An Abstract Study of Disjunction Connectives in Non-Classical Logics

Carles Noguera i Clofent

Artificial Intelligence Research Institute (IIIA - CSIC)
Campus of the Autonomous University of Barcelona
Bellaterra, Catalonia

Joint work with: Petr Cintula

Disjunction in Classical Logic

(PD) $\varphi \vdash_{\mathrm{CPC}} \varphi \vee \psi$ and $\psi \vdash_{\mathrm{CPC}} \varphi \vee \psi$

PCP If $\Gamma, \varphi \vdash_{\mathrm{CPC}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{CPC}} \chi$, then $\Gamma, \varphi \vee \psi \vdash_{\mathrm{CPC}} \chi$.

Disjunction in Classical Logic

(PD) $\varphi \vdash_{\text{CPC}} \varphi \vee \psi$ and $\psi \vdash_{\text{CPC}} \varphi \vee \psi$

PCP If $\Gamma, \varphi \vdash_{\text{CPC}} \chi$ and $\Gamma, \psi \vdash_{\text{CPC}} \chi$, then $\Gamma, \varphi \vee \psi \vdash_{\text{CPC}} \chi$.

The same holds for many other logics: IPC, Ł, FL_{ew}, BL, ...

Disjunction in Classical Logic

(PD) $\varphi \vdash_{\mathrm{CPC}} \varphi \vee \psi$ and $\psi \vdash_{\mathrm{CPC}} \varphi \vee \psi$

PCP If $\Gamma, \varphi \vdash_{\mathrm{CPC}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{CPC}} \chi$, then $\Gamma, \varphi \vee \psi \vdash_{\mathrm{CPC}} \chi$.

The same holds for many other logics: IPC, Ł, FL_{ew}, BL, ...

(PD) and PCP could be equivalently formulated as:

$\Gamma, \varphi \vdash_{\mathrm{CPC}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{CPC}} \chi$, if and only if, $\Gamma, \varphi \vee \psi \vdash_{\mathrm{CPC}} \chi$.

Disjunction in Classical Logic

(PD) $\varphi \vdash_{\mathrm{CPC}} \varphi \vee \psi$ and $\psi \vdash_{\mathrm{CPC}} \varphi \vee \psi$

PCP If $\Gamma, \varphi \vdash_{\mathrm{CPC}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{CPC}} \chi$, then $\Gamma, \varphi \vee \psi \vdash_{\mathrm{CPC}} \chi$.

The same holds for many other logics: IPC, Ł, FL_{ew}, BL, ...

(PD) and PCP could be equivalently formulated as:

$\Gamma, \varphi \vdash_{\mathrm{CPC}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{CPC}} \chi$, if and only if, $\Gamma, \varphi \vee \psi \vdash_{\mathrm{CPC}} \chi$.

Dummett in '*The Logical Basis of Metaphysics*, HUP, 1991' says about (a weaker variant of) PCP:

> *If this law does not hold, the operator $\vee$ could not legitimately be called disjunction operator.*

A problem

Theorem

In FL_e, *the lattice connective* $\vee$ *does not satisfy the* PCP *(it would entail* $\varphi \vee \psi \vdash (\varphi \wedge \mathbf{1}) \vee (\psi \wedge \mathbf{1})$*).*

A problem

Theorem

In FL_e, the lattice connective $\vee$ does not satisfy the PCP (it would entail $\varphi \vee \psi \vdash (\varphi \wedge \mathbf{1}) \vee (\psi \wedge \mathbf{1})$).

A solution of this problem:

Theorem

The connective $\vee'$ defined as $\varphi \vee' \psi = (\varphi \wedge \mathbf{1}) \vee (\psi \wedge \mathbf{1})$ satisfies

(PD) $\varphi \vdash (\varphi \wedge \mathbf{1}) \vee (\psi \wedge \mathbf{1})$ *and* $\psi \vdash (\varphi \wedge \mathbf{1}) \vee (\psi \wedge \mathbf{1})$

PCP *If* $\Gamma, \varphi \vdash \chi$ *and* $\Gamma, \psi \vdash \chi$*, then* $\Gamma, (\varphi \wedge \mathbf{1}) \vee (\psi \wedge \mathbf{1}) \vdash \chi$.

