Tomáš Lávička
Carles Noguera

A new hierarchy of (in)finitary logics in abstract algebraic logic

Abstract. In this article we investigate infinitary propositional logics from the perspective of their completeness properties in abstract algebraic logic. It is well known that every finitary logic is complete with respect to its relatively (finitely) subdirectly irreducible models. We identify two syntactical notions formulated in terms of (completely) intersection-prime theories that follow from finitarity and are sufficient conditions for the aforementioned completeness properties. We construct all the necessary counterexamples to show that all these properties define pairwise different classes of logics. Consequently, we obtain a new hierarchy of logics going beyond the scope of finitarity.

Keywords: Abstract algebraic logic, consequence relations, infinitary logics, completeness properties

1. Introduction

A big part of the literature on non-classical propositional logics is devoted to systems that, just like classical logic, are *finitary*, in the sense that whenever a proposition follows from a set of premises, it must also from a *finite* subset of these premises. Such restriction is due to the fact that finitarity is a technically convenient assumption that substantially simplifies the necessary mathematical framework. Moreover, it may also be argued, from a more philosophical point of view, that if mathematical logic is supposed to model *correct reasoning*, then it should provide systems that, like a finite rational being, can only perform finitely-many inference steps to justify a proposition. However, beyond that motivation, one can as well find many natural examples of infinitary logics in the literature, i.e. systems where a proposition may follow from an infinite set of premises, but not from any of its finite subsets, or equivalently, systems that need infinitary inference when presented in terms of a proof calculus.[1]. A prominent one is the infinitely-valued Łukasiewicz logic $Ł_\infty$ [13]. Therefore, abstract algebraic logic, a dis-

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[1]In this paper we always use 'infinitary' to denote this property. We do not consider logics with infinitely long sentences, typically obtained by means of infinite conjunctions or disjunctions, also called *infinitary logics* in the corresponding literature

cipline that intends to provide a very general and encompassing approach to the study of non-classical logics, cannot be restricted only to finitary logics.

The contribution to the study of infinitarity systems in abstract algebraic logic contained in this paper is mostly concerned with their completeness properties. Mainly we focus on relatively (finitely) subdirectly irreducible models of a given logic and call the corresponding completeness properties RSI-*completeness* and RFSI-*completeness*. We study them via a syntactical property called *intersection-prime extension property* (IPEP), that says that, for a given logic, the family of *intersection-prime* theories (theories that cannot be decomposed as the intersection of two strictly larger theories) form a basis of the closure system of all theories. This property was first introduced in [5] where it turn out to be very useful for the study of general disjunctions, since intersection-prime filters coincide with the usual notion of prime filter (a filter such that, if it contains a disjunction of two formulas, then it also contains one of the disjuncts). Moreover, the IPEP also turned out to be an essential property for the characterization of semilinear logics (logics complete with respect to a semantics of linearly ordered matrices) obtained in [6].

The present paper stems from the master thesis [12], devoted to the study of a hierarchy of (in)finitary logics given the mentioned completeness properties, the IPEP and a variation of this property. Namely, we propose a natural strengthening of IPEP, called *completely intersection-prime extension* property, CIPEP, which says that completely intersection-prime theories (those that cannot be decomposed as the intersection of an *arbitrary* family of strictly larger theories) form a basis.

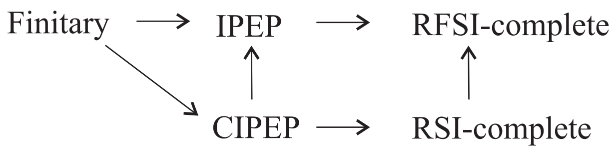

Figure 1. Inclusions between the defined classes of logics

These properties define corresponding classes of logics that extend that of finitary logics as depicted in Figure 1. IPEP (resp. CIPEP) are sufficient syntactical conditions for logic to be RFSI-complete (resp. RSI-complete). The natural question is whether this picture does indeed give a new meaningful hierarchy of logics in abstract algebraic logic, i.e. whether the classes are pairwise different or they collapse.

The main contribution of this paper is that it gives a complete answer to that question. Namely, we will prove that all the mentioned classes are pairwise different, by producing three examples of *infinitary* logics:

1. a logic with the CIPEP,
2. a logic with the IPEP which is not RSI-complete, and
3. an RSI-complete logic without the IPEP.

The paper is organized as follows. Section 2 gives some necessary preliminaries from abstract algebraic logic. In Section 3.1 we consider a notion of completeness with respect to surjective evaluations that allows as to show that the infinitely valued Łukasiewicz logic $Ł_\infty$ and infinitely valued product logic Π_∞ are infinitary logics with the CIPEP. In Section 3.2 we present an example of the second kind, a logic with the IPEP which is not RSI-complete, as a variant of the implicational fragment of Gödel–Dummet fuzzy logic G [8] enriched with ω-many truth constants. Finally, Section 3.3 describes in details a rather involved example of an RSI-complete logic without the IPEP. One may argue the last example is too unnatural. However, we may argue that it has interesting features, besides showing the corresponding separation of classes. On the one hand, most of its connectives behave like predicates of a first-order language. On the other hand, the logic is presented semantically as a consequence relation of one particular matrix, where the universe of its algebraic reduct consists of two parts, one of them encoding the syntax of the logic. Moreover, in spite of the technicalities involved, this example still belongs, like the previous ones, to the family of *weakly implicative logics* introduced in [2].

2. Preliminaries

2.1. Basic notions

In this subsection we briefly recall the definitions and fix the notations of the basic notions of abstract algebraic logic that will be needed in the paper (for comprehensive monographs and a survey see [7, 9, 10, 11]). A *propositional language* $\mathcal{L}$ is any type (with no restriction on the cardinality), by $\boldsymbol{Fm}_{\mathcal{L}}$ we denote the free term algebra over an arbitrary (but fixed) infinite countable set of variables in the language $\mathcal{L}$, by $Fm_{\mathcal{L}}$ we denote its universe. For any sets of formulas Γ, Δ and a formula φ we often write 'Γ, Δ', and 'Γ, φ' for, respectively, '$\Gamma \cup \Delta$', and '$\Gamma \cup \{\varphi\}$'.

An *$\mathcal{L}$-consecution* is a pair $\Gamma \rhd \varphi$. Given a set of $\mathcal{L}$-consecutions L, we write $\Gamma \vdash_{\mathrm{L}} \varphi$ rather than $\Gamma \rhd \varphi \in \mathrm{L}$. A *consequence relation* L in the language $\mathcal{L}$ is a set of $\mathcal{L}$-consecutions satisfying:

- If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathrm{L}} \varphi$. (Reflexivity)
- $\Delta \vdash_{\mathrm{L}} \varphi$ and $\Delta \subseteq \Gamma$ then $\Gamma \vdash_{\mathrm{L}} \varphi$ (Monotonicity)
- If $\Delta \vdash_{\mathrm{L}} \psi$ for each $\psi \in \Gamma$ and $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Cut)

Moreover *logic* is a structural consequence relation; i.e. a consequence relation with the following additional condition:

- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each $\mathcal{L}$-substitution σ. (Structurality)

Finally, a logic is *finitary* if it satisfies the following condition:

- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathrm{L}} \varphi$. (Finitarity)

We write $\Gamma \vdash_{\mathrm{L}} \Delta$ when $\Gamma \vdash_{\mathrm{L}} \varphi$ for every $\varphi \in \Delta$. A *theory* of a logic L is a set of formulas closed under the consequence relation. By Th(L) we denote the set of all theories of L, which is a complete lattice. By $\mathrm{Th}_{\mathrm{L}}(\Gamma)$ we denote the theory generated by Γ.

An $\mathcal{L}$-matrix is a pair $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$, where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra (the *algebraic reduct* of the matrix) and $F \subseteq A$ is a subset called the *filter* of the matrix. Given a class $\mathbb{K}$ of $\mathcal{L}$-matrices, a semantical consequence relation is defined as: $\Gamma \models_{\mathbb{K}} \varphi$ iff for each $\langle \boldsymbol{A}, F \rangle \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e (i.e. a homomorphism $e\colon \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{A}$) such that $e[\Gamma] \subseteq F$, we have $e(\varphi) \in F$. Clearly, $\models_{\mathbb{K}}$ is a logic. Moreover, if $\mathbb{K}$ is a finite set of finite $\mathcal{L}$-matrices, the logic $\models_{\mathbb{K}}$ is known to be finitary.

Given a matrix $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$, we say that a congruence θ of $\boldsymbol{A}$ is *compatible* with F iff for each $a, b \in A$, if $\langle a, b \rangle \in \theta$ and $a \in F$, then $b \in F$. Compatible congruences with F form a complete sublattice of the lattice of all congruences of $\boldsymbol{A}$, and thus there is a maximum congruence compatible with F, which is called the *Leibniz congruence* of $\mathbf{A}$ and denoted as $\Omega_{\boldsymbol{A}}(F)$. We say that $\mathbf{A}$ is a *reduced matrix* if $\Omega_{\boldsymbol{A}}(F) = Id_{\boldsymbol{A}}$.

A matrix $\mathbf{A}$ is a *model* of L if $\vdash_{\mathrm{L}} \subseteq \models_{\{\mathbf{A}\}}$. The class of (reduced) matrix models of a logic L is denoted as $\mathbf{MOD}(\mathrm{L})$ (or $\mathbf{MOD}^*(\mathrm{L})$ respectively). Both classes give complete semantics for any logic L; however it is common to consider meaningful subclasses of reduced models which may provide stronger completeness theorems. A matrix $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ is *relatively (finitely) subdirectly irreducible in* $\mathbf{MOD}^*(\mathrm{L})$, in symbols $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ ($\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$), if it cannot be decomposed as a non-trivial subdirect product of an arbitrary (finite non-empty) family of matrices from $\mathbf{MOD}^*(\mathrm{L})$.

Filter of a matrix $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ is an L-*filter* provided $\mathbf{A}$ is a model of L. By $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ we denote the set of all L-filters over $\boldsymbol{A}$; $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is also a complete lattice and hence it induces a closure operator denoted as $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}$ (we write simply Fi when the logic and the algebra are clear from the context).

In this paper we will consider some logics belonging to the following implication-based class introduced in [2]:

DEFINITION 2.1 (Weakly implicative logic). *Let* L *be a logic in a language* $\mathcal{L}$. *We say that* L *is a* weakly implicative logic *if there is a binary connective* $\to$ *(primitive or definable by a formula of two variables in* $\mathcal{L}$*) such that:*

(R) $\vdash_{\mathrm{L}} \varphi \to \varphi$

(MP) $\varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi$

(T) $\varphi \to \psi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \to \chi$

(sCng) $\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} c(\chi_1, \dots, \chi_i, \varphi, \dots, \chi_n) \to c(\chi_1, \dots, \chi_i, \psi, \dots, \chi_n)$ *for each* $\langle c, n\rangle \in \mathcal{L}$ *and each* $0 \leq i < n$.

We say that L *is a* Rasiowa-implicative logic *if, moreover, it satisfies:*

(W) $\varphi \vdash_{\mathrm{L}} \psi \to \varphi$

A very useful property of weakly implicative logics is that they enjoy an easy characterization of the Leibniz congruence via the connective $\to$:

PROPOSITION 2.2. *Given a weakly implicative logic* L *and a matrix* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$, *we have:* $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$ *if and only if* $\{a \to b, b \to a\} \subseteq F$. *In particular,* $\langle \boldsymbol{A}, F\rangle$ *is reduced if* $\{a \to b, b \to a\} \subseteq F$ *implies* $a = b$ *for every* $a, b \in A$.

Weakly implicative logics are a subclass of *protoalgebraic logics*, that will also be used in this paper. Protoalgebraic logics (as explained in [4]) can be introduced by requiring the same conditions as Definition 2.1, but for a generalized implication given an arbitrary set of formulas with two variables (possibly infinite, possibly with parameters). The monograph [7] is the most extensive reference for the rich theory of protoalgebraic logics.

