Abstract Algebraic Logic: an introduction

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Outline

Logic is the science that studies correct reasoning.

The are many kinds of correct reasoning, hence many logics:

- Classical logic
- Modal logics
- Intuitionistic logic
- Linear logic
- Fuzzy logics
- Relevance logics
- Paraconsistent logics
- Non-monotonic logics
- ...

Algebraic Logic is the subdiscipline of Mathematical Logic which studies logical systems (classical and non-classical) by using tools from Universal Algebra.

Logic	Algebraic counterpart
Classical logic	Boolean algebras
Modal logics	Modal algebras
Intuitionistic logic	Heyting algebras
Linear logics	Commutative residuated lattices
Fuzzy logics	Semilinear residuated lattices
Relevance logics	Commutative contractive residuated lattices
...	...

Abstract Algebraic Logic provides a unified theoretical framework to deal with classical logic and (most of) non-classical logics.

- What does it mean for a class of algebras to be the algebraic counterpart of a logic?
- In which ways can a class of algebras be linked to a logic?
- How to find the algebraic counterpart of a logic?
- How do algebraic properties translate into logical properties and vice versa?

Outline

Basic syntactical elements

- Language: $\mathcal{L} = \{\wedge, \vee, \rightarrow, \top, \bot\}$ (primitive connectives)
- Variables: Denumerable set X
- Set of all formulae: $Fm(X)$
- Algebra of formulae:
  $\mathbf{Fm}(X) = \langle Fm(X), \wedge^{\mathbf{Fm}(X)}, \vee^{\mathbf{Fm}(X)}, \rightarrow^{\mathbf{Fm}(X)}, \top^{\mathbf{Fm}(X)}, \bot^{\mathbf{Fm}(X)} \rangle$
- Substitutions: Endomorphisms of $\mathbf{Fm}(X)$.
- Defined connectives: $\neg\varphi = \varphi \rightarrow \bot$,
  $\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$

Hilbert-style calculus IPC

Axioms:

A0. $\top$
A1. $\varphi \to (\psi \to \varphi)$
A2. $\varphi \to (\psi \to \varphi \wedge \psi)$
A3. $\varphi \wedge \psi \to \varphi$
A4. $\varphi \wedge \psi \to \psi$
A5. $\varphi \to \varphi \vee \psi$
A6. $\psi \to \varphi \vee \psi$
A7. $\varphi \vee \psi \to ((\varphi \to \delta) \to ((\psi \to \delta) \to \delta))$
A8. $(\varphi \to \psi) \to ((\varphi \to (\psi \to \delta)) \to (\varphi \to \delta))$
A9. $\bot \to \varphi$

Rule of inference: Modus Ponens

$$\frac{\varphi, \varphi \to \psi}{\psi}$$

Given $\Gamma \cup \{\varphi\} \subseteq Fm(X)$,
$\Gamma \vdash_{\mathrm{IPC}} \varphi$ iff there is a proof of φ from Γ in the calculus IPC.

1. $\varphi \vdash_{\mathrm{IPC}} \varphi$ (Reflexivity)
2. If $\Gamma \subseteq \Delta$ and $\Gamma \vdash_{\mathrm{IPC}} \varphi$, then $\Delta \vdash_{\mathrm{IPC}} \varphi$ (Monotonicity)
3. If $\Gamma \vdash_{\mathrm{IPC}} \varphi$ and for all $\psi \in \Gamma$ $\Delta \vdash_{\mathrm{IPC}} \psi$, the $\Delta \vdash_{\mathrm{IPC}} \varphi$ (Cut)
4. If $\Gamma \vdash_{\mathrm{IPC}} \varphi$ and σ is a substitution, then $\sigma[\Gamma] \vdash_{\mathrm{IPC}} \sigma(\varphi)$ (Structurality)
5. If $\Gamma \vdash_{\mathrm{IPC}} \varphi$, then there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\mathrm{IPC}} \varphi$ (Finitarity)

Theorem ((Global) Deduction Theorem)

For every set of formulae $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_{\mathrm{IPC}} \psi \text{ iff } \Gamma \vdash_{\mathrm{IPC}} \varphi \to \psi$$

Semantical interpretations:

- Brower-Heyting-Kolmogorov interpretation
- Kripke frames
- Topological interpretation
- Category-theoretical interpretation
- Game semantics
- Algebraic semantics

Definition

An algebra $\mathcal{A} = \langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \top^{\mathcal{A}}, \bot^{\mathcal{A}} \rangle$ is a Heyting algebra if

1. $\langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \top^{\mathcal{A}}, \bot^{\mathcal{A}} \rangle$ is a bounded distributive lattice
2. for every $a, b \in A$, $a \rightarrow^{\mathcal{A}} b$ is a relative pseudo-complement of a and b, i.e. for every $c \in A$
$$a \wedge^{\mathcal{A}} c \leq b \text{ iff } c \leq a \rightarrow^{\mathcal{A}} b$$

where the relation $\leq$ is the lattice ordering. $\rightarrow^{\mathcal{A}}$ is the residuum of $\wedge^{\mathcal{A}}$.

Let $\mathbb{HA}$ be the class of all Heyting algebras.