A bigger problem

Theorem

In the implication fragment of Gödel-Dummet logic we cannot define any connective $\vee$ *satisfying* (PD) *and* PCP.

A bigger problem

Theorem

In the implication fragment of Gödel-Dummet logic we cannot define any connective $\vee$ *satisfying* (PD) *and* PCP*.*

A solution of this problem:

Theorem

The 'connective' $\{(\varphi \to \psi) \to \psi, (\psi \to \varphi) \to \varphi\}$ *satisfies*

$$
\begin{array}{lllll}
(\mathrm{PD})_\varphi & \varphi \vdash (\varphi \to \psi) \to \psi & \textit{and} & \varphi \vdash (\psi \to \varphi) \to \varphi \\
(\mathrm{PD})_\psi & \psi \vdash (\varphi \to \psi) \to \psi & \textit{and} & \psi \vdash (\psi \to \varphi) \to \varphi \\
\mathrm{PCP} & \textit{If } \Gamma, \varphi \vdash \chi \textit{ and } \Gamma, \psi \vdash \chi\textit{, then} & & \\
& \qquad \Gamma, (\varphi \to \psi) \to \psi, (\psi \to \varphi) \to \varphi \vdash \chi. & &
\end{array}
$$

An even bigger problem

Theorem

In FL *no finite set of formulae of two variables defines any 'connective' satisfying* (PD) *and* PCP.

An even bigger problem

Theorem

In FL *no finite set of formulae of two variables defines any 'connective' satisfying* (PD) *and* PCP*.*

BUT there is still a solution of this problem:

Theorem

The following 'connective' satisfies both (PD) *and* PCP

$\{\gamma_1(\varphi \wedge \mathbf{t}) \vee \gamma_2(\psi \wedge \mathbf{t}) \mid$ *where* γ_1, γ_2 *are iterated conjugates*$\}$

An iterated conjugate of φ is a formula $\gamma_{\alpha_1}(\gamma_{\alpha_2}(\ldots\gamma_{\alpha_n}(\varphi)\ldots))$ where $\gamma_{\alpha_i} = \lambda_{\alpha_i}(\varphi) = (\alpha_i \backslash \varphi \& \alpha_i) \wedge \mathbf{t}$ or $\gamma_{\alpha_i} = \rho_{\alpha_i}(\varphi) = (\alpha_i \& \varphi / \alpha_i) \wedge \mathbf{t}$ for some formulae α_i.

A little bit of history

- Verdú. *Lògiques distributives i booleanes*, Stochastica, 3 (1979), 97–108.
- Dzik. *On the content of lattices of logics part 1: The representation theorem for lattices of logics*, Reports on Mathematical Logic, 13 (1981), 17–28.
- Czelakowski. *Logical matrices, primitive satisfaction and finitely based logics*, Studia Logica, 42 (1983), 89–104.
- Czelakowski. *Remarks on finitely based logics*, in Proceedings of the Logic Colloquium 1983. Springer, 1984, pp. 147–168.
- Font, Verdú. *Algebraic logic for classical conjunction and disjunction*, Studia Logica, 50 (1991), 391–419.
- Font, Jansana. *A General Algebraic Semantics for Sentential Logics*, Springer-Verlag, 1996.
- **Czelakowski. *Protoalgebraic Logic*, Kluwer, 2000.**

Definition and useful conventions

Let $\nabla(p, q, \overrightarrow{r})$ be a set of formulae. We write

$$\varphi \nabla \psi = \bigcup \{\nabla(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{F}m^{\leq \omega}\}.$$

$$\Sigma_1 \nabla \Sigma_2 = \bigcup \{\varphi \nabla \psi \mid \varphi \in \Sigma_1, \psi \in \Sigma_2\}$$

Definition and useful conventions

Let $\nabla(p, q, \overrightarrow{r})$ be a set of formulae. We write

$$\varphi \nabla \psi = \bigcup \{\nabla(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{F}m^{\leq \omega}\}.$$

$$\Sigma_1 \nabla \Sigma_2 = \bigcup \{\varphi \nabla \psi \mid \varphi \in \Sigma_1, \psi \in \Sigma_2\}$$

A logic is a (Tarski-style) structural consequence relation

need not be finitary!!