2.2. Intersection-prime filters and classes of infinitary logics

In this subsection we recall definitions for filters from [7, 5] that allow to consider infinitary logics with good completeness properties with respect to the aforementioned classes of reduced matrix models.

Given a logic L and a filter $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, we say that F is *intersection-prime* if it is finitely meet-irreducible, i.e. there is no pair of filters $F_1, F_2 \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ such that $F = F_1 \cap F_2$ and $F \subsetneq F_1, F_2$. Similarly, we say that F is *completely intersection-prime* if it is meet-irreducible, i.e. whenever $F = \bigcap_{i\in I} F_i$ for a family $\{F_i \mid i \in I\} \subseteq \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, there is $i_0 \in I$ such that $F = F_{i_0}$.

It is well known [7, Proposition 1.3.4.] that

- $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ iff F is *intersection-prime* in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$,
- $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ iff F is *completely intersection-prime* in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$.

Recall that a family $\mathcal{B} \subseteq \mathcal{C}$ is a *basis* of a closure system $\mathcal{C}$ if for every $X \in \mathcal{C}$ there is a $\mathcal{D} \subseteq \mathcal{C}$ such that $X = \bigcap \mathcal{D}$ (which can be equivalent formulated as an extension property: for every $Y \in \mathcal{C}$ and every $a \in A \setminus Y$ there is $Z \in \mathcal{B}$ such that $Y \subseteq Z$ and $a \notin Z$).

Using these notions one can define the following properties:[2]

Definition 2.3. *We say that* L *has the* (completely) intersection-prime extension property*,* (C)IPEP *for short, if the (completely) intersection-prime theories form a basis of* Th(L)*. Finally, we say that a logic* L *is* R(F)SI-*complete if* $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{R(F)SI}}}$.

Let us formulate two straightforward observations:

Proposition 2.4. *For every logic* L *we have:*

1. *If* L *has the* CIPEP*, then it has the* IPEP*.*
2. *If* L *is* RSI-*complete, then it is* RFSI-*complete.*

The properties in Definition 2.3 determine corresponding classes of logics that include that of finitary logics, as described by the next proposition.

Proposition 2.5. *For every logic* L *we have:*

1. *If* L *is finitary, then it has the* CIPEP*.*
2. *If* L *has the* CIPEP*, then it is* RSI-*complete.*
3. *If* L *has the* IPEP*, then it is* RFSI-*complete.*

Proof. The first claim is proved in [7, Corollary 1.3.3.]). The third claim is proved in [5, Lemma 2.6], and the second one is shown analogously. □

Figure 1 depicts the inclusions between classes of logics stated by the two previous propositions. It is important to stress that this hierarchy does not exhaust completely the class of all propositional logic. Indeed, one can show that there exist non-RFSI-complete logics. An example of such a logic can be found in [5, Example 3.12]. This logic is shown to have a protodisjunction satisfying the PCP (*proof by cases property*) while not satisfying the sPCP (*strong proof by cases property*) . From Section 4.4 of [5] it can be extracted that for any protoalgebraic RFSI-complete logic, the two proof by cases properties must coinncide. Therefore, since this logic is weakly implicative (and hence protoalgebraic), it cannot be RFSI-complete.

[2] The IPEP and the RFSI-completeness were already introduced explicitly in [5, Definition 2.5].

3. Hierarchy of infinitary logics

Our aim is to show that the notions introduced in the previous section yield a hierarchy of finitary and infinitary propositional logics by showing that all the classes are pairwise different. This will be achieved by means of the results proved and the examples built in the following subsections.

3.1. Surjective completeness

In this subsection we provide a semantical characterization of CIPEP and IPEP for protoalgebraic logics via what we call a *surjective completeness.* We use this characterization to show that there exist infinitary logics with the CIPEP (and thus also with the IPEP).

We first refine the notion of semantical consequence given by a class of matrices considering only surjective evaluations and describe its behavior.

DEFINITION 3.1 (Surjective semantical consequence). *A formula φ is a* surjective semantical consequence *of a set Γ of formulas w.r.t. a class $\mathbb{K}$ of $\mathcal{L}$-matrices if for each $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and each surjective $\boldsymbol{A}$-evaluation e (surjective as a function $e : \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{A}$), we have $e(\varphi) \in F$ whenever $e[\Gamma] \subseteq F$; we denote it by $\Gamma \models^{s}_{\mathbb{K}} \varphi$.*

First notice that $\models_{\mathbb{K}} \subseteq \models^{s}_{\mathbb{K}}$. Moreover, it is easy to show that $\models^{s}_{\mathbb{K}}$ is a consequence relation. However, it is not necessarily structural, and hence not necessarily a logic, as shown by the following example.

EXAMPLE 3.2. *Consider a language $\mathcal{L}$ with one binary connective $\bullet$ and an $\mathcal{L}$-algebra $\boldsymbol{A} = \langle \{1, a, 0\}, \bullet\rangle$ with the operation given by*

$\bullet$	1	a	0
1	1	1	0
a	a	1	a
0	0	0	0

consider L *defined as $\models^{s}_{\mathbf{A}}$, where $\mathbf{A} = \langle \boldsymbol{A}, \{1\}\rangle$. Further let $\Gamma = \{p \bullet q\} \cup \{r \in Var \mid r \neq p, r \neq q\}$. It is easy to show that $\Gamma \vdash_{\mathrm{L}} p$; in fact, there is no surjective evaluation satisfying Γ. Now take the substitution σ such that $\sigma(q) = p$, $\sigma(p) = p$ and $\sigma(r) = q$ for any other variable r. We have*

$\sigma[\Gamma] = \{p \bullet p, q\}$. *Notice that* $\sigma[\Gamma] \nvdash_{\mathrm{L}} \sigma(p) = p$; *indeed take any evaluation* e *such that* $e(p) = a$, $e(q) = 1$ *and* $e(r) = 0$ *for some other variable* r.

We use the notion of cardinality of L to characterize sufficient conditions under which $\models^s_{\mathbb{K}} = \models_{\mathbb{K}}$, hence conditions under which $\models^s_{\mathbb{K}}$ is indeed a logic.

Definition 3.3. *The cardinality of a consequence relation* L, card(L), *is the smallest cardinal* κ *such that for each* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ *we have: if* $\Gamma \vdash_{\mathrm{L}} \varphi$, *then there is* $\Gamma_0 \subseteq \Gamma$ *with* $|\Gamma_0| < \kappa$ *such that* $\Gamma_0 \vdash_{\mathrm{L}} \varphi$.

Observe that a logic L is *finitary* if $\mathrm{card}(\mathrm{L}) \leq \omega$.

Proposition 3.4. *Let* κ *be an infinite cardinal and* $\mathbb{K}$ *a class of* $\mathcal{L}$*-matrices. Assume that* $|Var| = \kappa$, $\mathrm{card}(\models^s_{\mathbb{K}}) \leq \kappa$, *and* $|A| \leq \kappa$ *for each* $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$. *Then,* $\models^s_{\mathbb{K}} = \models_{\mathbb{K}}$ *and, in particular,* $\models^s_{\mathbb{K}}$ *is structural.*

Proof. The inclusion $\supseteq$ trivially holds always. Suppose that $\Gamma \models^s_{\mathbb{K}} \varphi$. Then we obtain a set $\Gamma' \subseteq \Gamma$ of cardinality less than κ such that $\Gamma' \models^s_{\mathbb{K}} \varphi$. We claim that $\Gamma' \models_{\mathbb{K}} \varphi$ and consequently also $\Gamma \models_{\mathbb{K}} \varphi$. Consider any $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and any evaluation e on $\boldsymbol{A}$ such that $e[\Gamma'] \subseteq F$. Since $\Gamma' \cup \{\varphi\}$ contains less than κ variables, we can easily find a surjective evaluation e' which coincides with e on all variables occurring in $\Gamma' \cup \{\varphi\}$. Obviously, we have $e'[\Gamma'] \subseteq F$ and thus also $e(\varphi) = e'(\varphi) \in F$. □

Further, in the next proposition, we refine usual completeness results using the surjective consequence relation.

Proposition 3.5. *Let* L *be a logic. Then:*

$$\mathrm{L} = \models^s_{\mathbf{MOD}(\mathrm{L})} = \models^s_{\mathbf{MOD}^*(\mathrm{L})}.$$

Moreover,

1. *if* L *has the* IPEP, *then* $\mathrm{L} = \models^s_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}}$,
2. *if* L *has the* CIPEP, *then* $\mathrm{L} = \models^s_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}}$.

Proof. It is enough to observe that the evaluations used in the proof of completeness w.r.t. reduced models (Lindenbaum–Tarski construction) are in fact surjective (cf. proof of Proposition 2.5). □

Given a matrix $\langle A, F\rangle \in \mathbf{MOD}(\mathrm{L})$, note that the set of filters $[F, A] = \{G \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A}) \mid F \subseteq G\}$ can be seen as an interval in the lattice of L-filters over the algebra $\boldsymbol{A}$. We will need the following result (that follows from [7, Theorem 1.1.8]).

Proposition 3.6. *Let* L *be a protoalgebraic logic. Take* $\langle A, F\rangle, \langle B, G\rangle \in \mathbf{MOD}(\mathrm{L})$ *and let* $h\colon \langle A, F\rangle \to \langle B, G\rangle$ *be a strict surjective homomorphism. Then the mapping* $\boldsymbol{h}$ *defined as* $\boldsymbol{h}(H) = h[H]$ *is an isomorphism between* $[F, A]$ *and* $[G, B]$.

Next we prove the characterization theorem for the CIPEP and the IPEP via surjective evaluations.

Proposition 3.7. *Let* L *be a protoalgebraic logic. Then:*

1. L *has the* CIPEP *if and only if* $\mathrm{L} = \models^{s}_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}}$.
2. L *has the* IPEP *if and only if* $\mathrm{L} = \models^{s}_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}}$.

Proof. We prove only the first part of the theorem (the second one is identical). $\Rightarrow$: This implication is given by Proposition 3.5.
$\Leftarrow$: Suppose $\Gamma \nvdash_{\mathrm{L}} \varphi$. There is $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ and a surjective evaluation $e\colon Fm_{\mathcal{L}} \to A$. We know that $T = e^{-1}[F]$ is an L-theory, in symbols $\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle \in \mathbf{MOD}(\mathrm{L})$. By Proposition 3.6 we obtain an isomorphism $\boldsymbol{e}$ between $[T, Fm_{\mathcal{L}}]$ and $[F, A]$. It easily follows that T is $\cap$-irreducible (because F is $\cap$-irreducible). Moreover, since $\Gamma \subseteq T$ and $\varphi \notin T$, we are done. □

By means of the usual submatrix operator $\mathbf{S}$ we can show in the next proposition that any logic given as the consequence relation of a class of matrices is, actually, a surjective semantical consequence.

Proposition 3.8. *For every class of* $\mathcal{L}$*-matrices* $\mathbb{K}$ *we have:* $\models_{\mathbb{K}} = \models^{s}_{\mathbf{S}(\mathbb{K})}$.

Proof. Suppose that $\Gamma \not\models_{\mathbb{K}} \varphi$. Then there is $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and an evaluation e such that $e[\Gamma] \subseteq F$ and $e(\varphi) \notin F$. Take the theory $T = e^{-1}[F]$. We only need to prove that $\langle e[\boldsymbol{Fm}_{\mathcal{L}}], e[T]\rangle$ is a submatrix of $\langle \boldsymbol{A}, F\rangle$. Clearly $e[\boldsymbol{Fm}_{\mathcal{L}}]$ is a subalgebra of $\boldsymbol{A}$ and, since e is strict, it follows that $e[T] = e[Fm_{\mathcal{L}}] \cap F$. The converse direction follows easily from the fact that any evaluation on a submatrix is also an evaluation on the original matrix. □

As an easy consequence of the previous proposition and Proposition 3.7, we obtain useful sufficient condition for a logic to have the CIPEP (resp. IPEP).