Theorem

Let $\mathcal{A}$ be an $\mathcal{L}$-algebra. $\mathcal{A} \in \mathbb{HA}$ iff the following equations hold in $\mathcal{A}$:

- E1 $x \to x \approx \top$
- E2 $\top \to x \approx x$
- E3 $x \to (y \to z) \approx (x \to y) \to (x \to z)$
- E4 $(x \to y) \to ((y \to x) \to y) \approx (y \to x) \to ((x \to y) \to x)$
- E5 $x \to x \vee y \approx \top$, $y \to x \vee y \approx \top$
- E6 $(x \to y) \to ((y \to z) \to (x \vee y \to z)) \approx \top$
- E7 $x \wedge y \to x \approx \top$, $x \wedge y \to y \approx \top$
- E8 $(x \to y) \to ((x \to z) \to (x \to y \wedge z)) \approx \top$
- E9 $\bot \to x \approx \top$

$\mathbb{HA}$ is a variety.

Given $\mathcal{A} \in \mathbb{HA}$, an $\mathcal{A}$-evaluation is a homomorphism $e : \mathbf{Fm}(X) \to \mathcal{A}$.

Definition

Given $\Gamma \cup \{\varphi\} \subseteq Fm(X)$, we define the consequence relation $\Gamma \models_{\mathbb{HA}} \varphi$ iff for every $\mathcal{A} \in \mathbb{HA}$ and every $\mathcal{A}$-evaluation e: if $e[\Gamma] \subseteq \{\top^{\mathcal{A}}\}$, then $e(\varphi) = \top^{\mathcal{A}}$.

1. $\varphi \models_{\mathbb{HA}} \varphi$ (Reflexivity)
2. If $\Gamma \subseteq \Delta$ and $\Gamma \models_{\mathbb{HA}} \varphi$, then $\Delta \models_{\mathbb{HA}} \varphi$ (Monotonicity)
3. If $\Gamma \models_{\mathbb{HA}} \varphi$ and for all $\psi \in \Gamma$ $\Delta \models_{\mathbb{HA}} \psi$, the $\Delta \models_{\mathbb{HA}} \varphi$ (Cut)
4. If $\Gamma \models_{\mathbb{HA}} \varphi$ and σ is a substitution, then $\sigma[\Gamma] \models_{\mathbb{HA}} \sigma(\varphi)$ (Structurality)
5. If $\Gamma \models_{\mathbb{HA}} \varphi$, then there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \models_{\mathbb{HA}} \varphi$ (Finitarity)

Algebraic completeness

Theorem

For every $\Gamma \cup \{\varphi\} \subseteq Fm(X)$, $\Gamma \vdash_{\mathrm{IPC}} \varphi$ *iff* $\Gamma \models_{\mathbb{HA}} \varphi$.

Soundness: If $\Gamma \vdash_{\mathrm{IPC}} \varphi$, then $\Gamma \models_{\mathbb{HA}} \varphi$.

- Heyting algebras satisfy the axioms of IPC.
- Heyting algebras satisfy Modus Ponens: $\mathcal{A} \in \mathbb{HA}$, e $\mathcal{A}$-evaluation. If $e(\varphi) = \top^{\mathcal{A}}$ and $e(\varphi \to \psi) = \top^{\mathcal{A}}$, then $e(\psi) = \top^{\mathcal{A}}$ (because of $\top \to x \approx x$).

Completeness: If $\Gamma \models_{\mathbb{HA}} \varphi$, then $\Gamma \vdash_{\mathrm{IPC}} \varphi$.

- A theory is a set $T \subseteq Fm(X)$ closed under $\vdash_{\mathrm{IPC}}$ (if $T \vdash_{\mathrm{IPC}} \varphi$, then $\varphi \in T$). $Th(\mathrm{IPC})$: set of all theories of the logic IPC.
- Given a theory T, we define the relation $\Omega(T)$ by: $\langle \varphi, \psi \rangle \in \Omega(T)$ iff $T \vdash_{\mathrm{IPC}} \varphi \leftrightarrow \psi$.
- $\Omega(T)$ is a congruence of $\mathbf{Fm}(X)$.
- $\Omega(T)$ is compatible with T, i.e. for every φ and ψ

  $$\text{if } \langle \varphi, \psi \rangle \in \Omega(T) \text{ and } \varphi \in T, \text{ then } \psi \in T.$$
- $\Omega(T)$ is the greatest congruence of $\mathbf{Fm}(X)$ compatible with T.

- For every formula φ, $\langle \varphi, \top \rangle \in \Omega(T)$ iff $\varphi \in T$
- $\Omega(T)$ is the interderivability relation modulo T, i.e.:

$$\langle \varphi, \psi \rangle \in \Omega(T) \text{ iff } T, \varphi \vdash_{\mathrm{IPC}} \psi \text{ and } T, \psi \vdash_{\mathrm{IPC}} \varphi$$

- Lindenbaum-Tarski algebra: $\mathbf{Fm}(X)/\Omega(T) \in \mathbb{HA}$.

- Assume that $\Gamma \nvdash_{\mathrm{IPC}} \varphi$
- Let T be the theory generated by Γ.
- $\varphi \notin T$.
- Consider the algebra $\mathbf{Fm}(X)/\Omega(T)$ and the $\mathbf{Fm}(X)/\Omega(T)$-evaluation $e(p) = p/\Omega(T)$.
- For every formula ψ, $e(\psi) = \psi/\Omega(T)$.
- $\psi/\Omega(T) = \top/\Omega(T)$ iff $\psi \in T$.
- $e[\Gamma] \subseteq \{\top^{\mathbf{Fm}(X)/\Omega(T)}\}$ and $e(\varphi) \neq \top^{\mathbf{Fm}(X)/\Omega(T)}$.
- $\Gamma \not\models_{\mathbb{HA}} \varphi$.