Definition and useful conventions

Let $\nabla(p, q, \overrightarrow{r})$ be a set of formulae. We write

$$\varphi \nabla \psi = \bigcup \{\nabla(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{F}m^{\leq\omega}\}.$$

$$\Sigma_1 \nabla \Sigma_2 = \bigcup \{\varphi \nabla \psi \mid \varphi \in \Sigma_1, \psi \in \Sigma_2\}$$

A logic is a (Tarski-style) structural consequence relation

need not be finitary!!

A set T is a theory in L if $T \vdash_{\mathrm{L}} \varphi$ implies $\varphi \in T$

$\mathrm{Th}(\mathrm{L})$ is the set of all theories in L

Definition

Let L be a logic in $\mathcal{L}$ and $\boldsymbol{A}$ be an $\mathcal{L}$-algebra. A set $F \subseteq A$ is called L-filter on $\boldsymbol{A}$ if:

$T \vdash_{\mathrm{L}} \varphi$ implies that for each $\boldsymbol{A}$-evaluation e if $e[T] \subseteq F$ then $e(\varphi) \in F$

- Filters on $\boldsymbol{A}$ form a closure system.
- By $\mathrm{Fi}(X)$ we denote the filter generated by X.
- Filters over $\mathrm{Fm}_{\mathcal{L}}$ are theories.
- By $\mathrm{Th}_{\mathrm{L}}(\Gamma)$ we denote the theory generated by Γ.

Generalized disjunctions

A (parameterized) set of formulae ∇ is a (p-)protodisjunction if:

$$\text{(PD)} \quad \varphi \vdash_{\mathrm{L}} \varphi \nabla \psi \quad \text{and} \quad \psi \vdash_{\mathrm{L}} \varphi \nabla \psi$$

Generalized disjunctions

A (parameterized) set of formulae ∇ is a (p-)protodisjunction if:

(PD) $\quad \varphi \vdash_{\mathrm{L}} \varphi \nabla \psi \quad$ and $\quad \psi \vdash_{\mathrm{L}} \varphi \nabla \psi$

We will consider the following three properties:

wPCP	$\varphi \vdash_{\mathrm{L}} \chi$	and	$\psi \vdash_{\mathrm{L}} \chi$	implies	$\varphi \nabla \psi \vdash_{\mathrm{L}} \chi$
PCP	$\Gamma, \varphi \vdash_{\mathrm{L}} \chi$	and	$\Gamma, \psi \vdash_{\mathrm{L}} \chi$	implies	$\Gamma, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi$
sPCP	$\Gamma, \Sigma \vdash_{\mathrm{L}} \chi$	and	$\Gamma, \Pi \vdash_{\mathrm{L}} \chi$	implies	$\Gamma, \Sigma \nabla \Pi \vdash_{\mathrm{L}} \chi$

Generalized disjunctions

A (parameterized) set of formulae ∇ is a (p-)protodisjunction if:

(PD) $\varphi \vdash_{\mathrm{L}} \varphi \nabla \psi$ and $\psi \vdash_{\mathrm{L}} \varphi \nabla \psi$

We will consider the following three properties:

wPCP	$\varphi \vdash_{\mathrm{L}} \chi$	and	$\psi \vdash_{\mathrm{L}} \chi$	implies	$\varphi \nabla \psi \vdash_{\mathrm{L}} \chi$
PCP	$\Gamma, \varphi \vdash_{\mathrm{L}} \chi$	and	$\Gamma, \psi \vdash_{\mathrm{L}} \chi$	implies	$\Gamma, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi$
sPCP	$\Gamma, \Sigma \vdash_{\mathrm{L}} \chi$	and	$\Gamma, \Pi \vdash_{\mathrm{L}} \chi$	implies	$\Gamma, \Sigma \nabla \Pi \vdash_{\mathrm{L}} \chi$

Clearly: sPCP $\Rightarrow$ PCP $\Rightarrow$ wPCP

Theorem

For finitary logics: sPCP $\Leftrightarrow$ PCP $\not\Leftarrow$ wPCP
But in general: sPCP $\not\Leftarrow$ PCP

Generalized disjunctions

A (parameterized) set of formulae ∇ is a (p-)protodisjunction if:

(PD) $\varphi \vdash_{\mathrm{L}} \varphi \nabla \psi$ and $\psi \vdash_{\mathrm{L}} \varphi \nabla \psi$