Corollary 3.9. *Let* L *be a protoalgebraic logic and suppose that* $\mathbb{K}$ *is a class of* $\mathcal{L}$*-matrices such that* $\mathrm{L} = \models_{\mathbb{K}}$. *Then:*

- *if* $\mathbf{S}(\mathbb{K}) \subseteq \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$, *then* L *has the* CIPEP,
- *if* $\mathbf{S}(\mathbb{K}) \subseteq \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$, *then* L *has the* IPEP.

As a corollary, we will show that two prominent fuzzy logics (see e.g. [3]), namely the infinitely-valued Łukasiewicz logic Ł_∞ and the infinitely-valued product fuzzy logic Π_∞, have the CIPEP (and consequently also the IPEP).

COROLLARY 3.10. *The infinitely-valued Łukasiewicz logic* Ł_∞ *enjoys the* CIPEP.[3]

PROOF. Let us first recall a definition of Ł_∞. The logic Ł_∞ has three connectives $\to$, $\&$ and $\overline{0}$ and is given as the semantical consequence of the following matrix $\mathbf{A} = \langle \boldsymbol{A}, \{1\}\rangle$, where $\boldsymbol{A} = \langle [0,1], \to, \&, \overline{0}\rangle$ such that $a \to b = \min\{1 - a + b, 1\}$, $a \,\&\, b = \max\{a + b - 1, 0\}$, and $\overline{0} = 0$.

Claim 1: There are only two Ł-filters on any subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$, namely $\{1\}$ and the trivial filter B.
Proof (of Claim 1): it is enough to realize that $p \vdash_{\text{Ł}} p \,\&\, p$, thus any filter which contains $a < 1$ also contains 0. Moreover Ł_∞ is closed under *modus ponens* which implies that any filter is an upset of $[0, 1]$. This conludes the claim 1.

Claim 2: $\mathbf{S}(\mathbf{A}) \subseteq \mathbf{MOD}^*(\text{Ł}_\infty)_{\mathrm{RSI}}$.
Proof (of Claim 2): Since Ł_∞ is an equivalential logic,[4] we obtain $\mathbf{S}(\mathbf{A}) \subseteq \mathbf{MOD}^*(\text{Ł}_\infty)$. Further, following claim 1, we have proved claim 2.

The rest then follows by Corollary 3.9. □

COROLLARY 3.11. *The product fuzzy logic* Π_∞ *has the* CIPEP.

PROOF. Again we start by recalling the definition of Π_∞. Π_∞ is given by the matrix $\mathbf{A} = \langle \boldsymbol{A}, \{1\}\rangle$, where $\boldsymbol{A} = \langle [0,1] \to, \&, \overline{0}\rangle$ such that $a \to b = 1$ if $a \leq b$ and $a \to b = b/a$ otherwise; $a \,\&\, b = a \cdot b$ and $\overline{0} = 0$. The proof of this corollary proceeds similarly as the previous one. We can again prove Claim 2 using an analog of Claim 1 and then argue by Corollary 3.9.

Claim 1: There are at most three Π_∞-filters on any subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$, namely $\{1\}$, $B \setminus \{0\}$, and the trivial filter B. □

THEOREM 3.12. *There is a logic which is not finitary and has the* CIPEP.

PROOF. Both Π_∞ and Ł_∞ are well known to be infinitary and, by the previous two corollaries, they have the CIPEP. □

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[3] We thank our colleague Petr Cintula for pointing at Łukasiewicz logic as an example of an infinitary logic with the CIPEP.

[4] Reduced models in equivalential logics are closed under submatrices.

3.2. Non RSI-complete logic with the IPEP

The aim of this subsection is to separate the top layer of Figure 1 from the lower one. Consider a language $\mathcal{L} = \{\rightarrow\} \cup \{\bar{q} \mid q \in (0,1] \cap \mathbb{Q}\}$.[5] For every $0 < q \leq 1$ define an $\mathcal{L}$-algebra $\boldsymbol{A}_q$ with domain $[0,q]$, $a \rightarrow b = q$ if $a \leq b$ and $a \leq b = b$ otherwise (i.e. it is a Gödel implication); and for constants $\bar{r}^{\boldsymbol{A}_q} = \min\{r, q\}$. Define $\mathbb{K} = \{\mathbf{A}_q = \langle \boldsymbol{A}_q, \{q\}\rangle \mid 0 < q \leq 1\}$ and let L be the consequence relation of this class of matrices, i.e. $\mathrm{L} = \models_{\mathbb{K}}$. The idea behind this definition is to have a logic with as many filters as possible. Moreover the presence of constants will enable us to prove that there are no subdirectly irreducible matrices in $\mathbf{MOD}^*(\mathrm{L})$.

We start with an easy observation:

Observation 3.13. L *is Rasiowa-implicative.*

Let $\mathbb{K}^a$ denote the class of algebraic reducts of matrices in $\mathbb{K}$. As a consequence we obtain that $\mathbf{ALG}^*(\mathrm{L}) = \mathbf{GQ}(\mathbb{K}^a)$, i.e. $\mathbf{ALG}^*(\mathrm{L})$ is the generalized quasivariety generated by $\mathbb{K}^a$.

Let us use the following characterization for generation of generalized quasivarieties (see e.g. [1, 14]):

$$\mathbf{GQ}(\mathbb{K}) = \mathbf{UISP}(\mathbb{K}),$$

where $\mathbf{U}(\mathbb{M}) = \{\boldsymbol{A} \mid \text{if } \boldsymbol{B} \in \mathbf{S}(A) \text{ is a countably generated then } \boldsymbol{B} \in \mathbb{M}\}$.

Given any two matrices $\mathbf{A}_q$ and $\mathbf{A}_r$ with $g < r$ and given any $\mathbf{A}_r$-evaluation e, we define $\mathbf{A}_q$-evaluation e^q as follows: $e^q(p) = \min\{e(p), q\}$ for each variable p.

Proposition 3.14. *Consider two matrices $\mathbf{A}_q$ and $\mathbf{A}_r$ such that $q < r$. Then for every $\mathbf{A}_r$-evaluation and every formula φ:*

1. *$e(\varphi) \in [q, r]$ iff $e^q(\varphi) = q$,*
2. *if $e(\varphi) \leq q$ then $e(\varphi) = e^q(\varphi)$.*

Proof. Easy induction on the complexity of formulas. □

For every $\mathcal{L}$-algebra $\boldsymbol{A}$ we will denote the set of all elements in A bigger (w.r.t. the canonical order induced by $\rightarrow$) than $\bar{q}^{\boldsymbol{A}}$ as $F_q^{\boldsymbol{A}}$, in symbols $F_q^{\boldsymbol{A}} = \{a \in A \mid \bar{q}^{\boldsymbol{A}} \leq^{\boldsymbol{A}} a\}$.

Corollary 3.15. *For any $\boldsymbol{A}_r$ and $q < r$, $F_q^{\boldsymbol{A}_r} = [q, r]$ is an* L*-filter on $\boldsymbol{A}_r$. In particular, $\mathbf{A}_r$ is $\cap$-reducible.*

[5] In this subsection we will denote rational numbers by the letters: q, r, s.

Proof. Suppose that $\Gamma \vdash_{\mathrm{L}} \varphi$ and $e[\Gamma] \subseteq [q,r]$. From the previous proposition we know that $e^q[\Gamma] \subseteq \{q\}$ and, since $\mathbf{A}_q$ is a model, we obtain $e^q(\varphi) = q$ and again, from the previous proposition, $e(\varphi) \in [q,r]$. □

Proposition 3.16. *The unique reduced matrix based on every algebra in* $\mathbf{SP}(\mathbb{K}^a)$ *is* $\cap$*-reducible.*

Proof. Let $\boldsymbol{B} \in \mathbf{SP}(\mathbb{K}^a)$. $\boldsymbol{B}$ is a subalgebra of some direct product of algebras $\boldsymbol{C} = \prod_{i\in I} \boldsymbol{C}_i$ for $\boldsymbol{C}_i \in \mathbb{K}^a$. The only filter that makes $\boldsymbol{B}$ reduced is $\{\bar{1}^{\boldsymbol{C}}\}$; we show it is $\cap$-reducible: it is easy to observe that for any system of filters $F_i \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{C}_i)$ we have $\prod_{i\in I} F_i \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{C})$. In particular, if we choose $F_i = F_q^{\boldsymbol{C}_i}$, then $\prod_{i\in I} F_i = F_q^{\boldsymbol{C}}$ is a filter on $\boldsymbol{C}$.

Define $Z = \{q \in (0,1) \mid$ *there is some* $\boldsymbol{C}_i$ *with domain* $[0,r]$ *and* $q < r\}$. Observe that for every $q \in Z$ we have $\{\bar{1}^{\boldsymbol{C}}\} \subsetneq F_q^{\boldsymbol{C}}$ and moreover $\{\bar{1}^{\boldsymbol{C}}\} = \bigcap_{q\in Z} F_q^{\boldsymbol{C}}$.

Further $F_q^{\boldsymbol{B}} = F_q^{\boldsymbol{C}} \cap B$ is an L-filter on $\boldsymbol{B}$ and, since, for every $q \in Z$: $\bar{1}^{\boldsymbol{C}} \neq \bar{q}^{\boldsymbol{C}} = \bar{q}^{\boldsymbol{B}} \in F_q^{\boldsymbol{B}}$ we conclude $\{\bar{1}^{\boldsymbol{C}}\} \subsetneq F_q^{\boldsymbol{B}}$ and finally $\{\bar{1}^{\boldsymbol{C}}\} = \bigcap_{q\in Z} F_q^{\boldsymbol{B}}$. Thus $\langle \boldsymbol{B}, \bar{1}^{\boldsymbol{B}}\rangle$ is $\cap$-reducible. □

Now we are heading towards the same claim for $\mathbf{UISP}(\mathbb{K}^a)$. We first show some properties of chains in $\mathbf{SP}(\mathbb{K}^a)$:

Lemma 3.17. *For any chain*[6] $\boldsymbol{A} \in \mathbf{SP}(\mathbb{K}^a)$ *and any* $a \in A$ *such that* $a < \bar{1}^{\boldsymbol{A}}$ *there is some* $q \in (0,1)$ *such that* $a < \bar{q}^{\boldsymbol{A}} < \bar{1}^{\boldsymbol{A}}$.

Proof. Start with $\boldsymbol{A}$ a subalgebra of a direct product of algebras $\boldsymbol{B} = \prod_{i\in I} \boldsymbol{B}_i$. Let us have $a \in A$ such that $a < \bar{1}^{\boldsymbol{A}}$. Clearly there is i such that $a(i) < \bar{1}^{\boldsymbol{A}_i}$ and consequently some $q \in (0,1)$ such that $a(i) < \bar{q}^{\boldsymbol{A}_i} < \bar{1}^{\boldsymbol{A}_i}$. From linearity we know that either $a < \bar{q}^{\boldsymbol{A}}$ or $\bar{q}^{\boldsymbol{A}} \leq a$ is true. Clearly, the second possibility would lead to contradiction. Thus, since obviously $\bar{q}^{\boldsymbol{A}} < \bar{1}^{\boldsymbol{A}}$, we are done. □

Proposition 3.18. *The unique reduced matrix based on every algebra in* $\mathbf{UISP}(\mathbb{K}^a)$ *is* $\cap$*-reducible.*

Proof. In pursuit of a contradiction suppose that there is $\boldsymbol{A}$ in $\mathbf{UISP}(\mathbb{K}^a)$ such that $\langle \boldsymbol{A}, \{\bar{1}^{\boldsymbol{A}}\}\rangle$ is $\cap$-irreducible. First note that this implies that $\boldsymbol{A}$ is linear with maximum element $\bar{1}^{\boldsymbol{A}}$.

Claim 1: *For every* $a \in A$ *such that* $a < \bar{1}^{\boldsymbol{A}}$ *there is* $q \in (0,1)$ *such that* $a < \bar{q}^{\boldsymbol{A}} < \bar{1}^{\boldsymbol{A}}$.

