *is finite and parameter-free, we have:*
  5. *if* ∇ *has the* PCP*, then* $\Rightarrow$ *has the* SLP.

[16] This is proved in [6, Lemma 4.15] under the assumption of transferred PCP. Moreover, in [6, Proposition 4.29] it is shown that the PCP transfers in all protoalgebraic logics (actually the proof of this proposition is restricted to countable languages, but it can be proved in full generality analogously to the proof of the transfer theorem of the SLP).

Proof Parts 1 and 3 are straightforward corollaries of the previous lemma (the fact that the PEP implies the sPCP is proved in [6, Proposition 4.14]).

2. From $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{L}} \chi$ we obtain, using $(\mathrm{DMP}^{\nabla}_{\Rightarrow})$, $\Gamma, \varphi \nabla \psi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \varphi \nabla \psi, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} \chi$; SLP completes the proof.
4. From $\Gamma, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} \chi$ we obtain $\Gamma, (\varphi \Rightarrow \psi) \nabla (\psi \Rightarrow \varphi) \vdash_{\mathrm{L}} \chi$ (due to sPCP). Knowing that L satisfies $(\mathrm{P}^{\nabla}_{\Rightarrow})$ we obtain $\Gamma \vdash_{\mathrm{L}} \chi$.
5. It is proved in the same way as 4 using (repeatedly, but finitely many times) the PCP instead of the sPCP (cf. [6, Lemma 3.10]). □

Observe that, while the LEP and the PEP are equivalent in the presence of $(\mathrm{DMP}^{\nabla}_{\Rightarrow})$ and $(\mathrm{P}^{\nabla}_{\Rightarrow})$, the situation is not the same for their metarule counterparts, the SLP and the PCP. Indeed, in 4 we had to use the sPCP; we can only guarantee the equivalence of the SLP and the PCP when the implication is given by a finite parameter-free set of formulae.

Theorem 4 (Disjunction-based characterization of semilinear logics) *Let* L *be a logic with a weak p-implication* $\Rightarrow$ *and a p-protodisjunction* ∇ *satisfying* $(\mathrm{DMP}^{\nabla}_{\Rightarrow})$. *Then the following are equivalent:*[17]

1. $\Rightarrow$ *is semilinear in* L*,*
2. L *satisfies* $(\mathrm{P}^{\nabla}_{\Rightarrow})$ *and has the* PEP *w.r.t.* ∇*,*
3. L *satisfies* $(\mathrm{P}^{\nabla}_{\Rightarrow})$*, has the* IPEP *and any of the following conditions holds:*
   - *3a.* L *has the* PCP *w.r.t.* ∇*,*
   - *3b.* L *has the* sPCP *w.r.t.* ∇*,*
   - *3c.* L *has the transferred* sPCP *w.r.t.* ∇*,*
   - *3d.* L *has the transferred* PCP *w.r.t.* ∇*,*
   - *3e.* ∇*-prime filters coincide with* $\cap$*-prime filters in each* $\mathcal{L}$*-algebra.*
   - *3f.* ∇*-prime theories coincide with* $\cap$*-prime theories.*

Proof The equivalence of 1 and 2 is easy: indeed, $(\mathrm{P}^{\nabla}_{\Rightarrow})$ holds by Proposition 6; using Theorem 3, we obtain the LEP and from this, by Proposition 7, the PEP. Conversely, from the PEP we obtain the LEP (by Proposition 7) and, using Theorem 3, this entails semilinearity.

From $(\mathrm{P}^{\nabla}_{\Rightarrow})$ and the PEP we obtain the LEP (by Proposition 7) and from that we have the IPEP (by Theorem 3). We know from [6, Proposition 4.14] that the PEP implies the sPCP, so we have justified that 2 implies 3b. From [6, Theorems 4.7 and 4.17] we obtain (in presence of the IPEP) the equivalence of 3a, 3b, 3c, and 3d. From 3d, we obtain 3e by [6, Lemma 4.15], which trivially implies 3f. Finally, thanks to the IPEP, we have that 3f implies 2. □

Analogously to $\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$, one can also define $\mathbf{MOD}^{p}_{\nabla}(\mathrm{L})$ as the class of reduced models of L whose filter is ∇-prime. We could expand the previous theorem with equivalent conditions involving $\mathbf{MOD}^{p}_{\nabla}(\mathrm{L})$, but this would require stronger side-conditions (namely, requiring ∇ to be finite and parameter-free; cf. [6, Proposition 4.23]), so we prefer to include them only as properties of semilinear p-disjunctional logics.