Definition

Let $Eq(X)$ be the set of $\mathcal{L}$-equations (expressions of the form $\varphi \approx \psi$, where $\varphi, \psi \in Fm(X)$). Given $\Pi \cup \{\varphi \approx \psi\} \subseteq Eq(X)$, we define the equational consequence $\Pi \models_{\mathbb{HA}} \varphi \approx \psi$ iff for every $\mathcal{A} \in \mathbb{HA}$ and every $\mathcal{A}$-evaluation e: if $e(\alpha) = e(\beta)$ for every $\alpha \approx \beta \in \Pi$, then $e(\varphi) = e(\psi)$.

1. For every $\Gamma \cup \{\varphi\} \subseteq Fm(X)$,
   $\Gamma \vdash_{\mathrm{IPC}} \varphi$ iff $\{\psi \approx \top \mid \psi \in \Gamma\} \models_{\mathbb{HA}} \varphi \approx \top$
2. For every $\Pi \cup \{\varphi \approx \psi\} \subseteq Eq(X)$,
   $\Pi \models_{\mathbb{HA}} \varphi \approx \psi$ iff $\{\alpha \leftrightarrow \beta \mid \alpha \approx \beta \in \Pi\} \vdash_{\mathrm{IPC}} \varphi \leftrightarrow \psi$
3. For every $\varphi \in Fm(X)$,
   $\varphi \vdash_{\mathrm{IPC}} \varphi \leftrightarrow \top$ and $\varphi \leftrightarrow \top \vdash_{\mathrm{IPC}} \varphi$
4. For every $\varphi, \psi \in Fm(X)$,
   $\varphi \approx \psi \models_{\mathbb{HA}} \varphi \leftrightarrow \psi \approx \top$ and $\varphi \leftrightarrow \psi \approx \top \models_{\mathbb{HA}} \varphi \approx \psi$

Translations:

- $\tau : \varphi \mapsto \varphi \approx \top$
- $\rho : \alpha \approx \beta \mapsto \alpha \leftrightarrow \beta$

Heyting algebras are the algebraic counterpart of Intuitionistic Logic.

Filters

Definition

Let $\mathcal{A}$ be a Heyting algebra. A set $F \subseteq A$ is a filter iff:

1. $\top \in F$
2. if $a, b \in F$, then $a \wedge b \in F$
3. if $a \in F$ and $a \leq b$, then $b \in F$

Definition

Let $\mathcal{A}$ be a Heyting algebra. A set $F \subseteq A$ is an implicative filter iff:

1. $\top \in F$
2. if $a, a \to b \in F$, then $b \in F$

Definition

Let $\mathcal{A}$ be a Heyting algebra. A set $F \subseteq A$ is a logical filter iff for every $\Gamma \cup \{\varphi\} \subseteq Fm(X)$ for every e $\mathcal{A}$-evaluation, if $\Gamma \vdash_{\mathrm{IPC}} \varphi$ and $e[\Gamma] \subseteq F$, then $e(\varphi) \in F$.

filters = implicative filters = logical filters

$\mathcal{A} \in \mathbb{HA}$, $F \subseteq A$ filter. A congruence θ of $\mathcal{A}$ is compatible with F iff:

$$\text{if } \langle a, b \rangle \in \theta \text{ and } a \in F, \text{ then } b \in F$$

Proposition

Given $\mathcal{A} \in \mathbb{HA}$ and a filter $F \subseteq A$, we define a relation:

$$\Omega_{\mathcal{A}}(F) = \{\langle a, b \rangle \in A^2 \mid a \rightarrow b, b \rightarrow a \in F\}$$

Then $\Omega_{\mathcal{A}}(F)$ is a congruence of $\mathcal{A}$, $F = \top/\Omega_{\mathcal{A}}(F)$. Moreover $\Omega_{\mathcal{A}}(F)$ is compatible with F and it is the greatest one with this property.

Proposition

Let $\mathcal{A}$ be a Heyting algebra and θ a congruence of $\mathcal{A}$. Then $\top/\theta$ is a filter of $\mathcal{A}$.

Proposition

Let $\mathcal{A}$ be a Heyting algebra and θ a congruence of $\mathcal{A}$. Then $\Omega_{\mathcal{A}}(\top/\theta) = \theta$.

Theorem

$\mathcal{A} \in \mathbb{HA}$. $\Omega_{\mathcal{A}}$ is an isomorphism between the lattice of filters and the lattice of congruences of $\mathcal{A}$.

Theorem

$\mathcal{A} \in \mathbb{HA}$, $F \subseteq A$ *filter. Then for every* $a, b \in A$: $\langle a, b \rangle \in \Omega_{\mathcal{A}}(F)$ *iff for every formula* $\varphi(x, \overrightarrow{z})$, *and* $\overrightarrow{c} \in A^{<\omega}$ *we have* $\varphi^{\mathcal{A}}(a, \overrightarrow{c}) \in F$ *iff* $\varphi^{\mathcal{A}}(b, \overrightarrow{c}) \in F$.

$\Omega_{\mathcal{A}}(F)$ is called the Leibniz congruence.

Definition

Given $\mathcal{A} \in \mathbb{HA}$ and a set $B \subseteq A$, $Fi_{\mathcal{A}}(B)$ is the minimum filter of $\mathcal{A}$ containing B.

