On expansions of WNM t-norm based logics with truth-constants[*]

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Abstract

This paper focuses on completeness results about generic expansions of propositional Weak Nilpotent Minimum (WNM) logics with truth-constants. Indeed, we consider algebraic semantics for expansions of these logics with a set of truth-constants $\{\bar{r} \mid r \in C\}$, for a suitable countable $C \subseteq [0,1]$, and provide a full description of completeness results when (i) the t-norm is a Weak Nilpotent Minimum satisfying the *finite partition property* and (ii) the set of truth-constants *covers* all the unit interval in the sense that each interval of the partition contains values of C in its interior.

Keywords: Monoidal t-norm based Logic (MTL), Nilpotent minimum Logic (NM), Weak nilpotent minimum logics (WNM), Rational t-norm based logic, completeness results.

1 Introduction

T-norm based fuzzy logics are basically logics of *comparative truth*. In fact, the residuum $\Rightarrow$ of a (left-continuous) t-norm $*$ satisfies the condition $x \Rightarrow y = 1$ if, and only if, $x \leq y$ for all $x, y \in [0,1]$. This means that a formula $\varphi \to \psi$ is a logical consequence of a theory if the truth degree of ψ is at least as high as the truth degree of φ in any interpretation which is a model of the theory. Indeed, the logic of continuous t-norms as it is presented in Hájek's seminal book [21], only deals with valid formulae and deductions taking 1 as the only truth value to be preserved by inference (in the sense of yielding true consequences from true premises for each interpretation). This line is followed by the majority of logical papers written from then

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[*]This paper is an updated and shortened version of the manuscript "On expansions of t-norm based logics with truth-constants" that was written in 2006 as a kind of survey, but also with new results and proofs, in principle intended to appear in a book gathering contributions presented at the Linz Seminar 2005 but eventually never edited. Since then, new developments on this topic have been published [15, 16] using or referring to results from this paper. Although it might be felt as a sort of circular referencing, for the sake of being complete, we will also refer them when suitable.

in the setting of many-valued systems of mathematical fuzzy logic. But, in general, these *truth-preserving* logics do not exploit in depth neither the idea of comparative truth nor the potentiality of dealing with explicit partial truth that a many-valued logic setting offers.

There are however two approaches that aim at remedying these shortcomings. On the one hand, the idea of comparative truth is pushed forward in the logics called *truth-degree preserving* logics in [18, 3] where a deduction is valid if, and only if, the degre of truth of the premises is less or equal than the degree of truth of conclusion, so in fact what they preserve are lower bounds of truth values. Actually, since Gödel logic is the only t-norm based logic enjoying the classical deduction-detachment theorem, it is the only case where both notions of logic coincide. On the other hand, in some situations one might be also interested to explicitly represent and reason with intermediate degrees of truth. A way to circumvent this possible problem while keeping the truth preserving framework is to introduce truth-constants into the language.

This paper sticks to this latter approach, which in fact goes back to Pavelka [33], where he built a propositional many-valued logic which turned out to be equivalent to the expansion of Łukasiewicz logic by adding a truth constant $\bar{r}$ into the language for each real $r \in [0, 1]$, together with a number of additional axioms. In this way the expanded language allows one to have formulae of the kind $\bar{r} \to \varphi$ which, when evaluated to 1, express that the truth value of φ is greater or equal than r. Pavelka's logic, with a form of infinitary notion of completeness, was later further developed by Novák et al. [30, 31] and simplified by Hájek [21].

More recently, an alternative approach, based on traditional algebraic semantics, has been considered to study completeness results (in the usual sense) for expansions of t-norm based logics with countably-many truth-constants. Indeed, after [21], only the case of Łukasiewicz logic was known to be finite strong complete. Using this algebraic approach, the expansion of Gödel (and of some t-norm based logic related to the Nilpotent Minimum t-norm) with rational truth-constants and the expansion of Product logic with countable sets of truth-constants have been respectively studied in [14] and in [34]. The basic cases of Łukasiewicz, Gödel and Product logics have been extended to the more general case of logics of continuous t-norms which are finite ordinal sums of the three basic components in [12]. Moreover, in [5] the case of expanding Łukasiewicz logic with irrational truth-constants has been addressed. In these papers, the issue of canonical standard completeness (that is, completeness with respect to the standard algebra where the truth-constants are interpreted as their own values) for these logics has been determined. Also, special attention has been paid to the fragment of formulae of the kind $\bar{r} \to \varphi$, where φ is a formula without additional truth-constants. Actually, this kind of formulae have been extensively considered in other frameworks for reasoning with partial degrees of truth, like in Novák's evaluated syntax formalism based on Łukasiewicz Logic (see e.g. [32]), in Gerla's framework of abstract fuzzy logics [19] or in fuzzy logic programming (see e.g. [36]). More recently, similar formulae are also being used in systems of fuzzy description logic (see e.g. [26]).