[6]The order is induced by $\to$ and the unique filter that makes $\boldsymbol{A}$ reduced, i.e. the filter $\{\bar{1}^{\boldsymbol{A}}\}$.

Proof (of Claim 1): Let $\langle a\rangle$ be the subalgebra generated by the element a. Since it is countable generated, we have

$$i : \langle a\rangle \simeq \boldsymbol{B} \hookrightarrow \prod_{i\in I} \boldsymbol{B}_i.$$

Further, since $\boldsymbol{B}$ is a chain (due to the isomorphism i) and $\boldsymbol{B} \in \mathbf{SP}(\mathbb{K}^a)$, we can find the desired q by applying i and Lemma 3.17.

Claim 2: *$F_q^{\boldsymbol{A}}$ is a filter on $\boldsymbol{A}$ for every $q \in (0,1)$.*

Proof (of Claim 2): Suppose $\Gamma \vdash_{\mathrm{L}} \varphi$ and $e[\Gamma] \subseteq F_q^{\boldsymbol{A}}$. It is clear that $e[\boldsymbol{Fm}_{\mathcal{L}}]$ is a countably generated subalgebra of $\boldsymbol{A}$ thus we have

$$i : e[\boldsymbol{Fm}_{\mathcal{L}}] \simeq \boldsymbol{B} \hookrightarrow \prod_{i\in I} \boldsymbol{B}_i.$$

For any $\psi \in \Gamma$ we have $\bar{q}^{\boldsymbol{A}} \leq^{\boldsymbol{A}} e(\psi)$. Since i is an isomorphism, also $i(\bar{q}^{\boldsymbol{A}}) = \bar{q}^{\boldsymbol{B}} \leq^{\boldsymbol{B}} i(e(\psi))$. We know that $F_q^{\boldsymbol{B}}$ is a filter on $\boldsymbol{B}$ (see the proof of Proposition 3.16), which implies $\bar{q}^{\boldsymbol{B}} \leq^{\boldsymbol{B}} i(e(\varphi))$. Thus it follows that $\bar{q}^{\boldsymbol{A}} \leq^{\boldsymbol{A}} e(\varphi)$, as we wanted.

To finish the proof observe that if $\boldsymbol{A}$ is not trivial then it is, by Claim 1, infinite. Define $Z = \{q \in (0,1) \mid \bar{q}^{\boldsymbol{A}} < \bar{1}^{\boldsymbol{A}}\}$. Now, using both claims, we can easily decompose $\{\bar{1}^{\boldsymbol{A}}\}$ by means of the collection of $F_q^{\boldsymbol{A}}$ ranging over Z. □

THEOREM 3.19. *There is a logic with the* IPEP *which is not* RSI-*complete.*

PROOF. Proposition 3.18 tells us that the logic L defined in this section has in fact no subdirectly irreducible reduced model (except, of course, for the trivial one). Thus, in particular, it is not RSI-complete.

On the other hand we can easily argue, using Corollary 3.9, that L has the IPEP. □

3.3. An RSI-complete logic without IPEP

As we have seen in the previous section, when we want to determine whether a given logic is RSI-complete or RFSI-complete, the notions of CIPEP or IPEP are useful sufficient conditions. It is enough to check whether the logic satisfies one of these extension properties (or finitarity). The rest of this paper is devoted to the problem of separating the classes of logics with the IPEP from RFSI-complete logics, and the classes of logics with the CIPEP from RSI-complete logics. This will be achieved by producing a single example, rather difficult to construct, of an RSI-complete logic which does not enjoy the IPEP. This way we prove that CIPEP and IPEP are not trivial notions, which, as conclusion, allows us to obtain a hierarchy of infinitary logics.

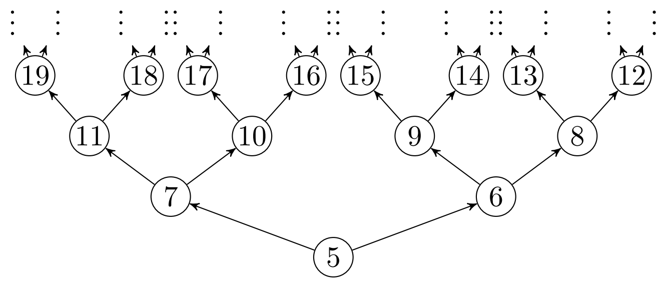

Figure 2. The tree T with its enumeration f

3.3.1. Introducing the example

We are going to describe RSI-complete *weakly implicative* logic L, which does not belong to he IPEP class.

Our logic will be given semantically by a suitable matrix $\langle \boldsymbol{A}, F\rangle$. This approach will manifest useful in proving of RSI-completeness: we only need to check that the matrix is reduced and F is completely $\cap$-prime filter in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, i.e. that $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$.

In order to falsify the IPEP in L, we will implement a full binary tree of height ω into $\boldsymbol{A}$. The motivation is that every node in the tree is $\wedge$-reducible (e.g. can be expressed as a meet of its two immediate successors). To benefit from this idea we make sure that every node of the tree is uniquely determined by some theory of L. To this end, we add a unary connective for each node s, which will allow us to capture (within the logic) which nodes are above s. Interestingly enough, in order to follows through we need to let the logic 'know' something about itself, that is, we include in the algebra also some substantial subset of $Fm_{\mathcal{L}}(\{p\})$ (algebra of formulas in one variable p). Therefore, an interesting feature of this logic is that its semantics is partially based on its own syntax.

Let $\mathrm{T} = \langle T, \leq_{\mathrm{T}}\rangle$ be the full binary tree of height ω. Note that we can view T as the collection of all functions which has some natural number n as a domain and its range is subset of 2, where $\leq_{\mathrm{T}}$ is the inclusion order ($\emptyset$ is the root of this tree). Moreover let $f : \mathbb{N} \setminus \{1,2,3,4\} \to \mathrm{T}$ be the natural enumeration of T (i.e. enumerating layer after layer). Note that in this setting $f(5)$ is the root of T. Whenever we mention function f we refer to this enumeration. The reason why we start enumerating from number 5 will be explained in the upcoming paragraphs.

Let us next focus on the language of L. Let $\mathcal{L} = \{\overline{0}, \to, B\} \cup \{B_i \mid i \geq_{\mathbb{N}}$

5}, where $\overline{0}$ is a nullary connective, $\rightarrow$ is binary and the rest are unary connectives; read B_i as 'bigger than the node $f(i)$'. Moreover we define a nullary connective $\overline{1}_0$ as $\overline{0} \rightarrow \overline{0}$. Again we define B_i only for natural numbers greater than 5 for the same reason as before.

As we said we want also to take some subset of $Fm_{\mathcal{L}}(\{p\})$ as part of the algebra. However, instead of speaking about $Fm_{\mathcal{L}}(\{p\})$, we will rather work with its representation via codes, capitalizing on the possibility of coding the syntax. We will do it in the following manner. We know that we can assign a unique natural number to any finite sequence of natural numbers. In this way there are some natural numbers which represent some finite sequence of natural numbers (the represented sequence is uniquely determined). We will use variables $w, w', w_1, w_2, \dots$ for these codes. If w is the code of a sequence $a_1, \dots, a_n$ of natural numbers, we write $w = \langle a_1, \dots, a_n \rangle$. For the *concatenation* of two codes, $w = \langle a_1, \dots, a_n \rangle$ and $w' = \langle b_1, \dots, b_m \rangle$, we write $w * w' = \langle a_1, \dots, a_n, b_1, \dots, b_m \rangle$. As it is usual when coding the syntax, we assign a specific meaning to each number: number $1 \in \mathbb{N}$ is used to code $\overline{0}$; $2 \in \mathbb{N}$ codes $\overline{1}_0$; $3 \in \mathbb{N}$ codes the variable p; $4 \in \mathbb{N}$ is used to code implication $\rightarrow$, namely, we define a *generalized concatenation*, denoted as $w \odot w'$, in such a way that the number 4 plays also a rôle of brackets, in particular we can presume $w \odot w'$ starts with 4 i.e. $w \odot w' = \langle 4, a_1, \dots, a_n \rangle$ (this will be convenient latter). Finally any $i \geq_{\mathbb{N}} 5$ will be used to code B_i. Now it should be apparent that we start enumerating the tree from number 5, because we need to free four numbers for other codification purposes.

Next we recursively define a set C of codes reflecting the desired subset of $Fm_{\mathcal{L}}(\{p\})$. We call these codes *proper*. We write $\ulcorner \varphi \urcorner$ for the code of the formula $\varphi \in Fm_{\mathcal{L}}(\{p\})$. The meaning of each code is written in the right part of the definition.

Definition 3.20 (Proper Codes). *The set* C *is defined to be the least set of codes such that:*

1. *(a)* $\{\langle i \rangle \mid i \geq_{\mathbb{N}} 5\} \subseteq \mathrm{C}$, $\quad \ulcorner B_i(p) \urcorner$
   (b) $\{\langle i, 1 \rangle \mid i \geq_{\mathbb{N}} 5\} \subseteq \mathrm{C}$, $\quad \ulcorner B_i(\overline{0}) \urcorner$
   (c) $\{\langle i, 2 \rangle \mid i \geq_{\mathbb{N}} 5\} \subseteq \mathrm{C}$, $\quad \ulcorner B_i(\overline{1}_0) \urcorner$
2. *moreover for every* $w_1, w_2 \in \mathrm{C}$ *also*
   (a) $\{\langle i \rangle * w_1 \mid i \geq_{\mathbb{N}} 5\} \subseteq \mathrm{C}$, $\quad \ulcorner B_i(\varphi) \urcorner$, *where* $\ulcorner \varphi \urcorner = w_1$
   (b) $\langle 1 \rangle \odot w_1 \in \mathrm{C}$ *and* $w_1 \odot \langle 1 \rangle \in \mathrm{C}$, $\quad \ulcorner \overline{0} \rightarrow \varphi \urcorner$, *where* $\ulcorner \varphi \urcorner = w_1$
   (c) $\langle 2 \rangle \odot w_1 \in \mathrm{C}$ *and* $w_1 \odot \langle 2 \rangle \in \mathrm{C}$, $\quad \ulcorner \overline{1}_0 \rightarrow \varphi \urcorner$, *where* $\ulcorner \varphi \urcorner = w_1$
   (d) $\langle 3 \rangle \odot w_1 \in \mathrm{C}$ *and* $w_1 \odot \langle 3 \rangle \in \mathrm{C}$, $\quad \ulcorner p \rightarrow \varphi \urcorner$, *where* $\ulcorner \varphi \urcorner = w_1$

(e) $w_1 \odot w_2 \in \mathrm{C}$. $\ulcorner \varphi \to \psi \urcorner$, *where* $\ulcorner \varphi \urcorner = w_1$ *and* $\ulcorner \psi \urcorner = w_2$

Note that in the definition we can easily see which subset of $Fm_{\mathcal{L}}(\{p\})$ is actually encoded. We denote this subset as ${Fm_{\mathcal{L}}}^p$ (for example $p, p \to p \notin {Fm_{\mathcal{L}}}^p$). Moreover notice that we can choose the coding in such a way that for every proper code w we can unambiguously determine whether it is of the base form (i.e. exactly one of 1. (a), (b) or (c) from the previous definition) or a composed form. In the latter case, we can unambiguously determinate from which unique proper codes, and in which unique way, it has been produced.

Let $\mathbb{N}^+$ denote the set of all natural numbers without 0, i.e. $\mathbb{N}^+ = \mathbb{N}\setminus\{0\}$. Now we are ready to define the matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ and the logic $\mathrm{L} = \models_{\mathbf{A}}$.