[17] Note that we could move the assumption that L satisfies $(\mathrm{DMP}^{\nabla}_{\Rightarrow})$ into claim 1 and the equivalencies would still hold. But, since our goal is to characterize semilinearity, we prefer to formulate it with $(\mathrm{DMP}^{\nabla}_{\Rightarrow})$ as a precondition.

Proposition 8 *Let* L *be a logic with a weak semilinear p-implication* $\Rightarrow$ *and a p-disjunction* ∇*. Then we have the following:*

1. L *has the* PEP *and the transferred* sPCP *w.r.t.* ∇*,*
2. ∇*-prime,* $\cap$*-prime, and* $\Rightarrow$*-linear filters coincide in each* $\mathcal{L}$*-algebra, and*
3. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}} = \mathbf{MOD}^p_\nabla(\mathrm{L}) = \mathbf{MOD}^\ell_\Rightarrow(\mathrm{L})$.

As an easy corollary of Theorem 4 we obtain that in many logics (including all logics with a Rasiowa semilinear implication given by a binary connective $\to$ such that $p \vdash (p \to q) \to q$) we can define a strong disjunction from implication.

Proposition 9 *Let* L *be a logic with a weak semilinear (p-)implication* $\Rightarrow$*. Let us define* $\nabla = (p \Rightarrow q) \Rightarrow q \cup (q \Rightarrow p) \Rightarrow p$*. Then* ∇ *is a strong (p-)disjunction iff* $\varphi \vdash_{\mathrm{L}} (\varphi \Rightarrow \psi) \Rightarrow \psi$ *and* $\psi \vdash_{\mathrm{L}} (\varphi \Rightarrow \psi) \Rightarrow \psi$*.*

Next we provide a characterization of semilinearity using the notions of distributivity and framality of the lattice of filters (or congruences).

Definition 8 A logic L is *filter-distributive* if for each $\mathcal{L}$-algebra, the lattice $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is distributive. A logic L is *filter-framal* if for each $\mathcal{L}$-algebra, the lattice $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is a frame, i.e., for each $\mathcal{F} \cup \{G\} \subseteq \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ holds:

$$G \cap \bigvee_{F \in \mathcal{F}} F = \bigvee_{F \in \mathcal{F}} (G \cap F).$$

We omit the prefix 'filter-' whenever the defining property holds for $\boldsymbol{A} = \boldsymbol{Fm}_{\mathcal{L}}$.

Theorem 5 (Semilinearity and distributivity) *Let* L *be a logic with a weak p-implication* $\Rightarrow$ *and a p-protodisjunction* ∇ *satisfying the* IPEP*,* (DMP$^\nabla_\Rightarrow$) *and* (P$^\nabla_\Rightarrow$)*. Then the following are equivalent:*

1. $\Rightarrow$ *is semilinear in* L*.*
2. L *is filter-framal.*
3. L *is distributive.*

If furthermore L *is weakly algebraizable, then we can add the following conditions:*

4. *For each* $\mathcal{L}$*-algebra* $\boldsymbol{A}$*, the lattice of all congruences* Θ *of* $\boldsymbol{A}$ *such that* $\boldsymbol{A}/\Theta \in \mathbf{ALG}^*(\mathrm{L})$ *is a frame.*
5. *The lattice of all congruences* Θ *of* $\boldsymbol{Fm}_{\mathcal{L}}$ *such that* $\boldsymbol{Fm}_{\mathcal{L}}/\Theta \in \mathbf{ALG}^*(\mathrm{L})$ *is distributive.*

Proof The first part is an easy corollary of Theorem 4 and [6, Theorems 4.27, 4.28, and 4.17]. The second part follows from the well known relation between filters and relative congruences on weakly algebraizable logics (see e.g. [8]). □

Our next goal is to give a syntactical characterization of semilinear logics using the notion of ∇-form of a rule of an axiomatic system (see below). We obtain a rather strong theorem that allows to reduce the question of semilinarity to the (much simpler) question of derivability of certain consecutions.