This paper, always within the algebraic semantics approach, is meant as a follow-up of the paper [12] where the focus was on extensions of BL logic. Here we focus on standard completeness results for expansions with truth-constants of logics of weak nilpotent minimum t-norms[1] (WNM t-norms for short) in a general setting. The latter are axiomatic extensions of the WNM logic, an extension of the MTL logic introduced in [13] which is complete

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[1]A weak nilpotent t-norm $*$ is a left-continuous t-norm satisfying $x * y = \min(x, y)$ whenever $x * y > 0$, for all $x, y \in [0, 1]$.

with respect to all standard algebras defined by weak nilpotent t-norms. More specifically, extending first results in [14], we provide a full description of completeness results for the expansions of logics of WNM t-norms $*$ with a set of truth-constants $\{\overline{r} \mid r \in C\}$, for a suitable countable subalgebra $\mathcal{C}$ of the standard algebra $[0,1]_*$ when: (i) the WNM t-norm $*$ has the *finite partition property* (see Def. 15), and (ii) the set of truth-constants C *covers* all the unit interval in the sense that each component (for continuous case) or interval of the partition (for the WNM case) contains at least one value of C in its interior. Many of the results about expansions of WNM logics are formally presented here for the first time. We notice that in [15] completeness results with respect to rational semantics are provided for a wide family of expansions of t-norm logics with truth-constants, including WNM logics.

The paper is structured as follows. After this introduction, we provide the necessary background in the next three sections. In Section 2, we give the general definitions of t-norm based logics we will deal with in the paper, the notion of standard completeness and general results for axiomatic extensions of these logics, the equivalence between different kinds of standard completeness and properties of the corresponding algebraic varieties (the embeddability properties playing an important role). In Section 3 we overview known completeness results for logics of WNM t-norms and in Section 4 we introduce the general definitions of the expansions of t-norm logics with truth-constants and their algebraic counterpart. In Section 5 we study the structure and relevant algebraic properties of the expanded linearly ordered WNM-algebras, which are needed to obtain the completeness results described in Section 6. Section 7 deals with completeness results when restricting the language to evaluated formulae. Section 8 provides some complexity results for some WNM t-norm based logics expanded with truth-constants. We finish with some concluding remarks.

2   Preliminaries

The weakest logic that we will consider in this paper is the Monoidal T-norm based Logic (MTL). It is defined in [13] by means of a Hilbert-style calculus in the language $\mathcal{L} = \{\&, \rightarrow, \wedge, \overline{0}\}$ of type $\langle 2, 2, 2, 0 \rangle$. The only inference rule is *modus ponens* and the axiom schemata are the following (taking $\rightarrow$ as the least binding connective):

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(A2) $\varphi \& \psi \rightarrow \varphi$
(A3) $\varphi \& \psi \rightarrow \psi \& \varphi$
(A4) $\varphi \wedge \psi \rightarrow \varphi$
(A5) $\varphi \wedge \psi \rightarrow \psi \wedge \varphi$
(A6) $\varphi \& (\varphi \rightarrow \psi) \rightarrow \varphi \wedge \psi$
(A7a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \& \psi \rightarrow \chi)$
(A7b) $(\varphi \& \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(A8) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
(A9) $\overline{0} \rightarrow \varphi$

The usual defined connectives are introduced as follows:

$$\begin{aligned}
\varphi \vee \psi &:= ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi); \\
\varphi \leftrightarrow \psi &:= (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi); \\
\neg \varphi &:= \varphi \rightarrow \overline{0}; \\
\overline{1} &:= \neg \overline{0}.
\end{aligned}$$

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Axiom schema	Name
$\neg\neg\varphi \to \varphi$	Involution (Inv)
$\neg\varphi \lor ((\varphi \to \varphi\&\psi) \to \psi)$	Cancellation (C)
$\varphi \to \varphi\&\varphi$	Contraction (Con)
$\varphi \land \psi \to \varphi\&(\varphi \to \psi)$	Divisibility (Div)
$\varphi \land \neg\varphi \to \overline{0}$	Pseudocomplementation (PC)
$(\varphi\&\psi \to \overline{0}) \lor (\varphi \land \psi \to \varphi\&\psi)$	Weak Nilpotent Minimum (WNM)
$\varphi \lor \neg\varphi$	Excluded Middle (EM)

Table 1: Some usual axiom schemata in fuzzy logics.

Tables 1 and 2 collect some axiom schemata and the axiomatic extensions of MTL that they define.[2]

Logic	Additional axiom schemata
SMTL	(PC)
ΠMTL	(C)
IMTL	(Inv)
WNM	(WNM)
NM	(Inv) and (WNM)
BL	(Div)
SBL	(Div) and (PC)
Ł	(Div) and (Inv)
Π	(Div) and (C)
G	(Con)
CPC	(EM)

Table 2: Some axiomatic extensions of MTL obtained by adding the corresponding additional axiom schemata.