Definition 3.21 (The logic L). *Suppose that all the elements in* C, $\mathbb{N}^+$ *and* $\{0, 1_0\}$ *are mutually distinct objects and denote* $\mathrm{D} = \{w \in \mathrm{C} \mid \exists w' \in \mathrm{C}\,(w = w' \odot w')\}$. *We define:*

$$A = \mathrm{C} \cup \mathbb{N} \cup \{1_0\} \text{ and } F = \mathrm{D} \cup \mathbb{N}^+ \cup \{1_0\}.$$

For every $n, n' \in \mathbb{N}^+$ *and every* $w, w' \in \mathrm{C}$ *the operations of the algebra* $\boldsymbol{A}$ *are given by the tables 1 and 2.*

	Operation $\to$
$\mathbb{N}^+$, $\mathbb{N}^+$	$n \to n' = n$ if $n = n'$ and $n \to n' = 0$ otherwise
$\mathbb{N}^+$, C	$n \to w = \langle 3\rangle \odot w$ and $w \to n = w \odot \langle 3\rangle$
$\mathbb{N}^+$, 0, 1_0	$n \to 0 = 0 \to n = 0$ and $n \to 1_0 = 1_0 \to n = 0$
C, C	$w \to w' = w \odot w'$
C, 0	$0 \to w = \langle 1\rangle \odot w$ and $w \to 0 = w \odot \langle 1\rangle$
C, 1_0	$1_0 \to w = \langle 2\rangle \odot w$ and $w \to 1_0 = w \odot \langle 2\rangle$
0, 1_0	$0 \to 0 = 1_0 \to 1_0 = 1_0$ and $1_0 \to 0 = 0 \to 1_0 = 0$

Table 1. Interpretation of the connective $\to$

	Operations B_i
$\mathbb{N}^+$	$B_i(n) = n$ if $f(n) \geq_{\mathrm{T}} f(i)$ and $B_i(n) = \langle i\rangle$ otherwise
C	$B_i(w) = \langle i\rangle * w$
0, 1_0	$B_i(0) = \langle i, 1\rangle$ and $B_i(1_0) = \langle i, 2\rangle$

Table 2. Interpretation of B_i's

Moreover for every $a \in A$*: if* $a \in \{1, 2, 3, 4\}$*, then* $B(a) = a$*, else* $B(a) = B_5(a)$*. The constant* $\overline{0}$ *is interpreted as* 0*. We define* L *as the logic of this matrix, i.e.* $\mathrm{L} = \models_{\mathbf{A}}$*.*

Observe that the set of 'true' codes D consists of codes of the form $\ulcorner \varphi \to \varphi \urcorner$ for some formula $\varphi \in Fm_{\mathcal{L}}{}^p$. In order to get better acquainted with this matrix, let us describe how certain formulas are evaluated. Consider a formula φ and an evaluation e. Then, if for every subformula $B_i(\chi)$ of φ we have $f(e(\chi)) \geq_{\mathrm{T}} f(i)$ (i.e. the node corresponding to $e(\chi)$ is above the node corresponding to i), then $e(\varphi) \in \mathbb{N} \cup \{1_0\}$; otherwise $e(\varphi) \in \mathrm{C}$.

Example 3.22. *Consider an evaluation e and a formula*

$$\varphi = B_9(q \to q) \to q$$

- *if* $e(q) = 14$*, then* $e(\varphi) = 14$*;*
- *if* $e(q) = 6$ *(or* $8, 13, 17, \ldots$*), then* $e(\varphi) = \langle 9 \rangle \odot \langle 3 \rangle$*, i.e. it is the code* $\ulcorner B_9(p) \to p \urcorner$*;*
- *if* $e(q) = 0$*, then* $e(\varphi) = \langle 9, 2 \rangle \odot \langle 1 \rangle$*, i.e. it is the code* $\ulcorner B_9(\overline{1}_0) \to \overline{0} \urcorner$*;*
- *if* $e(q) = w \in \mathrm{C}$*, then* $e(\varphi) = (\langle 9 \rangle * (w \odot w)) \odot w$*, i.e. it is the code* $\ulcorner B_9(\psi \to \psi) \to \psi \urcorner$ *where* $\psi \in Fm_{\mathcal{L}}{}^p$ *and* $\ulcorner \psi \urcorner = w$*.*

From now on we will only talk about codes (we will not be mentioning to which formulas from $Fm_{\mathcal{L}}{}^p$ they correspond to). The fact that L is a weakly implicative logic follows easily from the next observation (note that it relies on the fact that we included some codes, namely the set D, in the filter).

Observation 3.23. *For every* $a, b \in A$ *it holds:* $a \to b \in F$ *iff* $a = b$*.*

Corollary 3.24. L *is a weakly implicative logic and* $\langle \boldsymbol{A}, F \rangle$ *is reduced.*

3.3.2. Failure of the IPEP

In this section we prove that L does not satisfy the IPEP. The proof of this claim is divided into two main parts. The goal of each part is to prove a key property of the logic: *upward persistency* (Proposition 3.29) and *infimum property* (Proposition 3.38). To prove each one of these two properties we will need several lemmata first. Also during this section we will introduce many notational conventions which should help in reading the upcoming rather technical parts.

Convention 3.25. *Every formula mentioned in this section is assumed to contain only the variable p. Moreover whenever we mention evaluations e and e' without further specifying we assume the following: $e(p) \geq_{\mathbb{N}} 5$, $e'(p) \geq_{\mathbb{N}} 5$ and $f(e(p)) \leq_{\mathrm{T}} f(e'(p))$.*

According to this convention, e and e' are evaluations that both interpret p in the tree, in such a way that the interpretation given by e' is above that given by e. The upcoming series of lemmata can be viewed as some kind of *upward persistency* (see Lemma 3.29) in the tree.

Lemma 3.26. *For every formula φ and every evaluations e, e' it holds: if $e(\varphi) \geq_{\mathbb{N}} 1$, then $e(\varphi) = e(p)$ and $e'(\varphi) = e'(p)$.*

Proof. We prove it by induction over the complexity of φ. The base step where $\varphi = p$ (or $\varphi = \overline{0}$) is obvious. Induction step:

- If $\varphi = B_i(\psi)$ and $e(\varphi) \geq_{\mathbb{N}} 1$, then obviously $e(\psi) \geq_{\mathbb{N}} 1$ and, therefore, by induction assumption, $e(\psi) = e(p)$ and $e'(\psi) = e'(p)$. It is also clear that $f(i) \leq_{\mathrm{T}} f(e(\psi)) \leq_{\mathrm{T}} f(e'(\psi))$ (because $f(e(p)) \leq_{\mathrm{T}} f(e'(p))$). We can thus conclude that $e(\varphi) = e(\psi)$ and $e'(\varphi) = e'(\psi)$; the rest is easy.
- If $\varphi = B(\psi)$ and $e(\varphi) \geq_{\mathbb{N}} 1$, then $e(\psi) \geq_{\mathbb{N}} 1$ thus, by the induction assumption, $e'(\psi) = e'(p) \geq_{\mathbb{N}} 5$ and $e(\psi) = e(p) \geq_{\mathbb{N}} 5$. The rest follows because for any $n \in \mathbb{N}^+$ we have $B(n) = n$.
- Assume that $\varphi = \varphi_1 \to \varphi_2$ and $e(\varphi) \geq_{\mathbb{N}} 1$. Since $e(\varphi_1) = e(\varphi_2) \geq_{\mathbb{N}} 1$, we obtain the result simply from the induction assumption. □

Lemma 3.27. *For every formula φ and every evaluations e, e' it holds:*

1. *if $e(\varphi) = 0$, then $e'(\varphi) = 0$,*
2. *if $e(\varphi) = 1_0$, then $e'(\varphi) = 1_0$.*

Proof. We prove both cases simultaneously using induction over the complexity of the formula φ. The base step is again obvious.

- If $\varphi = B_i(\psi)$, it is trivial ($e(\varphi)$ can be neither 0 nor 1_0).
- If $\varphi = B(\psi)$, it is trivial for the same reasons.
- If $\varphi = \varphi_1 \to \varphi_2$ and $e(\varphi) = 0$, then we have the following possibilities:
  1. $e(\varphi_1) = 0$ and $e(\varphi_2) \geq_{\mathbb{N}} 1$ (or the other way around),
  2. $e(\varphi_1) = 1_0$ and $e(\varphi_2) \geq_{\mathbb{N}} 1$ (or the other way around),
  3. $e(\varphi_1) = 0$ and $e(\varphi_2) = 1_0$ (or the other way around),

4. $e(\varphi_1) \geq_{\mathbb{N}} 1$ and $e(\varphi_2) \geq_{\mathbb{N}} 1$.

Observe that the first three cases are easy to prove using the induction assumption and Lemma 3.26. The last case: again from Lemma 3.26 we get $e(\varphi_1) = e(\varphi_2) = e(p)$; thus, actually, $e(\varphi)$ cannot be 0.

- If $\varphi = \varphi_1 \to \varphi_2$ and $e(\varphi) = 1_0$, then both $e(\varphi_1)$ and $e(\varphi_2)$ are either 0 or 1_0. It follows from the induction assumption. □

LEMMA 3.28. *For every formulas φ and ψ, for every evaluations e, e' and for every $w \in \mathrm{C}$, it holds: if $e(\varphi) = e(\psi) = w$, then $e'(\varphi) = e'(\psi)$.*

PROOF. We prove it by induction over the complexity of the recursive set C. In the upcoming proof we will not deal with formulas of the form $B(\varphi')$ because the proof for these instances proceeds exactly in the same way as the proof for $B_i(\varphi')$.

1. (a) If $e(\varphi) = e(\psi) = \langle i \rangle$, then, obviously, $\varphi = B_i(\varphi')$ and $\psi = B_i(\psi')$ such that $e(\varphi') \not\geq_{\mathrm{T}} f(i)$ and $e(\psi') \not\geq_{\mathrm{T}} f(i)$, but $e(\varphi') \geq_{\mathbb{N}} 1$ and $e(\psi') \geq_{\mathbb{N}} 1$. We use Lemma 3.26 to derive: $e'(\varphi') = e'(\psi') = e'(p)$. The rest is straightforward.
   (b) If $e(\varphi) = e(\psi) = \langle i, 1 \rangle$, then $\varphi = B_i(\varphi')$, $\psi = B_i(\psi')$ and $e(\varphi') = e(\psi') = 0$. The rest is an easy consequence of Lemma 3.27.
   (c) If $e(\varphi) = e(\psi) = \langle i, 2 \rangle$, we do it similarly.
2. (a) If $e(\varphi) = e(\psi) = \langle i \rangle * w$, then $\varphi = B_i(\varphi')$, $\psi = B_i(\psi')$ and $e(\varphi') = e(\psi') = w$. The rest follows from the induction assumption.
   (b) If $e(\varphi) = e(\psi) = \langle 1 \rangle \odot w$, then $\varphi = \varphi_1 \to \varphi_2$, $\psi = \psi_1 \to \psi_2$ and $e(\varphi_1) = e(\psi_1) = 0$ and $e(\varphi_2) = e(\psi_2) = w$. The rest follows from the induction assumption. (The same proof applies to $w \odot \langle 1 \rangle$).
   (c) If $e(\varphi) = e(\psi) = \langle 2 \rangle \odot w$, we do it similarly.
   (d) If $e(\varphi) = e(\psi) = \langle 3 \rangle \odot w$, then $\varphi = \varphi_1 \to \varphi_2$, $\psi = \psi_1 \to \psi_2$ and $e(\varphi_1) = e(\psi_1) \geq_{\mathbb{N}} 1$ and $e(\varphi_2) = e(\psi_2) = w$. We apply Lemma 3.26 and the induction assumption.
   (e) If $e(\varphi) = e(\psi) = w_1 \odot w_2$, we do it similarly. □

These lemmata allow us to obtain the first of the two ingredients we need for disproving the IPEP:

PROPOSITION 3.29 (Upward persistency). *For every formula φ and any evaluations e, e', it holds: if $e(\varphi) \in F$, then $e'(\varphi) \in F$.*

Proof. Let us consider three possible cases. First, if $e(\varphi) \geq_{\mathbb{N}} 1$, it follows from Lemma 3.26. Second, if $e(\varphi) = 1_0$, it follows by Lemma 3.27. Finally, if $e(\varphi) = w \odot w$ for some $w \in \mathrm{C}$, it follows that $\varphi = \varphi_1 \to \varphi_2$ and $e(\varphi_1) = e(\varphi_2) = w$; then we just apply Lemma 3.28. □

Next we focus on the *infimum property*. We start with some notational conventions and shortcuts.