First we define the notion of ∇-form of a consecution, inspired by [7]. Let us fix a logic L with a p-protodisjunction ∇. For an $\mathcal{L}$-consecution $R = \Gamma \rhd \varphi$ we define the (set of) consecution(s) R^∇ as $\{\Gamma \nabla \chi \rhd \delta \mid \chi \in Fm_{\mathcal{L}} \text{ and } \delta \in \varphi \nabla \chi\}$.

Theorem 6 (Syntactic characterization of semilinearity) *Let* L *be a logic with a weak p-implication* $\Rightarrow$ *and a p-protodisjunction* ∇ *satisfying the* IPEP *and* $(\mathrm{DMP}^{\nabla}_{\Rightarrow})$ *and let* $\mathcal{AS}$ *be one of its presentations. Furthermore assume that*

$$\varphi \nabla \psi \vdash_{\mathrm{L}} \psi \nabla \varphi \qquad\qquad \varphi \nabla \varphi \vdash_{\mathrm{L}} \varphi$$

Then the following are equivalent:

1. L *is semilinear,*
2. ∇ *satisfies* $(\mathrm{P}^{\nabla}_{\Rightarrow})$ *and* $R^{\nabla} \subseteq \mathrm{L}$ *for each* $R \in \mathcal{AS}$,
3. ∇ *satisfies* $(\mathrm{P}^{\nabla}_{\Rightarrow})$ *and* $R^{\nabla} \subseteq \mathrm{L}$ *for each* $R \in \mathrm{L}$,

Proof It follows from Theorem 4 and [6, Theorem 4.5 and Proposition 4.6]. □

Corollary 4 *Let* L_1 *be a logic with a semilinear weak p-implication* $\Rightarrow$ *and* ∇ *a p-protodisjunction satisfying* $(\mathrm{DMP}^{\nabla}_{\Rightarrow})$. *Further let* L_2 *be an expansion of* L_1 *by a set* $\mathcal{C}$ *of consecutions such that* L_2 *has the* IPEP *and* $\Rightarrow$ *is one of its a weak p-implications. Then,* $\Rightarrow$ *is semilinear in* L_2 *iff* $R^{\nabla} \subseteq \mathrm{L}_2$ *for each* $R \in \mathcal{C}$.

Proof The left-to-right direction follows directly from Theorem 6. The prove the converse implication first observe that due to Theorem 4, ∇ has the sPCP in L_1 and so, by [6, Theorem 5.1], it has the sPCP also in L_2. If we show that L_2 proves $(\mathrm{P}^{\nabla}_{\Rightarrow})$, then it has the SLP (due to Proposition 7) and, as we assume IPEP, Theorem 3 will complete the proof.

From Proposition 6 we know that L_1 proves $(\mathrm{P}^{\nabla}_{\Rightarrow})$. We show that it is preserved in all expansions. Thus we need to check that $\vdash_{\mathrm{L}_2} \chi(\chi_1(\varphi, \psi, \overrightarrow{\delta_1}), \chi_2(\psi, \varphi, \overrightarrow{\delta_2}), \overrightarrow{\delta})$ for any:

- $\mathcal{L}_2$-formulae $\varphi, \psi, \overrightarrow{\delta_1}, \overrightarrow{\delta_2}, \overrightarrow{\delta}$
- $\mathcal{L}_1$-formula $\chi(p, q, \overrightarrow{r}) \in \nabla$
- $\mathcal{L}_1$-formulae $\chi_1(p, q, \overrightarrow{r_1}), \chi_2(p, q, \overrightarrow{r_2}) \in \Rightarrow$