Definition

Given $\mathcal{A} \in \mathbb{HA}$ we define a relation:

$$\Lambda_{\mathcal{A}} = \{\langle a, b\rangle \in A^2 \mid Fi_{\mathcal{A}}(a) = Fi_{\mathcal{A}}(b)\}$$

Proposition

For every $\mathcal{A} \in \mathbb{HA}$, the relation $\Lambda_{\mathcal{A}}$ is the identity and thus it is a congruence.

Definition

Given $\mathcal{A} \in \mathbb{HA}$ a filter $F \subseteq A$ we define a relation:

$$\Lambda_{\mathcal{A}}(F) = \{\langle a, b\rangle \in A^2 \mid Fi_{\mathcal{A}}(F, a) = Fi_{\mathcal{A}}(F, b)\}$$

Theorem

For every $\mathcal{A} \in \mathbb{HA}$ and every filter $F \subseteq A$, $\Omega_{\mathcal{A}}(F) = \Lambda_{\mathcal{A}}(F)$.

Axiomatic extensions of IPC: Intermediate logics

- $\mathrm{S} = \mathrm{IPC} + Ax$
- $\Gamma \vdash_{\mathrm{S}} \varphi$ iff $\Gamma \cup Ax \vdash_{\mathrm{IPC}} \varphi$
- $Alg(\mathrm{S}) = \{\mathcal{A} \in \mathbb{HA} \mid \mathcal{A}$ satisfies $\tau[Ax]\}$.
- $\Pi \models_{Alg(\mathrm{S})} \alpha \approx \beta$ if $\Pi \cup \tau[Ax] \models_{\mathbb{HA}} \alpha \approx \beta$
- We obtain the same relation between the logic and the algebraic semantics as before:
  1. $\Gamma \vdash_{\mathrm{S}} \varphi$ iff $\tau[\Gamma] \models_{Alg(\mathrm{S})} \tau(\varphi)$
  2. $\Pi \models_{Alg(\mathrm{S})} \varphi \approx \psi$ iff $\rho[\Pi] \vdash_{\mathrm{S}} \rho(\varphi \approx \psi)$
  3. $\varphi \vdash_{\mathrm{S}} \rho(\tau(\varphi))$ and $\rho(\tau(\varphi)) \vdash_{\mathrm{S}} \varphi$
  4. $\varphi \approx \psi \models_{Alg(\mathrm{S})} \tau(\rho(\varphi \approx \psi))$ and $\tau(\rho(\varphi \approx \psi)) \models_{Alg(\mathrm{S})} \varphi \approx \psi$

$Alg(\mathrm{S})$ is the algebraic counterpart of S.

Classical Logic

- CPC = IPC + $\varphi \vee \neg\varphi$
- $Alg(\mathrm{CPC}) = \mathbb{BA}$ (Boolean algebras)

Boolean algebras are the algebraic counterpart of Classical Logic.

Outline

Intuitionistic linear logic without exponentials

Language: $\mathcal{L} = \{\&, \wedge, \vee, \rightarrow, \overline{0}, \overline{1}, \bot, \top\}$

A1.	$\varphi \rightarrow \varphi$
A2.	$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
A3.	$\overline{1} \rightarrow (\varphi \rightarrow \varphi)$
A4.	$\overline{1}$
A5.	$\varphi \& \psi \rightarrow \psi \& \varphi$
A6.	$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \& \psi \rightarrow \chi)$
A6'.	$(\varphi \& \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
A7.	$\varphi \wedge \psi \rightarrow \varphi$, $\varphi \wedge \psi \rightarrow \psi$
A8.	$((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow \psi \wedge \chi))$
A9.	$\varphi \rightarrow \varphi \vee \psi$, $\psi \rightarrow \varphi \vee \psi$
A10.	$(\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi \rightarrow \chi)$
A11.	$\bot \rightarrow \varphi$, $\varphi \rightarrow \top$

Modus Ponens

Adjunction: $\frac{\varphi, \psi}{\varphi \wedge \psi}$

Definition

Given a formula φ, $Conj(\varphi)$ is the smallest set of formulae containing φ and $\overline{1}$ and closed under $\&$ and $\wedge$.

Theorem (Local Deduction Theorem)

For every set of formulae $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_{\mathrm{ILL}} \psi \text{ iff there is } \chi \in Conj(\varphi) \text{ such that } \Gamma \vdash_{\mathrm{ILL}} \chi \to \psi$$

Definition

An algebra $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \to^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}, \bot^{\mathcal{A}}, \top^{\mathcal{A}} \rangle$ is an ILL-algebra if

1. $\langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \top^{\mathcal{A}}, \bot^{\mathcal{A}} \rangle$ is a bounded lattice
2. $\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ is a commutative monoid
3. for every $a, b, c \in A$, $a \&^{\mathcal{A}} c \leq b$ iff $c \leq a \to^{\mathcal{A}} b$

Let $\mathbb{ILL}$ be the class of all ILL-algebras. It is a variety. They are also called bounded commutative residuated lattices.

Definition

Given formulae $\Gamma \cup \{\varphi\}$, we define the consequence relation $\Gamma \models_{\mathbb{ILL}} \varphi$ iff for every $\mathcal{A} \in \mathbb{ILL}$ and every $\mathcal{A}$-evaluation e: if $e(\psi) \geq \overline{1}^{\mathcal{A}}$ for every $\psi \in \Gamma$, then $e(\varphi) \geq \overline{1}^{\mathcal{A}}$.