The algebraic counterpart[3] of MTL logic is the variety $\mathbb{MTL}$ of the so-called *MTL-algebras*, which are structures defined as follows.

Definition 1 ([13]). *An* MTL-*algebra is an algebra* $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \to^{\mathcal{A}}, \land^{\mathcal{A}}, \lor^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ *of type* $\langle 2, 2, 2, 2, 0, 0 \rangle$ *such that:*

1. $\langle A, \land^{\mathcal{A}}, \lor^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ *is a bounded lattice.*

2. $\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ *is a commutative monoid with unit* $\overline{1}^{\mathcal{A}}$.

3. *The operations* $\&^{\mathcal{A}}$ *and* $\to^{\mathcal{A}}$ *form an adjoint pair:*
   $$\forall a, b, c \in A, \ a\&^{\mathcal{A}}b \le c \text{ iff } b \le a \to^{\mathcal{A}} c.$$

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[2]Of course, some of these logics were known well before MTL was introduced. We only want to point out that it is possible to present them as the axiomatic extensions of MTL obtained by adding the corresponding axioms to the Hilbert-style calculus for MTL given above.

[3]We assume some basic knowledge on Universal Algebra. All the undefined notions can be found in [4].

4. It satisfies the prelinearity equation:

$$(x \to^{\mathcal{A}} y) \vee^{\mathcal{A}} (y \to^{\mathcal{A}} x) = \overline{1}^{\mathcal{A}}$$

An additional (unary) negation operation is defined as $\neg^{\mathcal{A}} a := a \to^{\mathcal{A}} \overline{0}^{\mathcal{A}}$, for every $a \in A$. If the lattice order is total we will say that $\mathcal{A}$ is an MTL-*chain.*

For the sake of a simpler notation, from now on superscripts in the operations of the algebras will be omitted when they are clear from the context.

Given an MTL-algebra $\mathcal{A}$ and an element $a \in A$, we say that a is the (negation) *fixpoint* of $\mathcal{A}$ if, and only if, $a = \neg a$. It is easy to prove that there exists at most one fixpoint (see, for example, [24]). The sets of positive and negative elements of $\mathcal{A}$ are respectively defined as:

$$A_+ := \{a \in A \mid a > \neg a\} \qquad A_- := \{a \in A \mid a \leq \neg a\}$$

Consider the terms $p(x) := x \vee \neg x$ and $n(x) := x \wedge \neg x$. The next proposition is an easy but useful result describing these sets.

Proposition 2 ([27])**.** *Let $\mathcal{A}$ be an* MTL-*algebra. Then:*

- $A_+ = \{p(a) \mid a \in A, \neg a \neq \neg\neg a\}$.

- $A_- = \{n(a) \mid a \in A\}$.

Notice that $p(a)$ is the fixpoint if, and only if, $\neg a = \neg\neg a$.

Given an MTL-algebra $\mathcal{A}$, a *filter* is any set $F \subseteq A$ such that:

- $\overline{1}^{\mathcal{A}} \in F$,

- If $a \in F$ and $a \leq b$, then $b \in F$, and

- If $a, b \in F$, then $a\&b \in F$.

In the rest of the paper we will use the following notations:

- $Fi(\mathcal{A})$ will denote the set of proper filters of $\mathcal{A}$;

- given a filter $F \in Fi(\mathcal{A})$, $\overline{F}$ will denote the set $\{a \in A \mid \neg a \in F\}$;

- for each element $a \in A$, F_a will denote the filter generated by a, i.e. the minimum filter containing a.

Next proposition states the usual one-to-one correspondence between filters and congruences.

Proposition 3. *Let $\mathcal{A}$ be an* MTL-*algebra. For every filter $F \subseteq A$ we define $\Theta(F) := \{\langle a, b \rangle \in A^2 \mid a \leftrightarrow b \in F\}$, and for every congruence θ of $\mathcal{A}$ we define $Fi(\theta) := \{a \in A \mid \langle a, \overline{1} \rangle \in \theta\}$. Then Θ is an order isomorphism from the set of filters onto the set of congruences and Fi is its inverse.*

By virtue of this correspondence, we will do a notational abuse by writing $\mathcal{A}/F$ instead of $\mathcal{A}/\Theta(F)$, and for each $a \in A$, $[a]_F$ will denote the class of a in $\mathcal{A}/F$.