Note that we can assume w.l.o.g. that the variables in $\overrightarrow{r}, \overrightarrow{r_1}$ and $\overrightarrow{r_2}$ are all pairwise different. Since L_1 proves $(\mathrm{P}^{\nabla}_{\Rightarrow})$ and L_2 is one of its expansions, we can easily show that $\vdash_{\mathrm{L}_2} \chi(\chi_1(p, q, \overrightarrow{r_1}), \chi_2(q, p, \overrightarrow{r_2}), \overrightarrow{r})$. A suitable substitution completes the proof. □

The following corollary can be seen as an extension (for logics with a suitable disjunction) of [4, Corollary 4] from axiomatic extensions to axiomatic expansions.

Corollary 5 *Let* L_1 *be a weakly p-implicational logic with a semilinear implication* $\Rightarrow$ *and* ∇ *a p-protodisjunction satisfying* $(\mathrm{DMP}^{\nabla}_{\Rightarrow})$. *Then* $\Rightarrow$ *is semilinear in an* axiomatic *expansion* L' *of* L *provided that* L' *enjoys the* IPEP *and* $\Rightarrow$ *is still a weak p-implication in* L'.[18]

We conclude this section by using its results to obtain an axiomatization of the least extension of a given logic L where $\Rightarrow$ is semilinear; recall that we denote this logic by $\Rightarrow^{\ell}L$ and that, following [6], by L^{∇} we denote the weakest logic extending L where ∇ is a strong p-disjunction.

[18] Note that to check the last condition it suffices to show that L' proves (sCng) for its new connectives (due to [4, Proposition 3]).

Theorem 7 (Axiomatization of the least semilinear extension) *Let* L *be a logic with a weak p-implication* $\Rightarrow$ *and a p-protodisjunction* ∇ *satisfying the* IPEP *and* $(\mathrm{DMP}^{\nabla}_{\Rightarrow})$. *Then* $\Rightarrow^{\ell}L$ *is the extension of* L^{∇} *by* $(\mathrm{P}^{\nabla}_{\Rightarrow})$.

Proof Let us denote the extension of L^{∇} by $(\mathrm{P}^{\nabla}_{\Rightarrow})$ as $\hat{\mathrm{L}}$. Since $\hat{\mathrm{L}}$ is an axiomatic extension of L^{∇}, ∇ is a p-disjunction in $\hat{\mathrm{L}}$ as well (by [6, Theorem 5.1]) and $\hat{\mathrm{L}}$ has the IPEP (by [6, Lemma 2.8]). Thus $\Rightarrow$ is a weak semilinear p-implication in $\hat{\mathrm{L}}$ (by Theorem 4).

Let L' be any extension of L where $\Rightarrow$ is semilinear. Clearly L' proves $(\mathrm{DMP}^{\nabla}_{\Rightarrow})$ and so, by Theorem 4, L' proves $(\mathrm{P}^{\nabla}_{\Rightarrow})$ and has the sPCP w.r.t. ∇ and so $\mathrm{L}^{\nabla} \subseteq \mathrm{L}'$; thus $\hat{\mathrm{L}} \subseteq \mathrm{L}'$. □

We can improve this theorem by using [6, Theorem 5.6.] which gives, under some additional conditions, an axiomatization of L^{∇}.

Corollary 6 *Let* L *be a logic with a presentation* $\mathcal{AS}$, *a weak p-implication* $\Rightarrow$, *and a protodisjunction* ∇ *satisfying the* IPEP, $(\mathrm{DMP}^{\nabla}_{\Rightarrow})$ *and the following:*

$$\varphi \nabla \psi \vdash_{\mathrm{L}} \psi \nabla \varphi \qquad \varphi \nabla \varphi \vdash_{\mathrm{L}} \varphi \qquad \varphi \nabla (\psi \nabla \chi) \dashv\vdash_{\mathrm{L}} (\varphi \nabla \psi) \nabla \chi$$

Then $\Rightarrow^{\ell}L$ *is the extension of* L *by* $(\mathrm{P}^{\nabla}_{\Rightarrow})$ *and rules* $\{R^{\nabla} \mid R \in \mathcal{AS}\}$.