We will consider the following three properties:

wPCP	$\varphi \vdash_{\mathrm{L}} \chi$	and	$\psi \vdash_{\mathrm{L}} \chi$	implies	$\varphi \nabla \psi \vdash_{\mathrm{L}} \chi$
PCP	$\Gamma, \varphi \vdash_{\mathrm{L}} \chi$	and	$\Gamma, \psi \vdash_{\mathrm{L}} \chi$	implies	$\Gamma, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi$
sPCP	$\Gamma, \Sigma \vdash_{\mathrm{L}} \chi$	and	$\Gamma, \Pi \vdash_{\mathrm{L}} \chi$	implies	$\Gamma, \Sigma \nabla \Pi \vdash_{\mathrm{L}} \chi$

Clearly: $\mathrm{sPCP} \Rightarrow \mathrm{PCP} \Rightarrow \mathrm{wPCP}$

Theorem

For finitary logics: $\mathrm{sPCP} \Leftrightarrow \mathrm{PCP} \not\Leftarrow \mathrm{wPCP}$

But in general: $\mathrm{sPCP} \not\Leftarrow \mathrm{PCP}$

We define also transferred variants of these notions:

τ-wPCP, τ-PCP, and τ-sPCP

Example

Consider the non-distributive lattice *diamond*, with the domain $\{\bot, a, b, t, \top\}$, with t as central element, and the finitary logic given by all matrices over this algebra with a lattice filter.

Observe: $\Gamma \vdash \varphi$ iff $\bigwedge e[\Gamma] \leq e(\varphi)$ for every evaluation e.

$\vee$ is a protodisjunction with wPCP.

Assume now, for a contradiction, that it satisfies the PCP too. Then from $\varphi, \psi \vdash (\varphi \wedge \psi) \vee \chi$ and $\chi, \psi \vdash (\varphi \wedge \psi) \vee \chi$ we obtain $\varphi \vee \chi, \psi \vdash (\varphi \wedge \psi) \vee \chi$ and thus also (applying the PCP again) $\varphi \vee \chi, \psi \vee \chi \vdash (\varphi \wedge \psi) \vee \chi$ (a form of distributivity). Then, we reach a contradiction by observing that $a \vee b = t \vee b = \top$ while $(a \wedge t) \vee b = \bot \vee b = b$.

An finitary logic with a ∇ satisfying PCP but not sPCP

Example

Let $\boldsymbol{A}$ be a complete distributive lattice such that it is not a dual frame, i.e. there are elements $x_i \in A$ for $i \geq 0$ such that

$$\bigwedge_{i \geq 1} (x_0 \vee x_i) \not\leq x_0 \vee \bigwedge_{i \geq 1} x_i$$

expand the lattice language by constants $\{c_i \mid i \geq 0\} \cup \{c\}$ and define algebra $\boldsymbol{A}'$ in this language by setting $c_i^{\boldsymbol{A}'} = x_i$ and $c = \bigwedge_{i \geq 1} x_i$. Then we define the logic L in this language semantically given by the class of matrices $\{\langle \boldsymbol{A}', F \rangle \mid F$ is a principal lattice filter in $\boldsymbol{A}\}$.

Observe: $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\bigwedge_{\psi \in \Gamma} e(\psi) \leq e(\varphi)$ for each $\boldsymbol{A}$-evaluation e.

An finitary logic with a ∇ satisfying PCP but not sPCP

Example

First we show that $\vee$ enjoys the PCP: assume that for each e evaluation holds $(\bigwedge_{\delta\in\Gamma} e(\delta)) \wedge e(\varphi) \leq e(\chi)$ and $(\bigwedge_{\delta\in\Gamma} e(\delta)) \wedge e(\psi) \leq e(\chi)$, thus $[(\bigwedge_{\delta\in\Gamma} e(\delta)) \wedge e(\varphi)] \vee [(\bigwedge_{\delta\in\Gamma} e(\delta)) \wedge e(\psi)] \leq e(\chi)$, the distributivity of $\boldsymbol{A}$ completes the proof. Finally, by the way of contradiction, assume that $\vee$ enjoys the sPCP. Observe that: $c_0 \vdash_{\mathrm{L}} c_0 \vee c$ and $\{c_i \mid i \geq 1\} \vdash_{\mathrm{L}} c_0 \vee c$. Using the sPCP we obtain $\{c_0 \vee c_i \mid i \geq 1\} \vdash_{\mathrm{L}} c_0 \vee c$—a contradiction.