Given any class $\mathbb{K}$ of MTL-algebras, we denote its equational consequence as $\models_{\mathbb{K}}$, i.e. given a set of equations Λ and an equation $\varphi \approx \psi$ in the language $\mathcal{L}$, $\Lambda \models_{\mathbb{K}} \varphi \approx \psi$ means that for

every $\mathcal{A} \in \mathbb{K}$ and every evaluation e of the formulae in $\mathcal{A}$, $e(\varphi) = e(\psi)$ whenever $e(\alpha) = e(\beta)$ for every $\alpha \approx \beta \in \Lambda$. If $\Lambda = \emptyset$, then we will write $\models_{\mathbb{K}} \varphi \approx \psi$, instead of $\emptyset \models_{\mathbb{K}} \varphi \approx \psi$. When there is only one algebra in $\mathbb{K}$, say $\mathcal{A}$, we will write $\Lambda \models_{\mathcal{A}} \varphi \approx \psi$ instead of $\Lambda \models_{\{\mathcal{A}\}} \varphi \approx \psi$.

MTL is actually an algebraizable logic in the sense of Blok and Pigozzi (see [2]) and $\mathbb{MTL}$ is its equivalent algebraic semantics. This implies that any axiomatic extension L of MTL is also algebraizable and its equivalent algebraic semantics is the subvariety $\mathbb{L}$ of $\mathbb{MTL}$ defined by the translations of the axioms into equations. Therefore, any axiomatic extension L of MTL is strongly complete with respect the variety $\mathbb{L}$ of L-algebras.

Theorem 4 ([13]). *Let L be an axiomatic extension of MTL. For every set of formulae* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{L}} \varphi$ *if, and only,* $\{\psi \approx \overline{1} \mid \psi \in \Gamma\} \models_{\mathbb{L}} \varphi \approx \overline{1}$.

Taking into account that, for any axiomatic extension L of MTL, every L-algebra is representable as a subdirect product of L-chains, the above completeness result can be refined to consider the class of chains of the variety.

Corollary 5. *For every set of formulae* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{L}} \varphi$ *if, and only,* $\{\psi \approx \overline{1} \mid \psi \in \Gamma\} \models_{\{\mathrm{L}-chains\}} \varphi \approx \overline{1}$.

Moreover, a lot of expansions of MTL are also algebraizable. Indeed, let L be an axiomatic extension of MTL, let $\mathcal{L}'$ be a language extending $\mathcal{L}$, consider a set $\Sigma \subseteq Fm_{\mathcal{L}'}$ and let L' be the expansion of L obtained by adding the formulae of Σ as axiom schemata. Assume that for every new n-ary connective λ in the language $\mathcal{L}'$,

$$\{p_1 \leftrightarrow q_1, \ldots, p_n \leftrightarrow q_n\} \vdash_{\mathrm{L}'} \lambda(p_1, \ldots, p_n) \leftrightarrow \lambda(q_1, \ldots, q_n)$$

Then L$'$ is algebraizable and its equivalent algebraic semantics is the variety of algebras in the language $\mathcal{L}'$ axiomatized by the axioms of $\mathbb{L}$ plus the equations $\{\varphi \approx \overline{1} \mid \varphi \in \Sigma\}$. We call the members of this variety L$'$-*algebras*. In general, L$'$ needs not be a conservative expansion of L; in fact, we can extract from [2] the following criterion.

Proposition 6 ([2]). *Under the previous hypothesis,* L$'$ *is a conservative expansion of* L *if, and only if, every* L*-algebra is a subreduct of some* L$'$*-algebra.*

Some algebraizable expansions of the so far mentioned logics have been introduced in the literature. Among them, a remarkable set of expansions are those obtained by enriching the language with the projection connective Δ (see [1]). Namely, given any axiomatic extension L of MTL, the expansion L_Δ is defined by adding to the language a unary connective Δ, and adding to the Hilbert-style system of L the following axiom schemata:

$(\Delta 1) \quad \Delta\varphi \vee \neg\Delta\varphi$
$(\Delta 2) \quad \Delta(\varphi \vee \psi) \rightarrow (\Delta\varphi \vee \Delta\psi)$
$(\Delta 3) \quad \Delta\varphi \rightarrow \varphi$
$(\Delta 4) \quad \Delta\varphi \rightarrow \Delta\Delta\varphi$
$(\Delta 5) \quad \Delta(\varphi \rightarrow \psi) \rightarrow (\Delta\varphi \rightarrow \Delta\psi)$

and the rule of *necessitation* for Δ: from φ derive $\Delta\varphi$.