Finally, we notice that if ∇ is a p-disjunction, then $\mathrm{L}^{\nabla} = \mathrm{L}$ and L satisfies $(\mathrm{DMP}^{\nabla}_{\Rightarrow})$ (by Proposition 6). Thus, we can obtain an interesting corollary:

Corollary 7 *Let* L *be a logic satisfying the* IPEP *with a weak p-implication* $\Rightarrow$ *and a p-disjunction* ∇. *Then* $\Rightarrow^{\ell}L$ *is the extension of* L *by* $(\mathrm{P}^{\nabla}_{\Rightarrow})$.

Acknowledgements: The authors want to express their gratitude to the anonymous referee for his/her useful corrections and remarks. Both authors were supported by the project GA13-14654S of the Czech Science Foundation. Petr Cintula also acknowledges the support of RVO 67985807. Carles Noguera was also supported by the FP7-PEOPLE-2009-IRSES project MaToMUVI (PIRSES-GA-2009-247584).

References

1. Cintula, P.: Weakly implicative (fuzzy) logics I: Basic properties. *Archive for Mathematical Logic* **45**(6), 673–704 (2006)
2. Cintula, P., Hájek, P., Noguera, C. (eds.): Handbook of Mathematical Fuzzy Logic (in 2 volumes), *Studies in Logic, Mathematical Logic and Foundations*, vol. 37, 38. College Publications, London (2011)
3. Cintula, P., Noguera, C.: A note on natural extensions in abstract algebraic logic. *Studia Logica* **103**(4), 815–823 (2015)
4. Cintula, P., Noguera, C.: Implicational (semilinear) logics I: A new hierarchy. Archive for Mathematical Logic **49**(4), 417–446 (2010)
5. Cintula, P., Noguera, C.: A general framework for mathematical fuzzy logic. In: P. Cintula, P. Hájek, C. Noguera (eds.) Handbook of Mathematical Fuzzy Logic - Volume 1, *Studies in Logic, Mathematical Logic and Foundations*, vol. 37, pp. 103–207. College Publications, London (2011)
6. Cintula, P., Noguera, C.: The proof by cases property and its variants in structural consequence relations. Studia Logica **101**(4), 713–747 (2013)

7. Czelakowski, J.: Remarks on finitely based logics. In: Proceedings of the Logic Colloquium 1983. Vol. 1. Models and Sets, *Lecture Notes in Mathematics*, vol. 1103, pp. 147–168. Springer, Berlin (1984)
8. Czelakowski, J.: Protoalgebraic Logics, *Trends in Logic*, vol. 10. Kluwer, Dordrecht (2001)
9. Font, J.M., Jansana, R.: A General Algebraic Semantics for Sentential Logics, *Lecture Notes in Logic*, vol. 7, 2 edn. Association for Symbolic Logic, Ithaca, NY (2009). Freely downloadable from http://projecteuclid.org/euclid.lnl/1235416965
10. Font, J.M., Jansana, R., Pigozzi, D.L.: A survey of Abstract Algebraic Logic. Studia Logica **74**(1–2, Special Issue on Abstract Algebraic Logic II), 13–97 (2003)
11. Galatos, N., Jipsen, P., Kowalski, T., Ono, H.: Residuated Lattices: An Algebraic Glimpse at Substructural Logics, *Studies in Logic and the Foundations of Mathematics*, vol. 151. Elsevier, Amsterdam (2007)
12. Jansana, R.: Selfextensional logics with conjunction. Studia Logica **84**(1), 63–104 (2006)
13. Lávička, T.: *Classification of (In)finitary Logics,* Master thesis, Charles University in Prague (2015)
14. Łukasiewicz, J., Tarski, A.: Untersuchungen über den Aussagenkalkül. Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, cl. III **23**(iii), 30–50 (1930)
15. Raftery, J.G.: Order-algebraizable logics. Annals of Pure and Applied Logic **164**(3), 251–283 (2013)