Convention 3.30. *We use variables $n, n', n_1 \ldots$ for members of $\mathbb{N}^+$ and variables $s, s', s_1 \ldots$ for elements of T. Moreover we always consider variables for natural numbers and for elements of the tree connected if they are of the same form (e.g. when we use variables n' and s' we suppose $f(n') = s'$).*

Convention 3.31. *For a formula φ, we denote by φ^s the value of φ under an evaluation e such that $e(p) = n \in \mathbb{N}^+$ (i.e. $\varphi^s = e(\varphi)$, where $e(p) = n$ and, by the previous convention, $f(n) = s$). We also write $\varphi =_s \psi$ meaning that $\varphi^s = \psi^s$.*

Realize that thanks to these shortcuts we need not mention the enumeration function f and the evaluations e. Recall Convention 3.25 and note that, if we write φ^s, then we are again speaking only of evaluations e such that $e(p) \geq_{\mathbb{N}} 5$ (it follows from the definition of f).

Now we need to introduce several technical lemmata:

Lemma 3.32. *For every formula φ and any s_1 and s_2 such that $s_1 \leq_{\mathrm{T}} s_2$, it holds: if $\varphi^{s_1} = w$ for some $w \in \mathrm{C}$ and $\varphi^{s_2} \neq \varphi^{s_1}$, then there is a subformula $B_n(\psi)$ of φ such that $s_1 <_{\mathrm{T}} s \leq_{\mathrm{T}} s_2$.*

Proof. Suppose we are given s_1 and s_2 satisfying the conditions of this lemma. We prove the conclusion by induction over the complexity of φ. The base step holds trivially. For the induction step we consider the following cases:

- $\varphi = B_{n'}(\psi)$ and suppose $\varphi^{s_1} = w$ for some $w \in \mathrm{C}$ and $\varphi^{s_2} \neq \varphi^{s_1}$. Then either also ψ satisfies the conditions of this lemma and we are done by the induction assumption or $\psi^{s_1} \geq_{\mathbb{N}} 1$. From Lemma 3.26 we get $\psi^{s_1} = n_1$ and $\psi^{s_2} = n_2$. Finally, since $\varphi^{s_2} \neq \varphi^{s_1} = \langle n' \rangle$, we conclude that $s_1 <_{\mathrm{T}} s' \leq_{\mathrm{T}} s_2$.
- $\varphi = B(\psi)$ and suppose $\varphi^{s_1} = w$ for some $w \in \mathrm{C}$ and $\varphi^{s_2} \neq \varphi^{s_1}$. This case is very similar to the previous one; the only difference is that in this case the second possibility cannot happen.

- $\varphi = \varphi_1 \to \varphi_2$ and suppose $\varphi^{s_1} = w$ for some $w \in \mathrm{C}$ and $\varphi^{s_2} \neq \varphi^{s_1}$. There are many cases to discuss (based on the form of the code w - Definition 3.20), however all of them are easy to check (using Lemmas 3.26 and 3.27 and the induction assumption). For example $w = \langle 1 \rangle \odot w'$ for some $w' \in \mathrm{C}$, meaning that ${\varphi_1}^{s_1} = 0$ and ${\varphi_2}^{s_1} = w'$. By Lemma 3.27 ${\varphi_1}^{s_1} = {\varphi_1}^{s_2}$, therefore, since, $\varphi^{s_1} \neq \varphi^{s_2}$, we obtain ${\varphi_2}^{s_1} \neq {\varphi_2}^{s_2}$; the rest follows by the induction applied to φ_2. □

LEMMA 3.33. *If $B_n(\psi)$ is a subformula of φ and $s' \ngeq_{\mathrm{T}} s$, then $\varphi^{s'} = w$ for some $w \in \mathrm{C}$.*

PROOF. Because of Lemma 3.26, we know that $\psi^{s'}$ has one of these values: $n', 0, 1_0$, or w for some $w \in \mathrm{C}$ (thanks to the lemma we have excluded all other natural numbers). In all these cases we get $B_n(\psi)^{s'} = w'$ for some $w' \in \mathrm{C}$. The rest is easy (cf. comments right below the definition of the logic L). □

The next auxiliary lemma shows the relation between the presence of $\overline{0}$ in φ and certain values of φ^s.

LEMMA 3.34. *For every formula φ and any node s: $\overline{0}$ is a subformula of φ iff $\varphi^s \in \{0, 1_0\}$ or $\varphi^s = w$ for some $w = \langle a_1, \dots, a_n \rangle$ such that there is $1 \leq_{\mathbb{N}} i \leq_{\mathbb{N}} n$ such that $a_i \in \{1, 2\}$.*

PROOF. We prove it by induction over the complexity of φ. The base step is easy. Now let us write Prop0(φ) if $\varphi^s \in \{0, 1_0\}$ or $\varphi^s = w$ for some $w = \langle a_1, \dots, a_n \rangle$ such that there is $1 \leq_{\mathbb{N}} i \leq_{\mathbb{N}} n$ and $a_i \in \{1, 2\}$.

- If $\varphi = B_i(\psi)$, it is easy: $\overline{0}$ is a subformula of φ iff it is a subformula of ψ iff (by the induction assumption) Prop0(ψ) iff Prop0(φ).
- If $\varphi = B(\psi)$, it works similarly.
- Assume that $\varphi = \varphi_1 \to \varphi_2$. Then: $\overline{0}$ is a subformula of φ iff $\overline{0}$ is a subformula of φ_1 or φ_2 iff Prop0(φ_1) or Prop0(φ_2) iff Prop0(φ). □

LEMMA 3.35. *Let φ be a formula and take any $s_1, s_2 \in \mathrm{T}$. Suppose that $s = \inf\{s_1, s_2\}$.*[7] *Then:*

1. *if $\varphi^{s_1} \geq_{\mathbb{N}} 1$ and $\varphi^{s_2} \geq_{\mathbb{N}} 1$, then $\varphi^s \geq_{\mathbb{N}} 1$,*
2. *if $\varphi^{s_1} = \varphi^{s_2} = 0$ (resp. $\varphi^{s_1} = \varphi^{s_2} = 1_0$), then $\varphi^s = 0$ (resp. $\varphi^s = 1_0$),*

[7] $\inf\{s_1, s_2\}$ is the infimum of s_1 and s_2 w.r.t. $\leq_{\mathrm{T}}$. Note that it always exists.

3. *moreover any other combination of these values is not possible, i.e. the following cannot happen:*
   - $\varphi^{s_1} \geq_{\mathbb{N}} 1$ *and* $\varphi^{s_2} \in \{0, 1_0\}$,
   - $\varphi^{s_1} = 0$ *and* $\varphi^{s_2} = 1_0$.

PROOF. 1. By the way of contradiction suppose that $\varphi^{s_1} \geq_{\mathbb{N}} 1$ and $\varphi^{s_2} \geq_{\mathbb{N}} 1$ and $\varphi^{s} \ngeq_{\mathbb{N}} 1$. Therefore, $\varphi^s \in \{0, 1_0\}$ or $\varphi^s = w$ for some $w \in \mathrm{C}$, but the first possibility is not true because of Lemma 3.27, thus $\varphi^s = w$ for some $w \in \mathrm{C}$. Now we use twice Lemma 3.32 to obtain two subformulas of φ: $B_{n'_1}(\psi_1)$ and $B_{n'_2}(\psi_2)$ such that $s <_{\mathrm{T}} s'_1 \leq_{\mathrm{T}} s_1$ and $s <_{\mathrm{T}} s'_2 \leq_{\mathrm{T}} s_2$. Since s is the infimum of s_1, s_2, we obtain that s'_1 and s'_2 are $\leq_{\mathrm{T}}$-incomparable. Thus we obtain a contradiction from Lemma 3.33.

2. It follows by an analogous argument.

3. First point: we argue using Lemma 3.34. If $\varphi^{s_2} \in \{0, 1_0\}$ we obtain that $\overline{0}$ is a subformula of φ and thus $\varphi^{s_1} \ngeq_{\mathbb{N}} 1$.
   Second point: we prove it by induction over the complexity of φ. The base step is immediate.
   - If $\varphi = B_i(\psi)$ (or $= B(\psi)$) then we are done (φ^s can be neither 0 nor 1_0 for any s).
   - $\varphi = \varphi_1 \to \varphi_2$ and suppose $\varphi^{s_1} = 0$. Now there are two possibilities. First: ${\varphi_1}^{s_1} = 0$ and ${\varphi_2}^{s_1} \geq_{\mathbb{N}} 1$. From the induction assumption (and from the previous point) we get: ${\varphi_1}^{s_2} = w$ for some $w \in \mathrm{C}$ (or ${\varphi_1}^{s_2} = 0$). The case $= w$ is obvious. In the other in order to obtain $\varphi^{s_2} = 1_0$ we would need ${\varphi_2}^{s_2} = 0$, but it is not possible, by the previous point (because ${\varphi_2}^{s_1} \geq_{\mathbb{N}} 1$). Second: ${\varphi_1}^{s_1} = 0$ and ${\varphi_2}^{s_1} = 1_0$. From the induction assumption (and from the previous point) we obtain: ${\varphi_1}^{s_2} = w$ (or ${\varphi_1}^{s_2} = 0$) and ${\varphi_2}^{s_2} = w'$ (or ${\varphi_2}^{s_2} = 1_0$). However, in none of these cases we get $\varphi^{s_2} = 1_0$. □

LEMMA 3.36. *For every s, s' and every formulas φ, ψ: if $\varphi^s = 0$ and $\varphi^{s'} = \psi^{s'} = 0$ (or $\varphi^{s'} = \psi^{s'} = w$ for some $w \in \mathrm{C}$), then $\psi^s \neq 1_0$.*

PROOF. We prove it by induction over the complexity of formulas φ and ψ.

- $\varphi = p$: trivial.
- $\varphi = \overline{0}$: if we have $\varphi^{s'} = 0 = \psi^{s'}$, then we use Lemma 3.35 to conclude that $\psi^s \neq 1_0$.
- $\varphi = B_i(\varphi')$ (or $\varphi = B(\varphi')$): trivial.

- $\varphi = \varphi_1 \to \varphi_2$: we discuss two cases. First: if $\varphi^{s'} = 0 = \psi^{s'}$ then, we can again easily use Lemma 3.35 to obtain the conclusion. Second, if $\varphi^{s'} = w = \psi^{s'}$, we have that $\psi = \psi_1 \to \psi_2$. We must again deal with two possibilities:
  1. ${\varphi_1}^s = 0$ and ${\varphi_2}^s = n$. We argue that ${\psi_2}^s = n$ or ${\psi_2}^s = w'$ for some $w' \in \mathrm{C}$. Suppose not, i.e. ${\psi_2}^s = 0$ (or $= 1_0$), by Lemma 3.34 we can conclude that $\overline{0}$ is a subformula also of φ_2 and thus again, by the same lemma, ${\varphi_2}^s \neq n$, a contradiction. From this we infer that it cannot be the case that $\psi^s = 1_0$.
  2. ${\varphi_1}^s = 0$ and ${\varphi_2}^s = 1_0$. Suppose for contradiction that $\psi^s = 1_0$, i.e. ${\psi_1}^s = {\psi_2}^s = 0$ (or both are equal to 1_0). Since the preconditions of this lemma are satisfied for φ_2 and ψ_2 (because, by Lemma 3.35, neither $\psi_2^{s'} \geq_{\mathbb{N}} 1$ nor $\psi_2^{s'} = 1_0$), we obtain by the induction assumption that ${\varphi_2}^s \neq 1_0$, contradiction (the other case is similar). □

Now we have everything set up to prove the most important lemma of this section:

LEMMA 3.37. *For every formula φ, ψ and every s_1, s_2, it holds: if $\varphi =_{s_1} \psi$ and $\varphi =_{s_2} \psi$, then $\varphi =_s \psi$, where $s = \inf\{s_1, s_2\}$.*

PROOF. We prove it by induction over the complexity of φ and ψ.