This logic is algebraizable and its equivalent algebraic semantics is the variety of L_Δ-algebras, i.e. expansions with Δ of L-algebras satisfying the translation of the axioms

$(\Delta 1), \ldots, (\Delta 5)$ and the equation $\Delta \overline{1} \approx \overline{1}$. It is easy to prove that all L_Δ-algebras are representable as subdirect products of L_Δ-chains. The interpretation of the Δ connective in these chains is very simple, namely if $\mathcal{A}$ is an L_Δ-chain, then $\Delta^{\mathcal{A}}(\overline{1}^{\mathcal{A}}) = \overline{1}^{\mathcal{A}}$ and $\Delta^{\mathcal{A}}(a) = \overline{0}^{\mathcal{A}}$ for every $a \in A \setminus \{\overline{1}^{\mathcal{A}}\}$.

Proposition 7. *For every axiomatic extension* L *of* MTL, L_Δ *is a conservative expansion of* L.

Proof: It is obvious that every L-chain is the reduct of an L_Δ-chain (just take the same chain and define Δ in the only possible way for chains), thus we can apply Proposition 6. $\square$

Fuzzy Logic has always been interested in semantics defined over the real unit interval. Such kind of semantics can be found inside the class of MTL-algebras. Indeed, given a left-continuous t-norm $*$ and its residuum $\Rightarrow$ (defined as $a \Rightarrow b = \max\{c \mid a * c \leq b\}$), the algebra

$$[0,1]_* = \langle [0,1], *, \Rightarrow, \min, \max, 0, 1 \rangle$$

is an MTL-chain. Notice that $[0,1]_*$ is completely determined by the t-norm. Moreover, it is obvious that in every MTL-chain $\mathcal{A}$ over $[0,1]$, the operation $\&^{\mathcal{A}}$ is a left-continuous t-norm. We call these chains *standard* chains.

For some expansions of MTL their completeness with respect to chains can be improved to completeness with respect to standard algebras. This leads to the following *standard completeness properties*.[4]

Definition 8 ($\mathcal{R}$C, FS$\mathcal{R}$C, S$\mathcal{R}$C). *If a logic* L *is an algebraizable expansion of* MTL *in a language* $\mathcal{L}'$, *we say that* L *has the* (finitely) strong standard completeness property, *(F)S$\mathcal{R}$C for short, when for every (finite) set of formulae* $T \subseteq Fm_{\mathcal{L}'}$ *and every formula* φ *it holds that* $T \vdash_L \varphi$ *iff* $\{\psi \approx \overline{1} \mid \psi \in T\} \models_{\mathcal{A}} \varphi \approx \overline{1}$ *for every standard* L-*algebra* $\mathcal{A}$. *We say that* L *has the* standard completeness property, $\mathcal{R}$C for short, *when the equivalence is true for* $T = \emptyset$.

Of course, the S$\mathcal{R}$C implies the FS$\mathcal{R}$C, and the FS$\mathcal{R}$C implies the $\mathcal{R}$C. These completeness properties are preserved when taking fragments of the logics:

Proposition 9. *Suppose that* L' *is a conservative expansion of* L. *Then:*

- *If* L' *enjoys the* $\mathcal{R}$C, *then* L *enjoys the* $\mathcal{R}$C.

- *If* L' *enjoys the* FS$\mathcal{R}$C, *then* L *enjoys the* FS$\mathcal{R}$C.

- *If* L' *enjoys the* S$\mathcal{R}$C, *then* L *enjoys the* S$\mathcal{R}$C.

These completeness properties have usually been proved using some forms of embeddings of L-chains into standard L-chains. Actually, the S$\mathcal{R}$C has been proved for the following logics by showing that all countable chains[5] are embeddable into a standard one: MTL (in [25]), IMTL and SMTL (in [11]), G (in [9]) and WNM and NM (in [13]). In fact, as stated in next

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[4]For an extensive study of completeness properties in fuzzy logics see [8]. In particular, the reader can find the definitions and results reported in this section in that paper and references thereof.

[5]This and many similar statements in the paper must be understood as referring to non-trivial chains. It is clear that the trivial chain, i.e. the one that has just one element, cannot be embedded into a non-trivial algebra, but for the sake of readability we will omit this obvious restriction.

theorem, S$\mathcal{R}$C is equivalent to the embeddability of the subdirectly irreducible countable chains. As regards to the FS$\mathcal{R}$C, in some cases (see for instance [23, 21, 6] for Product, Łukasiewicz and BL logics), the result has been obtained by proving first that every chain of the equivalent variety semantics is partially embeddable into a standard algebra. For a long time, this condition was only known to be sufficient, but recently it has been proved that it is actually equivalent to the FS$\mathcal{R}$C.