1. $\Gamma \vdash_{\mathrm{ILL}} \varphi$ iff $\tau[\Gamma] \models_{\mathbb{ILL}} \tau(\varphi)$
2. $\Pi \models_{\mathbb{ILL}} \varphi \approx \psi$ iff $\rho[\Pi] \vdash_{\mathrm{ILL}} \rho(\varphi \approx \psi)$
3. $\varphi \vdash_{\mathrm{ILL}} \rho(\tau(\varphi))$ and $\rho(\tau(\varphi)) \vdash_{\mathrm{ILL}} \varphi$
4. $\varphi \approx \psi \models_{\mathbb{ILL}} \tau(\rho(\varphi \approx \psi))$ and $\tau(\rho(\varphi \approx \psi)) \models_{\mathbb{ILL}} \varphi \approx \psi$

- $\tau : \varphi \mapsto \varphi \wedge \overline{1} \approx \overline{1}$
- $\rho : \alpha \approx \beta \mapsto (\alpha \leftrightarrow \beta)$

$\mathbb{ILL}$ is the algebraic counterpart of Intuitionistic Linear Logic.

Definition

Let $\mathcal{A} \in \mathbb{ILL}$. A set $F \subseteq A$ is a filter iff:

1. $\overline{1} \in F$
2. if $a, b \in F$, then $a \& b \in F$
3. if $a, b \in F$, then $a \wedge b \in F$
4. if $a \in F$ and $a \leq b$, then $b \in F$

- $\Omega_{\mathcal{A}}(F) = \{\langle a, b\rangle \in A^2 \mid a \rightarrow b, b \rightarrow a \in F\}$
- $\Omega_{\mathcal{A}}$ is a bijection between the lattices of filters and congruences of $\mathcal{A}$.
- $\Lambda_{\mathcal{A}}$ is not a congruence in general for $\mathcal{A} \in \mathbb{ILL}$.

ILL^{-}: the fragment of ILL without $\overline{1}$.

Definition

An algebra $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \to^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \bot^{\mathcal{A}}, \top^{\mathcal{A}} \rangle$ is an ILL^{-}-algebra if

1. $\langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \top^{\mathcal{A}}, \bot^{\mathcal{A}} \rangle$ is a bounded lattice
2. $\langle A, \& \rangle$ is a commutative semigroup
3. for every $a, b, c \in A$, $a \&^{\mathcal{A}} c \leq b$ iff $c \leq a \to^{\mathcal{A}} b$

Let $\mathbb{ILL}^{-}$ be the class of all ILL^{-}-algebras. It is a variety.

Definition

Given $\mathcal{A} \in \mathbb{ILL}^{-}$, let $E(\mathcal{A})$ be the lattice filter generated by $\{a \to^{\mathcal{A}} a \mid a \in A\}$.

Lemma

If $\mathcal{A} \in \mathbb{ILL}^{-}$, *then* $E(\mathcal{A}) = \{a \in A \mid a \to a \leq a\}$.

Definition

Given formulae $\Gamma \cup \{\varphi\}$, we define the consequence relation $\Gamma \models_{\mathbb{ILL}^-} \varphi$ iff for every $\mathcal{A} \in \mathbb{ILL}^-$ and every $\mathcal{A}$-evaluation e: if $e(\psi) \in E(\mathcal{A})$ for every $\psi \in \Gamma$, then $e(\varphi) \in E(\mathcal{A})$.

1. $\Gamma \vdash_{\mathrm{ILL}^-} \varphi$ iff $\tau[\Gamma] \models_{\mathbb{ILL}^-} \tau(\varphi)$
2. $\Pi \models_{\mathbb{ILL}^-} \varphi \approx \psi$ iff $\rho[\Pi] \vdash_{\mathrm{ILL}^-} \rho(\varphi \approx \psi)$
3. $\varphi \vdash_{\mathrm{ILL}^-} \rho(\tau(\varphi))$ and $\rho(\tau(\varphi)) \vdash_{\mathrm{ILL}^-} \varphi$
4. $\varphi \approx \psi \models_{\mathbb{ILL}^-} \tau(\rho(\varphi \approx \psi))$ and $\tau(\rho(\varphi \approx \psi)) \models_{\mathbb{ILL}^-} \varphi \approx \psi$

- $\tau : \varphi \mapsto \varphi \wedge (\varphi \to \varphi) \approx \varphi \to \varphi$
- $\rho : \alpha \approx \beta \mapsto (\alpha \leftrightarrow \beta)$

$\mathbb{ILL}^-$ is the algebraic counterpart of ILL^-.

Classical modal logics

Local modal logic lK:

- Axioms of Classical Logic
- Modus Ponens
- (AK) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

Global modal logic gK: lK + necessitation rule $\frac{\varphi}{\Box\varphi}$

Theorem (Deduction theorems)

For every set of formulae $\Gamma \cup \{\varphi, \psi\}$*,*

1. $\Gamma, \varphi \vdash_{lK} \psi$ *iff* $\Gamma \vdash_{lK} \varphi \to \psi$
2. $\Gamma, \varphi \vdash_{gK} \psi$ *iff there is* n *such that* $\Gamma \vdash_{gK} \Box^0\varphi \wedge \ldots \wedge \Box^n\varphi \to \psi$

Definition

An algebra $\mathcal{A} = \langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \Box^{\mathcal{A}}, \top^{\mathcal{A}}, \bot^{\mathcal{A}} \rangle$ is a normal modal algebra if:

1. $\langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \top^{\mathcal{A}}, \bot^{\mathcal{A}} \rangle$ is a Boolean algebra
2. $\Box^{\mathcal{A}} \top^{\mathcal{A}} = \top^{\mathcal{A}}$
3. $\Box^{\mathcal{A}}(a \wedge^{\mathcal{A}} b) = \Box^{\mathcal{A}} a \wedge^{\mathcal{A}} \Box^{\mathcal{A}} b$

Let $\mathbb{MNA}$ be the class of all normal modal algebras.