- $\varphi = p$. It must hold that $\psi^{s_1} = n_1$ and $\psi^{s_2} = n_2$ and, by Lemma 3.35, we get $\psi^s = n$. Thus we have verified $\varphi =_s \psi$.
- $\varphi = \overline{0}$. It must hold that $\psi^{s_1} = 0$ and $\psi^{s_2} = 0$. Again, by Lemma 3.35, we get $\psi^s = 0$. Again we can easily conclude $\varphi =_s \psi$.
- $\varphi = B_i(\varphi')$. Let us inspect what values φ can take. Note that for any s, φ^s can be neither 0 nor 1_0. Therefore, we have the following possibilities:[8]

cases	value of φ^{s_1}	value of φ^{s_2}
(i)	n_1	n_2
(ii)	w	n_2
(iii)	n_1	w
(iv)	w_1	w_2

(i) From Lemma 3.35 we get $\varphi^s = n$ and, since also $\psi^{s_1} = n_1$ and $\psi^{s_2} = n_2$, we can use again Lemma 3.35 and get $\psi^s = n$, i.e. $\varphi =_s \psi$.

[8]Note that, thanks to Lemma 3.26, we know that n is the only natural number that can be the value of φ^s.

(ii) Since $\psi^{s_1} = w$, we can infer that $\psi = B_i(\psi')$ for some ψ'. Now it is not difficult to show that $\varphi' =_{s_1} \psi'$ and $\varphi' =_{s_2} \psi'$. Therefore, by the induction assumption, we get $\varphi' =_s \psi'$, thus clearly also $\varphi =_s \psi$. (Note that in case of $i = 5$ it can also happen $\psi = B(\psi')$, which however behaves in a similar way).

(iii) Identical.

(iv) Similar to (ii).

- $\varphi = B(\varphi')$: almost the same.
- $\varphi = \varphi' \to \varphi''$. Here we need to discuss even more cases (realize that we have already rejected a few possibilities in Lemma 3.35):[9]

cases	value of φ^{s_1}	value of φ^{s_2}
(i)	n_1	n_2
(ii)	n_1	w
(iii)	0	0
(iv)	0	w
(v)	1_0	1_0
(vi)	1_0	w
(vii)	w_1	w_2

(i) Again, easily using Lemma 3.35, we obtain $\varphi^s = n$ and $\psi^s = n$; in other words, $\varphi =_s \psi$.

(ii) Since $\varphi^{s_2} = w$ and $\varphi = \varphi' \to \varphi''$, we infer that also $\psi = \psi' \to \psi''$. We can argue that $\varphi'^{s_1} = n_1 = \psi'^{s_1}$, equivalently $\varphi' =_{s_1} \psi'$, and in the same way we get $\varphi'' =_{s_1} \psi''$. We can also derive $\varphi' =_{s_2} \psi'$ and $\varphi'' =_{s_2} \psi''$: we need to distinguish cases based on the code w, whether it is: $w_1 \odot w_2$, $\langle 1\rangle \odot w_1$, $\langle 2\rangle \odot w_1$ or $\langle 3\rangle \odot w_1$ (or some of its symmetric variants). All these cases are easy to check. Now we can apply the induction assumption and obtain $\varphi' =_s \psi'$ and $\varphi'' =_s \psi''$ and, hence, conclude $\varphi =_s \psi$.

(iii) As in (i), it follows easily from Lemma 3.35.

(iv) First, since $\varphi^{s_2} = w$, we infer $\psi = \psi' \to \psi''$. Now we need to cover two cases based on φ^{s_1}.

1. $\varphi'^{s_1} = 0$ and $\varphi''^{s_1} = n_1$: we argue that also $\psi'^{s_1} = 0$ and $\psi''^{s_1} = n_1$. First, if $\psi''^{s_1} = 0$ (or $= 1_0$), then, by Lemma 3.34, we would get that $\overline{0}$ is a subformula of ψ'' and, since $\psi''^{s_2} = \varphi''^{s_2}$, we

[9] This time we do not mention symmetric cases.

would obtain by the same lemma that $\overline{0}$ is also a subformula of φ''. Therefore again, by Lemma 3.34, we know $\psi''^{s_1} = n_1$. By Lemma 3.36, we get that $\psi'^{s_1} \neq 1_0$ thus it must be the case that $\psi'^{s_1} = 0$. Thus we can conclude $\varphi' =_{s_1} \psi'$ and $\varphi'' =_{s_1} \psi''$. It is easy to derive that $\varphi' =_{s_2} \psi'$ and $\varphi'' =_{s_2} \psi''$. The rest is an easy consequence of the induction assumption.

2. $\varphi'^{s_1} = 0$ and $\varphi''^{s_1} = 1_0$. If we show that also $\psi'^{s_1} = 0$ and $\psi''^{s_1} = 1_0$, we are done simply by using the induction assumption. For contradiction suppose it is not the case, i.e. $\psi'^{s_1} = 1_0$ and $\psi''^{s_1} = 0$ (note that using Lemma 3.34, as in the previous point, we have $\psi'^{s_1} \neq n_1$ and $\psi''^{s_1} \neq n_1$). We get a contradiction from Lemma 3.36 applied on ψ' and φ'.

(v) It is similar to (i) and (iii).

(vi) Again we first argue that also $\psi = \psi' \to \psi''$ (because $\psi^{s_2} = w$). Suppose $\varphi'^{s_1} = \varphi''^{s_1} = 0$. Again it is enough to argue that also $\psi'^{s_1} = \psi''^{s_1} = 0$. This is easy to prove, just realize that the only other possibility would be $\psi'^{s_1} = \psi''^{s_1} = 1_0$, which is by Lemma 3.36 not possible.

(vii) This case is a straightforward application of the induction assumption. □

Finally we are ready to obtain the second key component, the *infimum property*:

PROPOSITION 3.38 (Infimum property). *For every formula φ and every s_1, s_2, it holds: if $\varphi^{s_1} \in F$ and $\varphi^{s_2} \in F$, then also $\varphi^s \in F$, where $s = \inf\{s_1, s_2\}$.*

PROOF. For a contradiction suppose that $\varphi^s \notin F$ and both $\varphi^{s_1} \in F$ and $\varphi^{s_2} \in F$. First we use Lemma 3.27 to argue that $\varphi^s \neq 0$ which implies $\varphi^s = w$ for some $w \in \mathrm{C} \setminus \mathrm{D}$. Since $\varphi^s \neq \varphi^{s_1}$ and $\varphi^s \neq \varphi^{s_2}$, we can use Lemma 3.32 to infer that there are nodes s_1' and s_2' such that $s <_{\mathrm{T}} s_1' \leq_{\mathrm{T}} s_1$ and $s <_{\mathrm{T}} s_2' \leq_{\mathrm{T}} s_2$ and subformulas $B_{n_1'}(\psi_1)$ and $B_{n_2'}(\psi_2)$ of the formula φ . Then, since $s = \inf\{s_1, s_2\}$, we obtain that $s_1' \not\leq_{\mathrm{T}} s_2$ and $s_2' \not\leq_{\mathrm{T}} s_1$, therefore by Lemma 3.33 it follows $\varphi^{s_1} = w_1 \in \mathrm{D}$ and $\varphi^{s_2} = w_2 \in \mathrm{D}$. Thus it follows that $w_1 = w_1' \odot w_1'$ and $w_2 = w_2' \odot w_2'$ for some $w_1', w_2' \in \mathrm{C}$. We can now easily conclude that $\varphi = \varphi_1 \to \varphi_2$ and $\varphi_1 =_{s_1} \varphi_2$ and $\varphi_1 =_{s_2} \varphi_2$ thus by Lemma 3.37 also $\varphi_1 =_s \varphi_2$, contradiction (Observation 3.23). □

Now to disprove the IPEP we need to describe a suitable set of formulas Γ_0 and a formula φ such that $\Gamma_0 \nvdash_{\mathrm{L}} \varphi$ and for every theory $T \supseteq \Gamma_0$: if

$T \nvdash_{\mathrm{L}} \varphi$, then T is not finitely $\cap$-irreducible, i.e. there are two theories T_1 and T_2 strictly containing T such that $T = T_1 \cap T_2$.

DEFINITION 3.39 (Γ_0). *Let us enumerate all propositional variables. We then define* $\Gamma_0 = \{p_i \to p_j \mid i, j \in \mathbb{N}\} \cup \{B_5(p_1)\}$.

THEOREM 3.40. *The logic* L *does not satisfy the* IPEP.

PROOF. We will denote as e_a the evaluation that sends every variable to a fixed element $a \in A$. First observe that $\Gamma_0 \nvdash_{\mathrm{L}} \overline{0}$, which can be stated as follows: there is an evaluation e which satisfies Γ_0, i.e. $e[\Gamma_0] \subseteq F$. Moreover all the evaluations that satisfy Γ_0 are exactly of the form e_n for some $5 \leq_{\mathbb{N}} n \in A$. Next let us have a theory T containing Γ_0 such that $T \nvdash_{\mathrm{L}} \overline{0}$. Consider the set $\mathrm{Sat}(T) = \{a \in A \mid e_a[T] \subseteq F\}$. Note that $\mathrm{Sat}(T) \subseteq \mathbb{N}^+ \setminus \{1, 2, 3, 4\}$ and it contains all the evaluations satisfying T. Further define the set $\mathrm{TreeSat}(T) = \{f(n) \in \mathrm{T} \mid n \in \mathrm{Sat}(T)\}$.

We show that there is a $\leq_{\mathrm{T}}$-least element in $\mathrm{TreeSat}(T)$. $\mathbb{N}$ is well-ordered by $\leq_{\mathbb{N}}$, so let n_0 be the least element in $\mathrm{Sat}(T)$ and s_0 be its corresponding node in T. If s_0 was not the least element in $\mathrm{TreeSat}(T)$, then there would be some $s \in \mathrm{TreeSat}(T)$ such that $s_0 \not\leq_{\mathrm{T}} s$. Take $s' = \inf\{s_0, s\}$. The node s' is obviously in a lower layer (in the tree) than s_0, thus $f^{-1}(s') = n' <_{\mathbb{N}} n_0$. However, from the *infimum property* (Proposition 3.38), we know that $s' \in \mathrm{TreeSat}(T)$; indeed, let σ be the substitution such that $\sigma(q) = p$ for every $q \in \mathit{Var}$; since for every node s and any formula φ we have $e_n(\varphi) = \sigma(\varphi)^s$, we can conclude (by the infimum property) that $\sigma(\varphi)^{s'} \in F$ for every $\varphi \in T$ (because both $\sigma(\varphi)^{s_0} \in F$ and $\sigma(\varphi)^s \in F$). In particular $n' \in \mathrm{Sat}(T)$ – a contradiction with the fact that n_0 is the least element in $\mathrm{Sat}(T)$.