Definition 10. *Given two algebras $\mathcal{A}$ and $\mathcal{B}$ of the same language we say that $\mathcal{A}$ is* partially embeddable *into $\mathcal{B}$ when every finite partial subalgebra of $\mathcal{A}$ is embeddable into $\mathcal{B}$. Generalizing this notion to classes of algebras, we say that a class $\mathbb{K}$ of algebras is* partially embeddable *into a class $\mathbb{M}$ if every finite partial subalgebra of a member of $\mathbb{K}$ is embeddable into a member of $\mathbb{M}$.*

Theorem 11. *If* L *is an algebraizable axiomatic expansion of* MTL *(in particular if it is an axiomatic extension of* MTL*), then*

(i) L *has the* FS$\mathcal{R}$C *if, and only if, the class of* L*-chains is partially embeddable into the class of standard* L*-algebras whenever the language of* L *is finite.*

(ii) L *has the* S$\mathcal{R}$C *if, and only if, every countable chain of* $\mathbb{L}$ *is embeddable into a standard* L*-chain.*

Notice that in (i) the implication from right to left is true even if the language is infinite.

Sometimes standard completeness properties can be refined with respect to some subclass of standard algebras; sometimes it is even enough to consider only one standard algebra. When the standard completeness can be proved with respect to a particular standard algebra which is the intended semantics for the logic, we call it *canonical* standard completeness. As a matter of fact, the equivalencies in the previous theorem remain true when restricted to some subclass of standard algebras.

3   About the logics L$_*$ of a WNM t-norm $*$

The canonical standard completeness is a matter of special interest when one considers the logic of the variety generated by the algebra defined by one particular t-norm, because then this t-norm gives the intended semantics for the logic.

Definition 12. *Let $*$ be a left-continuous t-norm.* L$_*$ *will denote the axiomatic extension of* MTL *whose equivalent algebraic semantics is* $\mathbf{V}([0,1]_*)$, *the variety generated by* $[0,1]_*$.

By definition, for every left-continuous t-norm $*$, the logic L$_*$ enjoys the $\mathcal{R}$C restricted to $[0,1]_*$, i.e. the canonical $\mathcal{R}$C. But what about (canonical) FS$\mathcal{R}$C and S$\mathcal{R}$C properties for the logics L$_*$ for $*$ being a left-continuous non-continuous t-norm?

Unlike the continuous case, there is no general representation theorem for left-continuous t-norms. However, some particular families of left-continuous non-continuous t-norms are well studied and even some finite axiomatizations are known for their corresponding logics. Namely, axiomatic extensions of the WNM logic (in particular the logics L$_*$ for $*$ being a WNM t-norm) and the corresponding varieties of WNM-algebras are studied in [13, 28, 29]. Moreover the variety of NM algebras (the WNM-algebras such that the corresponding

negation is involutive) and their subvarieties are fully studied in [20]. Next, we summarize the main results of all these papers.

The operations in WNM-chains are easily described. Let $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ be a WNM-chain. Then for every $a, b \in A$:

$$a \&^{\mathcal{A}} b = \begin{cases} a \wedge^{\mathcal{A}} b & \text{if } a > \neg b, \\ \overline{0}^{\mathcal{A}} & \text{otherwise.} \end{cases}$$

$$a \rightarrow^{\mathcal{A}} b = \begin{cases} \overline{1}^{\mathcal{A}} & \text{if } a \leq b, \\ \neg^{\mathcal{A}} a \vee^{\mathcal{A}} b & \text{otherwise.} \end{cases}$$

In [13] it is shown that the operation $*$ in WNM-chains defined over the real unit interval $[0, 1]$ is given by a special kind of left-continuous t-norm. These t-norms are defined in the following way. If n is a negation function[6] and $a, b \in [0, 1]$, the operation $*_n$ defined as:

$$a *_n b = \begin{cases} \min\{a, b\} & \text{if } a > n(b), \\ 0 & \text{otherwise,} \end{cases}$$

is a left-continuous t-norm and its residuum is given by:

$$a \Rightarrow_n b = \begin{cases} 1 & \text{if } a \leq b, \\ \max\{n(a), b\} & \text{otherwise,} \end{cases}$$

for every $a, b \in [0, 1]$. Moreover, it fulfills $a \Rightarrow_n 0 = n(a)$. It is straightforward that $[0, 1]_{*_n} := \langle [0, 1], *_n, \Rightarrow_n, \min, \max, 0, 1 \rangle$ is a WNM-chain, and all WNM-chains over $[0, 1]$ are of this form.

Notice that a standard WNM-chain given by a negation function n is an NM-chain if, and only if, n is involutive, i.e. $n(n(a)) = a$ for every $a \in [0, 1]$. It follows from the study of such negations in [35] that there is only one standard NM-chain up to isomorphism, namely the one given by the negation $n(x) = 1 - x$. We will refer to it as $[0, 1]_{\mathrm{NM}}$. The left-continuous t-norm corresponding to this algebra was introduced by Fodor in [17]. On the other hand, observe that $[0, 1]_{\mathrm{G}}$ is the standard WNM-chain defined by the so-called Gödel negation n_G ($n_G(0) = 1$ and $n_G(x) = 0$ for every $x > 0$), and in fact, n_G it is the only negation defining a continuous WNM t-norm.