Syntactical characterization

Theorem

Let ∇ a commutative and idempotent p-protodisjunction. TFAE:

1. ∇ *satisfies* sPCP,
2. *whenever* $\Gamma \vdash_{\mathrm{L}} \varphi$ *we have also:* $\Gamma\nabla\chi \vdash_{\mathrm{L}} \varphi\nabla\chi$ *for each* χ.

This theorem was previously known for finitary logics and PCP.

Syntactical characterization

Theorem

Let ∇ a commutative and idempotent p-protodisjunction. TFAE:

1. ∇ *satisfies* sPCP,
2. *whenever* $\Gamma \vdash_{\mathrm{L}} \varphi$ *we have also:* $\Gamma\nabla\chi \vdash_{\mathrm{L}} \varphi\nabla\chi$ *for each* χ.

This theorem was previously known for finitary logics and PCP.

Theorem

TFAE:

1. *There is a (p-)protodisjunction satisfying* wPCP.
2. *For each (surjective) substitution* σ *and formulae* φ, ψ*:*

$$\mathrm{Th}_{\mathrm{L}}(\sigma\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\sigma\psi) = \mathrm{Th}_{\mathrm{L}}(\sigma[\mathrm{Th}_{\mathrm{L}}(\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\psi)]).$$

More definitions

$\mathrm{Th}(\mathrm{L})$ is both a closure system and a complete lattice. A theory is intersection-prime if it is finitely meet-irreducible in $\mathrm{Th}(\mathrm{L})$.

Definition

We say that L:

- is distributive if $\mathrm{Th}(\mathrm{L})$ is a distributive lattice
- is framal if $\mathrm{Th}(\mathrm{L})$ is a frame
- has the IPEP (intersection-prime extension property) if intersection-prime theories form a basis of $\mathrm{Th}(\mathrm{L})$, i.e. if $T \in \mathrm{Th}(\mathrm{L})$ and $\varphi \notin T$, there is an intersection-prime theory $T' \supseteq T$ such that $\varphi \notin T'$.

More definitions

$\mathrm{Th}(\mathrm{L})$ is both a closure system and a complete lattice. A theory is intersection-prime if it is finitely meet-irreducible in $\mathrm{Th}(\mathrm{L})$.

Definition

We say that L:

- is distributive if $\mathrm{Th}(\mathrm{L})$ is a distributive lattice
- is framal if $\mathrm{Th}(\mathrm{L})$ is a frame
- has the IPEP (intersection-prime extension property) if intersection-prime theories form a basis of $\mathrm{Th}(\mathrm{L})$, i.e. if $T \in \mathrm{Th}(\mathrm{L})$ and $\varphi \notin T$, there is an intersection-prime theory $T' \supseteq T$ such that $\varphi \notin T'$.

We define filter-distributivity/framality and τ-IPEP by demanding the defining conditions for $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ for each $\mathcal{L}$-algebra $\boldsymbol{A}$.

Finitary vs. IPEP logics

Theorem

Every finitary logic has IPEP *and NOT vice versa.*

Finitary vs. IPEP logics

Theorem

Every finitary logic has IPEP *and NOT vice versa.*

Example

Recall that $Ł_\infty$ has connectives $\to$, $\neg$ and is given by $\mathbf{A} = \langle\langle[0,1], \to^{\mathbf{A}}, \neg^{\mathbf{A}}\rangle, \{1\}\rangle$, where $x \to^{\mathbf{A}} y = \min\{1 - x + y, 1\}$ and $\neg^{\mathbf{A}} x = 1 - x$. It is well known that $Ł_\infty$ is not finitary.
If $T \nvdash_{Ł_\infty} \chi$, then there is an evaluation e such that $e[T] = \{1\}$ and $e(\chi) \neq 1$. We define $T' = e^{-1}[\{1\}]$. Obviously T' is a theory, $T \subseteq T'$ and $T' \nvdash_{Ł_\infty} \chi$. Assume that T' is not intersection-prime; thus there are formulae $\varphi, \psi \notin T'$ such that $T' = \mathrm{Th}_{Ł_\infty}(T, \varphi) \cap \mathrm{Th}_{Ł_\infty}(T, \psi)$. Assume without loss of generality that $e(\varphi) \leq e(\psi)$, so $e(\varphi \to \psi) = 1$ and so $\varphi \to \psi \in T'$. Thus $\psi \in \mathrm{Th}_{Ł_\infty}(T, \varphi)$ (because $\varphi, \varphi \to \psi \vdash_{Ł} \psi$) and thus $\psi \in T'$—a contradiction. Therefore, it has the IPEP.