1. $\Gamma \vdash_{gK} \varphi$ iff $\tau[\Gamma] \models_{\mathbb{MNA}} \tau(\varphi)$
2. $\Pi \models_{\mathbb{MNA}} \varphi \approx \psi$ iff $\rho[\Pi] \vdash_{gK} \rho(\varphi \approx \psi)$
3. $\varphi \vdash_{gK} \rho(\tau(\varphi))$ and $\rho(\tau(\varphi)) \vdash_{gK} \varphi$
4. $\varphi \approx \psi \models_{\mathbb{MNA}} \tau(\rho(\varphi \approx \psi))$ and $\tau(\rho(\varphi \approx \psi)) \models_{\mathbb{MNA}} \varphi \approx \psi$

- $\tau : \varphi \mapsto \varphi \approx \top$
- $\rho : \alpha \approx \beta \mapsto \alpha \leftrightarrow \beta$

Definition

Let $\mathcal{A} \in \mathbb{NMA}$. A set $F \subseteq A$ is a filter iff:

1. $\top \in F$
2. if $a, b \in F$, then $a \wedge b \in F$
3. if $a \in F$ and $a \leq b$, then $b \in F$
4. if $a \in F$, then $\Box a \in F$

- $\Omega_{\mathcal{A}}(F) = \{\langle a, b\rangle \in A^2 \mid a \to b, b \to a \in F\}$
- $\Omega_{\mathcal{A}}$ is a bijection between the lattices of filters and congruences of $\mathcal{A}$.

$\mathbb{NMA}$ is the algebraic counterpart of gK.

Problems in the algebraization of lK

- $\rho(\alpha \approx \beta) = \alpha \leftrightarrow \beta$ does not define Leibniz congruence in lK.
- Indeed: $p, q \vdash_{lK} p \leftrightarrow q$, but $p, q \nvdash_{lK} \Box p \leftrightarrow \Box q$.
- $\Omega_{\mathcal{A}}(F) = \{\langle a, b\rangle \in A^2 \mid \Box^n(a \leftrightarrow b) \in F$ for every $n \in \omega\}$.
- Moreover, we cannot use $\tau(\varphi) = \varphi \approx \top$.
- Indeed: there is not a single equation, or even a set of equations $E(p)$ in one variable, such that

$$\varphi \in T \text{ iff } \varphi/\Omega(T) \text{ is a solution of } E(p)$$

- Assume $\Gamma \nvdash_{lK} \varphi$.
- Let T be the theory generated by Γ.
- Consider $\Omega(T) = \{\langle \alpha, \beta \rangle \mid T \vdash_{lK} \Box^n(\alpha \leftrightarrow \beta)$ for every $n \in \omega\}$.
- Lindenbaum-Tarski algebra: $\mathbf{Fm}(X)/\Omega(T) \in \mathbb{NMA}$.
- $T/\Omega(T)$ is a lattice filter of $\mathbf{Fm}(X)/\Omega(T)$.
- Take the $\mathbf{Fm}(X)$-evaluation $e(p) = p/\Omega(T)$.
- For every $\psi \in T$, $e(\psi) \in T/\Omega(T)$, while $e(\varphi) \notin T/\Omega(T)$.
- lK needs a matrix semantics.

Definition

$\mathbb{NMM}$ is the class of logical matrices of the form $\langle \mathcal{A}, F \rangle$, where $\mathcal{A} \in \mathbb{NMA}$ and F is a lattice filter of $\mathcal{A}$.

Definition

Given formulae $\Gamma \cup \{\varphi\}$, we define the consequence relation $\Gamma \models_{\mathbb{NMM}} \varphi$ iff for every $\langle \mathcal{A}, F \rangle \in \mathbb{NMM}$ and every $\mathcal{A}$-evaluation e: if $e(\psi) \in F$ for every $\psi \in \Gamma$, then $e(\varphi) \in F$.

Theorem

For every $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{IK} \varphi$ iff $\Gamma \models_{\mathbb{NMM}} \varphi$.

$\mathbb{NMA}$ is the algebraic counterpart of IK, but the corresponing filters are the lattice filters.

Outline

A propositional logic is a pair $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ where $\mathcal{L}$ is a propositional language and $\vdash_{\mathrm{L}}$ satisfies:

1. Consequence relation:
   For every $\Gamma \cup \Delta \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,
   (a) $\varphi \vdash_{\mathrm{L}} \varphi$.
   (b) If $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathrm{L}} \varphi$.
   (c) If $\Gamma \vdash_{\mathrm{L}} \varphi$ and for every $\psi \in \Gamma$, $\Delta \vdash_{\mathrm{L}} \psi$, then $\Delta \vdash_{\mathrm{L}} \varphi$.
2. Structural:
   For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ and $\sigma \in \mathrm{Sub}_{\mathcal{L}}$ if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$.