Now let s_1, s_2 be the two distinct immediate successors of s_0. Obviously, $T \nvdash_{\mathrm{L}} B_{n_1}(p_1)$ and $T \nvdash_{\mathrm{L}} B_{n_2}(p_1)$ (this fact is witnessed by the evaluations e_{n_1} and e_{n_2}). Therefore, both $T_1 = \mathrm{Th}_{\mathrm{L}}(T \cup \{B_{n_1}(p)\})$ and $T_2 = \mathrm{Th}_{\mathrm{L}}(T \cup \{B_{n_2}(p)\})$ strictly contain T. Further we verify that for every formula φ:

$$\text{(1)} \qquad \text{if } T_1 \vdash_{\mathrm{L}} \varphi \text{ and } T_2 \vdash_{\mathrm{L}} \varphi, \text{ then } T \vdash_{\mathrm{L}} \varphi.$$

Suppose $T_1 \vdash_{\mathrm{L}} \varphi$ and $T_2 \vdash_{\mathrm{L}} \varphi$ and observe that we only need to show that $e_{n_0}(\varphi) \in F$ (this follows, again using the substitution σ, from the *upward persistency* (Proposition 3.29) and the fact that n_0 is the least element in $\mathrm{Sat}(T)$). Easily using σ and the *upward persistency* we infer $\sigma(\psi)^{s_1} \in F$ for every $\psi \in T_1$ which implies $\sigma(\varphi)^{s_1} \in F$; analogously we obtain $\sigma(\varphi)^{s_2} \in F$. The rest is an easy consequence of the *infimum property.*

In particular, the fact (1) tells us that the theory T is intersection-reducible ($T = T_1 \cap T_2$) which is exactly what we wanted. □

3.3.3. Proof of RSI-completeness

Finally we prove that L is RSI-complete. From Corollary 3.24 we know that $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ is reduced. Moreover, by definition, $\mathbf{A}$ is complete semantics for L. Therefore to prove RSI-completeness, it is enough to show that $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$. Let us now prove that F is $\cap$-irreducible in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$. To obtain this result we recursively define for every $w \in \mathrm{C}$ a corresponding formula φ_w written in a fixed variable p:

Definition 3.41 (Formulas φ_w). *We define formulas φ_w recursively as follows:*

1. (a) $\varphi_{\langle i\rangle} = B_i(B(p))$,
   (b) $\varphi_{\langle i,1\rangle} = B_i(0 \to B(p))$,
   (c) $\varphi_{\langle i,2\rangle} = B_i(\overline{0} \to (\overline{0} \to B(p)))$,
2. (a) $\varphi_{\langle i\rangle * w} = B_i(\varphi_w)$,
   (b) $\varphi_{\langle 1\rangle \odot w} = \overline{0} \to \varphi_w$ *and* $\varphi_{w\odot\langle 1\rangle} = \varphi_w \to \overline{0}$,
   (c) $\varphi_{\langle 2\rangle \odot w} = \overline{1} \to \varphi_w$ *and* $\varphi_{w\odot\langle 2\rangle} = \varphi_w \to \overline{1}$,
   (d) $\varphi_{\langle 3\rangle \odot w} = p \to \varphi_w$ *and* $\varphi_{w\odot\langle 3\rangle} = \varphi_w \to p$,
   (e) $\varphi_{w\odot w'} = \varphi_w \to \varphi_{w'}$.

Observation 3.42. *For every $w \in \mathrm{C}$ and every evaluation e such that $e(p) \in \{1, 2, 3, 4\}$, it holds that $e(\varphi_w) = w$.*

Proof. Easy induction over the complexity of the recursively defined set C. □

Note that in the previous proof we benefited from the connective B, namely from the fact that for any $i \in \{1, 2, 3, 4\}$ we have $B(i) = i$. In upcoming proofs we will usually tacitly use the next obvious statement:

Observation 3.43. *Let us have an evaluation e such that $e(p) \notin \mathbb{N}^+$. Then, for every $w \in \mathrm{C}$, it holds that $e(\varphi_w) \in \mathrm{C}$.*

Proof. Again an easy induction. □

Lemma 3.44. *For every $w \in \mathrm{C}$ and for every evaluation e we have:*

1. if $e(p) = 0$, then $e(\varphi_w) \neq \langle 5, 1\rangle$, $e(\varphi_w) \neq \langle 1\rangle \odot \langle 5, 1\rangle$ and $e(\varphi_w) \neq \langle 1\rangle \odot (\langle 1\rangle \odot \langle 5, 1\rangle)$.
2. if $e(p) = 1_0$, then $e(\varphi_w) \neq \langle 5, 2\rangle$, $e(\varphi_w) \neq \langle 1\rangle \odot \langle 5, 2\rangle$ and $e(\varphi_w) \neq \langle 1\rangle \odot (\langle 1\rangle \odot \langle 5, 2\rangle)$.

3. if $e(p) = w'$ for some $w' \in \mathrm{C}$, then $e(\varphi_w) \neq \langle 5\rangle * w'$, $e(\varphi_w) \neq \langle 1\rangle \odot (\langle 5\rangle * w')$ and $e(\varphi_w) \neq \langle 1\rangle \odot (\langle 1\rangle \odot (\langle 5\rangle * w'))$.

Proof. We prove only the first point; the others are even simpler. First, $e(\varphi_w) \neq \langle 5, 1\rangle$: obvious. Second, $e(\varphi_w) \neq \langle 1\rangle \odot \langle 5, 1\rangle$: if $w = \langle 3\rangle \odot w'$, then $e(\varphi_w) = \langle 1\rangle \odot e(\varphi_{w'})$, but from the first inequation we obtain the result; the same argument works for $w = \langle 1\rangle \odot w'$. For other codes it is obvious. Third, $e(\varphi_w) \neq \langle 1\rangle \odot (\langle 5, 1\rangle \odot \langle 1\rangle)$: similar but using the second inequation. □

Lemma 3.45. *For every $w_1, w_2 \in \mathrm{C}$ and for any evaluation e such that $e(p) = 0$ or $e(p) = 1_0$ or $e(p) = w$ for some $w \in \mathrm{C}$, we have: $e(\varphi_{w_1}) = e(\varphi_{w_2})$ iff $w_1 = w_2$.*

Proof. We prove this lemma only for evaluations e such that $e(p) = 0$, the other cases follow almost in the same way (they are only using different points from the previous lemma). This lemma is then proved by induction over the complexity of w_1, w_2 according to Definition 3.20:

1. (a) $w_1 = \langle i\rangle$: we get $e(\varphi_{w_1}) = \langle i, 5, 1\rangle$. Now it is easy to see that the conclusion follows for base step for w_2 (i.e. points 1.(a),(b),(c)). Moreover, for the induction step there is only one more complicated variant, namely (a) (for the rest obviously $e(\varphi_{w_1}) \neq e(\varphi_{w_2})$, because the value $e(\varphi_{w_2})$ is obtained by applying the operation $\odot$, therefore it starts with number 4): suppose $w_2 = \langle i\rangle * w$ for some $w \in \mathrm{C}$, but, by Lemma 3.44, $e(\varphi_w) \neq \langle 5, 1\rangle$, i.e. $e(\varphi_{w_1}) \neq e(\varphi_{w_2})$.
   (b) $w_1 = \langle i, 1\rangle$: $e(\varphi_{w_1}) = \langle i\rangle * (\langle 1\rangle \odot \langle 5, 1\rangle)$; we again deal only with the case 2.(a). So let $w_2 = \langle i\rangle * w$ for some $w \in \mathrm{C}$. However, by Lemma 3.44, $e(\varphi_w) \neq \langle 1\rangle \odot \langle 5, 1\rangle$ and therefore $e(\varphi_{w_1}) \neq e(\varphi_{w_2})$.
   (c) $w_1 = \langle i, 2\rangle$: $e(\varphi_{w_1}) = \langle i\rangle * (\langle 1\rangle \odot (\langle 1\rangle \odot \langle 5, 1\rangle))$. Again 2.(a): $w_2 = \langle i\rangle * w$ for some $w \in \mathrm{C}$. However, by Lemma 3.44, $e(\varphi_w) \neq \langle 1\rangle \odot (\langle 1\rangle \odot \langle 5, 1\rangle)$
2. (a) $w_1 = \langle i\rangle * w_1'$: $e(\varphi_{w_1}) = \langle i\rangle * e(\varphi_{w_1'})$ the base step for w_2 follows by the first part of this proof. Case 2.(a): $w_2 = \langle i\rangle * w_2'$ and $e(\varphi_{w_2}) = \langle i\rangle * e(\varphi_{w_2'})$. We obtain the result easily from the induction assumption. For the other cases we trivially get $e(\varphi_{w_1}) \neq e(\varphi_{w_2})$.
   (b) $w_1 = \langle 1\rangle \odot w_1'$: $e(\varphi_{w_1}) = \langle 1\rangle \odot e(\varphi_{w_1'})$. The base step for w_2 is trivial. Moreover the only interesting induction cases for w_2 are 2.(b) and 2.(d)-which are treated in the same way: we obtain $w_2 = \langle 1\rangle \odot w_2'$ and $e(\varphi_{w_2}) = \langle 1\rangle \odot e(\varphi_{w_2'})$ and apply the induction assumption. In the remaining cases we can easily conclude $e(\varphi_{w_1}) \neq e(\varphi_{w_2})$.

(c),(d),(e) Similar. □

For the next lemma we assume that the set C is ordered by $\leq_{\mathrm{C}}$ in this way: $w \leq_{\mathrm{C}} w'$ iff the natural number corresponding to w is smaller or equal to the one corresponding to w'. We then prove:

LEMMA 3.46. *For every $w, w' \in \mathrm{C}$ and every evaluation e sending p to w', we have $w' <_{\mathrm{C}} e(\varphi_w)$ and thus in particular $e(\varphi_w) \neq w'$.*

PROOF. Straightforward induction over the complexity of φ_w.[10] □

The last two lemmata are summarized in the following corollary:

COROLLARY 3.47. *For every $w \in \mathrm{C} \setminus D$: $\varphi_w \vdash_{\mathrm{L}} B_5(p)$.*

PROOF. It is enough to show that for any evaluation e such that $e(p) \in \mathrm{C} \cup \{1, 2, 3, 4, 0, 1_0\}$ we have $e(\varphi_w) \notin F$. If $e(p) \in \{1, 2, 3, 4\}$ we argue using observation 3.42. For other evaluations we distinguish two possible scenarios in which we could get $e(\varphi_w) \in F$. First, if $w = \langle 3 \rangle \odot w'$, we get $e(\varphi_w) = e(p) \to^{\mathbf{A}} e(\varphi_{w'})$. If $e(p) = 0$ or $e(p) = 1_0$, obviously $e(\varphi_w) \notin F$ and, if $e(p) = w''$, we conclude $e(\varphi_w) \notin F$ by Lemma 3.46. Second, assume that $w = w_1 \odot w_2$. Since $w_1 \neq w_2$, we can use Lemma 3.45 to obtain $e(\varphi_w) \notin F$. □

THEOREM 3.48. *The logic* L *is* RSI-*complete.*

PROOF. We show that F is $\cap$-irreducible in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$. Let us consider a non-trivial $F' \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ which strictly contains F. First observe that $0 \notin F'$ (because $\overline{0} \vdash_{\mathrm{L}} p$, which would imply that F' is trivial). It follows that there is some $w \in \mathrm{C} \setminus D$ which is also in F'. However, from the previous corollary, we know that $\varphi_w \vdash_{\mathrm{L}} B_5(p)$; thus, if we consider an evaluation e such that $e(p) = 1$, we obtain, by Observation 3.42, that $e(\varphi_w) = w \in F'$, which implies $\langle 5 \rangle \in F'$. Therefore any non-trivial filter strictly above F contains $\langle 5 \rangle$. In other words: F is $\cap$-irreducible. □

Hence, we have successfully proved that L is an RSI-complete logic (Theorem 3.48) without the IPEP (Theorem 3.40), thus finally showing the separation of all classes of logics studied in the paper. Therefore, we have obtained the new hierarchy of (in)finitary logics depicted in Figure 3.

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[10] Note that in this proof it is crucial that whenever we have a proper code w which is a result of any operation of the second part of Definition 3.20 applied to code w_1 (resp. to codes w_1, w_2), then $w_1 <_{\mathrm{C}} w$ (resp. both $w_1 <_{\mathrm{C}} w$ and $w_2 <_{\mathrm{C}} w$). This property is standard for coding of the syntax.

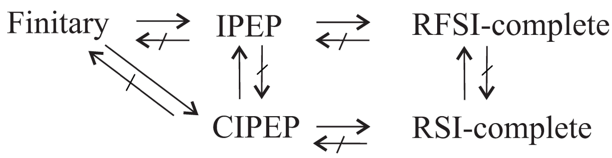

Figure 3. Separation of the classes in the hierarchy

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