Prime theories

Definition

A theory T is ∇-prime if it is consistent and $T \vdash \varphi \nabla \psi$ implies $T \vdash \varphi$ or $T \vdash \psi$.

∇ has the PEP if ∇-prime theories form a base of $\mathrm{Th}(\mathrm{L})$.

Theorem

If ∇ has PCP, then ∇-prime and intersection-prime theories coincide.

Theorem

Let L be a logic satisfying the IPEP. TFAE:

1. ∇ *has the* sPCP.
2. ∇ *has the* PCP.
3. ∇ *has the* PEP.

Disjunctions, distributivity, and framality

Theorem (Characterizations of PCP)

Let L *have* IPEP. *The following are equivalent:*

1. ∇ *enjoys the* PCP,
2. ∇ *enjoys the* wPCP *and the logic* L *is distributive,*
3. ∇ *enjoys the* wPCP *and the logic* L *is filter-distributive,*
4. ∇ *enjoys the* τ-PCP.

Disjunctions, distributivity, and framality

Theorem (Characterizations of PCP)

Let L *have* IPEP*. The following are equivalent:*

1. ∇ *enjoys the* PCP*,*
2. ∇ *enjoys the* wPCP *and the logic* L *is distributive,*
3. ∇ *enjoys the* wPCP *and the logic* L *is filter-distributive,*
4. ∇ *enjoys the* τ*-*PCP*.*

Theorem (Characterizations of sPCP)

The following are equivalent:

1. ∇ *enjoys the* sPCP*,*
2. ∇ *enjoys the* wPCP *and the logic* L *is framal,*
3. ∇ *enjoys the* wPCP *and the logic* L *is filter-framal,*
4. ∇ *enjoys the* τ*-*sPCP*.*

Protoalgebraic logics: stronger results

Theorem

Let L *be a protoalgebraic logic.*

- L *is distributive/framal IFF there is a p-protodisjunction* ∇ *which has* $\mathrm{PCP}/\mathrm{sPCP}$*.*
- *If* L *has* IPEP *and is distributive, then it is framal.*
- *If* L *has* IPEP *and is distributive, then it is filter-framal.*
- *If* ∇ *has* PCP*, then it has* τ*-*PCP*.*

Axiomatization of intersections of logics

Corollary

Let L *be a logic with the* IPEP, ∇ *a p-protodisjunction with* PCP, *and let* $\mathrm{L}_1, \mathrm{L}_2$ *be axiomatic extensions of* L *by sets of axioms* $\mathcal{A}_1$ *and* $\mathcal{A}_2$, *respectively. Then:*

$$\mathrm{L}_1 \cap \mathrm{L}_2 = \mathrm{L} + \{\varphi \nabla \psi \mid \varphi \in \mathcal{A}_1, \psi \in \mathcal{A}_2\}.$$

Note: we can safely always assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are written in disjoint sets of variables.

Theorem

Let L *be a logic with the* IPEP, ∇ *a p-protodisjunction with* PCP, *and* $\mathcal{C}$ *a set of positive clauses. Then:*

$$\models_{\{\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L}) \mid \mathbf{A} \models \mathcal{C}\}} = \mathrm{L} + \{\nabla_{\psi \in \Sigma_C} \psi \mid C \in \mathcal{C}\}.$$

P. Cintula, C. Noguera: *The Proof by Cases Property and its Variants in Structural Consequence Relations*. Submitted to the special issue of Studia Logica on Abstract Algebraic Logic.

P. Cintula, C. Noguera: *The Proof by Cases Property and its Variants in Structural Consequence Relations*. Submitted to the special issue of Studia Logica on Abstract Algebraic Logic.

Thank you
for your attention