L is finitary if for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ such that $\Gamma \vdash_{\mathrm{L}} \varphi$ there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathrm{L}} \varphi$.

Logical matrices

An $\mathcal{L}$-matrix is a pair $\mathbf{A} = \langle \mathcal{A}, D \rangle$ where $\mathcal{A}$ is an $\mathcal{L}$-algebra and D is a subset of A called the *filter* of $\mathbf{A}$.

The semantical consequence with respect to a class of matrices $\mathbb{K}$ is defined as $\Gamma \models_{\mathbb{K}} \varphi$ iff for each $\mathbf{A} \in \mathbb{K}$ and each $\mathbf{A}$-evaluation e we obtain $e(\varphi) \in D$ whenever $e[\Gamma] \subseteq D$.

$\langle \mathcal{L}, \models_{\mathbb{K}} \rangle$ is a logic.

$\mathbf{A}$ is a model of L if $\vdash_{\mathrm{L}} \subseteq \models_{\mathbf{A}}$.

Let $\mathbf{MOD}(\mathrm{L})$ be the class of all models of L.

Given an $\mathcal{L}$-algebra $\mathcal{A}$, a subset $F \subseteq A$ is an L-*filter* if $\langle \mathcal{A}, F \rangle \in \mathbf{MOD}(\mathrm{L})$. Let $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$ be the set of all L-filters over $\mathcal{A}$.

$Th(\mathrm{L})$ is the set of all theories of L (sets closed under $\vdash_{\mathrm{L}}$).
$Th(\mathrm{L}) = \mathcal{F}i_{\mathrm{L}}(Fm_{\mathcal{L}})$.

Ω can be seen as an operator on $Th(\mathrm{L})$.

The Leibniz Hierarchy

1. L is called protoalgebraic if Ω is monotone on $Th(\mathrm{L})$, i.e. for every $T_1, T_2 \in Th(\mathrm{L})$, if $T_1 \subseteq T_2$ then $\Omega(T_1) \subseteq \Omega(T_2)$.
2. L is called equivalential if Ω is monotone and commutes with inverse substitutions on $Th(\mathrm{L})$, i.e. for every $T \in Th(\mathrm{L})$ and every $\sigma \in \mathrm{Sub}_{\mathcal{L}}$, $\Omega(\sigma^{-1}[T]) = \sigma^{-1}[\Omega(T)]$.
3. L is called weakly algebraizable if Ω is monotone and injective on $Th(\mathrm{L})$.
4. L is called algebraizable if Ω is monotone and injective and it commutes with inverse substitutions on $Th(\mathrm{L})$.

The role of (definable) equivalencies in the hierarchy

- A logic $\mathcal{S}$ is equivalential iff there is a set of formulae $E(p,q)$ (called *equivalence set*) s.t.
  - $\vdash_{\mathcal{S}} E(p,p)$
  - $p, E(p,q) \vdash_{\mathcal{S}} q$
  - $E(p,q) \vdash_{\mathcal{S}} E(c(s_1 \dots s_{i-1}, p, \dots), c(s_1 \dots s_{i-1}, q, \dots))$ for each $\langle c, n\rangle \in \mathcal{L}$ and each $i \leq n$.
- A logic $\mathcal{S}$ is protoalgebraic iff there is a set of formulae $E(p,q,\overrightarrow{r})$ (called *parameterized equivalence set*) s.t.
  - $\vdash_{\mathcal{S}} E(p,p,\overrightarrow{r})$
  - $p, E(p,q,\overrightarrow{r}) \vdash_{\mathcal{S}} q$
  - $E(p,q,\overrightarrow{r}) \vdash_{\mathcal{S}} E(c(s_1 \dots s_{i-1}, p, \dots), c(s_1 \dots s_{i-1}, q, \dots), \overrightarrow{r})$ for each $\langle c, n\rangle \in \mathcal{L}$ and each $i \leq n$.

(Parameterized) equivalence sets define Leibniz congruence:

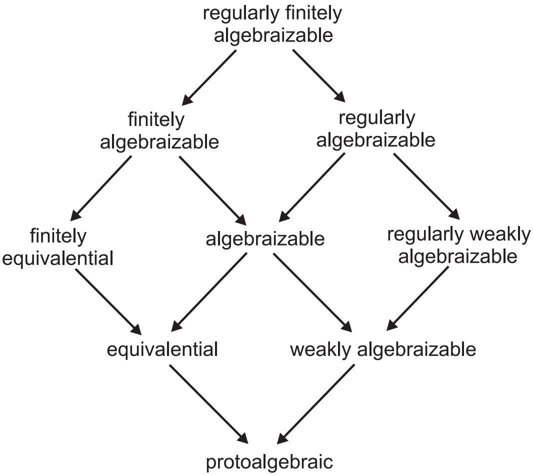
$\langle\alpha, \beta\rangle \in \Omega(T)$ if, and only, $T \vdash_{\mathrm{L}} E(\varphi, \psi)$.

$\langle\alpha, \beta\rangle \in \Omega(T)$ if, and only, $T \vdash_{\mathrm{L}} E(\varphi, \psi, \overrightarrow{\alpha})$ for every $\overrightarrow{\alpha}$

1. L is called *finitely equivalential (algebraizable)* if it is equivalential (algebraizable) with a finite equivalence set.
2. L is called *regularly weakly algebraizable* if it has a parameterized equivalence set satisfying the G-rule.
3. L is called *regularly (finitely) algebraizable* if it has a (finite) equivalence set satisfying the G-rule.