Since standard WNM-chains are completely determined by their negation functions, the study of L_* logics when $[0, 1]_*$ is a WNM-chain, requires some knowlegde on the properties of such negations functions, see [10], and [13] for generalizations to MTL and WNM chains.

Lemma 13 ([10]). *Let $\mathcal{A}$ be a MTL-chain. Then for every $a \in A$:*

   (i) $\neg a = \neg\neg\neg a$,

   (ii) $a \leq \neg\neg a$,

   (iii) $a = \neg\neg a$ *if, and only if, there is $b \in A$ such that $a = \neg b$ (in such a case a is called involutive),*

   (iv) $\neg\neg a = \min\{b \in A \mid a \leq b \text{ and } b = \neg\neg b\}$.

---

[6]A non-increasing function $n : [0, 1] \rightarrow [0, 1]$ is a negation function if $x \leq n(n(x))$ for any $x \in [0, 1]$ and $n(1) = 0$, (see [10]).
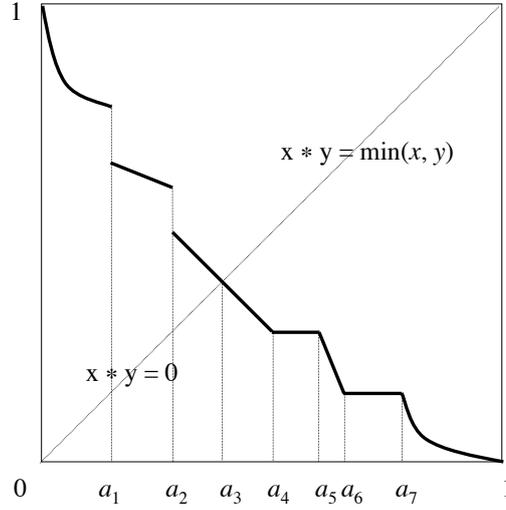
Figure 1: An example of WNM t-norm with a finite partition.

Moreover, when $A = [0, 1]$, $\neg$ is a left-continuous function.

The last result gives rise to the following definitions.

Definition 14. *Let $\mathcal{A}$ be a WNM-chain and let $a \in A$ be an involutive element. We define $I_a := \{b \in A \mid \neg b = \neg a\}$ and we call it the interval associated to a, where the negation function is constant with value $\neg a$. We say that a has a trivial associated interval when $I_a = \{a\}$.*

A weak negation function has a form of symmetry; roughly speaking: if we complete its graph by drawing vertical lines in the jumps, then the obtained graph is symmetric with respect to the diagonal $x = y$. Therefore, the constant intervals I_a in the positive part of the chain symmetrically correspond to jumps in the negative parts (and vice versa).

Definition 15. *We say that a WNM-chain $[0, 1]_{*_n}$, defined by a weak negation function $n : [0, 1] \to [0, 1]$, has the* finite partition property *if n is constant in a finite number of subintervals of $[0, 1]$. In such a case we define the associated finite partition by considering the set $X = \{0, 1\} \cup \{a \in (0, 1) \mid a$ is either the maximum or infimum of a non-trivial interval associated to an involutive element, a discontinuity of n, or the fixpoint\}. The family of intervals determined by X is called the* partition induced by the WNM t-norm $*_n$.

Notice that such a partition yields two kinds of intervals: those where the negation takes a constant value, and those where all the elements are involutive. As a matter of terminology, we call them *constant intervals* and *involutive intervals*, respectively. Figure 1 shows an example of a WNM t-norm with a fixpoint, a_3, and with a finite partition where the constant intervals are $[a_4, a_5]$ and $[a_6, a_7]$, while the involutive intervals are $[0, a_1]$, $[a_1, a_2]$, $[a_2, a_3]$, $[a_3, a_4]$, $[a_5, a_6]$ and $[a_7, 1]$.
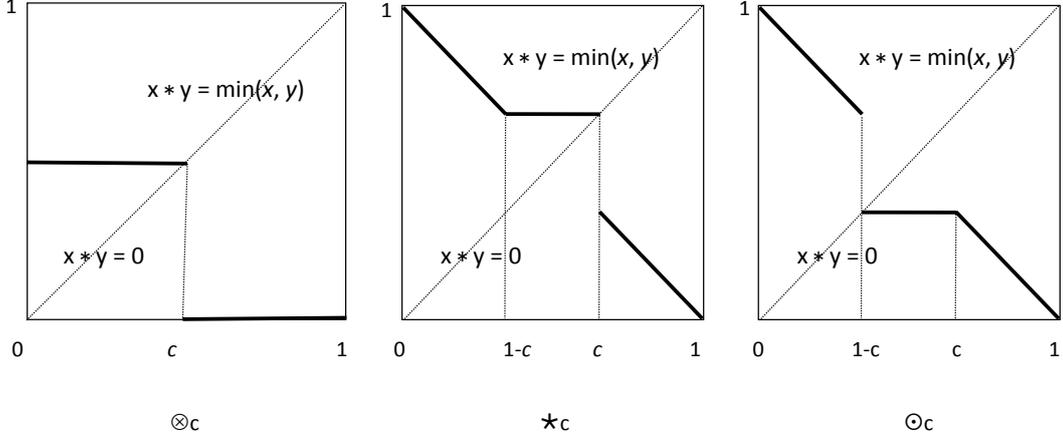
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Figure 2: Three parametric families of WNM t-norms with finite partition.