G-rule: $p, q \vdash_{\mathrm{L}} E(p, q)$.

Leibniz hierarchy



A matrix $\langle \mathcal{A}, F \rangle$ is reduced if $\Omega_{\mathcal{A}}(F) = Id_{\mathcal{A}}$.

Given a logic L, the class of its reduced models is denoted by $\mathbf{MOD}^*(\mathrm{L})$, and the class of algebraic reducts of $\mathbf{MOD}^*(\mathrm{L})$ is denoted by $\mathbf{ALG}^*(\mathrm{L})$.

Theorem

Let $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ *be a logic. The following are equivalent:*

1. L *is algebraizable.*
2. *For every* $\mathcal{L}$*-algebra* $\mathcal{A}$*,* $\Omega_{\mathcal{A}}$ *is a lattice isomorphism between* $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$ *and* $\mathcal{C}o_{\mathbf{ALG}^*(\mathrm{L})}(\mathcal{A})$ *which commutes with inverse homomorphisms.*
3. *There exists a set of formulae in two variables* $E(p,q)$ *and a set of equations in one variable* $\mathcal{E}(p) \subseteq \mathrm{Eq}_{\mathcal{L}}$ *such that:*
   - *For every* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ *we have:* $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\mathcal{E}[\Gamma] \models_{\mathbf{ALG}^*(\mathrm{L})} \mathcal{E}(\varphi)$*.*
   - $p \approx q \models_{\mathbf{ALG}^*(\mathrm{L})} \mathcal{E}[E(p,q)]$ *and* $\mathcal{E}[E(p,q)] \models_{\mathbf{ALG}^*(\mathrm{L})} p \approx q$
   - *For every* $\Pi \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}$ *we have:* $\Pi \models_{\mathbf{ALG}^*(\mathrm{L})} \varphi \approx \psi$ *iff* $E[\Pi] \vdash_{\mathrm{L}} E(\varphi,\psi)$
   - $p \vdash_{\mathrm{L}} E[\mathcal{E}(p)]$ *and* $E[\mathcal{E}(p)] \vdash_{\mathrm{L}} p$*.*

$\mathbf{ALG}^*(\mathrm{L})$ *is called* the equivalent algebraic semantics *of* L*.*

- IPC is regularly finitely algebraizable, $\mathbf{ALG}^*(\mathrm{IPC}) = \mathbb{HA}$.
- CPC is regularly finitely algebraizable, $\mathbf{ALG}^*(\mathrm{CPC}) = \mathbb{BA}$.
- ILL is finitely algebraizable (not regularly), $\mathbf{ALG}^*(\mathrm{ILL}) = \mathbb{ILL}$.
- ILL$^-$ is finitely algebraizable (not regularly), $\mathbf{ALG}^*(\mathrm{ILL}^-) = \mathbb{ILL}^-$.
- gK is regularly finitely algebraizable, $\mathbf{ALG}^*(gK) = \mathbb{NMA}$.
- lK is equivalential (not finitely, not algebraizable), $\mathbf{ALG}^*(lK) = \mathbb{NMA}$.

Bridge theorems

Definition

A variety $\mathbb{K}$ has the equationally definable principal congruences property (EDPC, for short) if there exists a finite set of equations in 4 variables

$$\{\sigma_i(x, y, z, w) \approx \tau_i(x, y, z, w) : i < n\}$$

such that for every $\mathcal{A} \in \mathbb{K}$ and every $a, b, c, d \in A$: $\langle c, d\rangle \in \Theta(\{\langle a, b\rangle\})$ if, and only if, $\sigma_i^{\mathcal{A}}(a, b, c, d) = \tau_i^{\mathcal{A}}(a, b, c, d)$ for every $i < n$.

Theorem

Let $\mathrm{S} = \langle \mathcal{L}, \vdash_{\mathrm{S}}\rangle$ be an algebraizable logic and let $\mathbb{K}$ be its equivalent algebraic semantics. Suppose that $\mathbb{K}$ is a variety. Then, S has a Global Deduction Theorem iff $\mathbb{K}$ has the EDPC.

Definition

A variety $\mathbb{K}$ has the congruence extension property (CEP, for short) if for every $\mathcal{A}, \mathcal{B} \in \mathbb{K}$ such that $\mathcal{B} \subseteq \mathcal{A}$ and every $\theta \in Con(\mathcal{B})$, there exists $\theta' \in Con(\mathcal{A})$ such that $\theta' \cap B^2 = \theta$.

Theorem

Let $\mathrm{S} = \langle \mathcal{L}, \vdash_\mathrm{S} \rangle$ be an algebraizable logic and let $\mathbb{K}$ be its equivalent algebraic semantics. Suppose that $\mathbb{K}$ is a variety. Then, S has a Local Deduction Theorem iff $\mathbb{K}$ has the CEP.

Final comments

- Other bridge theorems relate interpolation with amalgamation properties.
- Decidability of a logic can be shown by proving Finite Embeddability Property in the algebraic semantics.
- In finitary algebraizable logics the classification of axiomatic extensions (resp. finitary extensions) corresponds to the classification of subvarieties (resp. subquasivarieties) of the algebraic semantics.
- Abstract Algebraic Logic provides a powerful tool to study non-classical logics.
- Algebraic semantics is complementary to other semantical interpretations and proof-theoretical approaches.

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