Figure 2 shows three families of WNM t-norms with finite partition of at most three intervals, parametrized by a real number c: $c \in [0,1)$ for $\otimes_c$, $c \in [1/2, 1)$ for $\star_c$ and $c \in [1/2, 1]$ for $\odot_c$. As limit cases for the parameter c, we get two well-known t-norms, namely, $\otimes_0 = \odot_1 = \min$ and $\star_{1/2} = \odot_{1/2}$ is the Nilpotent Minimum t-norm. Moreover, for each family, for values c different from these limit cases, one gets isomorphic t-norms in each of three families, and hence yielding the same logics. Actually, up to isomorphims, the t-norms depicted in Figure 2 together with the limit cases gives all the WNM t-norms with at most three associated intervals.

An interesting observation is that in any standard WNM-chain $[0,1]_{*_n}$, if a is positive element then $F_a = [a, 1]$ and the elements of the quotient algebra $[0,1]_*/F_a$ are such that

$$[x]_{F_a} = \begin{cases} [1]_{F_a}, & \text{if } x \in F_a \ \ (\text{i.e. if } x \geq a) \\ [0]_{F_a}, & \text{if } x \in \overline{F_a} \ \ (\text{i.e. if } x \leq n(a)) \\ \{x\}, & \text{otherwise} \end{cases}$$

Therefore, the quotient algebra $[0,1]_*/F_a$ is isomorphic to another standard WNM-chain. If a belongs to a constant interval, then this standard chain has $I_1 \neq \{1\}$, see Figure 3.

To refer to the class of WNM t-norms and those with a finite partition we will use from now on the following notation:

WNM $= \{* \text{ is a weak nilpotent minimum t-norm}\}$

WNM-fin $= \{* \in \textbf{WNM} \mid * \text{ has a finite partition}\}$[7]

In [28, 29] the following results have been proved:

Theorem 16. *In the context of* L_* *logics for* $* \in \textbf{WNM}$, *the following statements hold:*

1. *If* $* \in \textbf{WNM-fin}$, *then the logic* L_* *is finitely axiomatizable (and a method for finding the axiomatization has been given).*

---

[7]Observe that this notation is somewhat analogous to the one used in [12] when studying expansions with truth-constants of logics based on continuous t-norm, namely **CONT** $= \{* \text{ is a continuous t-norm}\}$ and **CONT-fin** $= \{* \in \textbf{CONT} \mid * \text{ is an ordinal sum of finitely many basic components}\}$.

2. *If* $* \in$ **WNM-fin** *and* $I_1 \neq \{1\}$ *or* $*$ *has no positive constant intervals, then* L_* *has the canonical* SRC, *i.e. with respect to the algebra* $[0,1]_*$.

3. *If* $* \in$ **WNM-fin**, $*$ *has some positive constant interval and* $I_1 = \{1\}$, *then* L_* *has the* SRC *with respect to the class* $\{[0,1]_*, [0,1]_*/F_a\}$, *where* a *is the maximum involutive element such that* $I_a \neq \{a\}$. *Moreover, this result cannot be improved, i.e.* L_* *does not enjoy the* SRC *with respect to only one of these two algebras.*

4. *If* $* \notin$ **WNM-fin** *and* $I_1 \neq \{1\}$, *then* L_* *has the canonical* FSRC, *i.e. with respect to the class* $\{[0,1]_*\}$.

5. *If* $* \notin$ **WNM-fin** *and* $I_1 = \{1\}$, *then* L_* *has the* FSRC *with respect to the class* $\{[0,1]_*\} \cup \{[0,1]_*/F_a \mid a \text{ positive involutive element such that } I_a \neq \{a\}\}$.

6. *There are* $* \notin$ **WNM-fin** *for which* L_* *has not the* SRC.

Figure 3 shows on the left an example of a t-norm $*$ of **WNM-fin** falling under item 3 of the last theorem, where $I_1 = \{1\}$ and a is the maximum involutive element such that $I_a \neq \{a\}$, while on the right it shows the t-norm defining the standard algebra isomorphic to the quotient algebra $[0,1]_*/F_a$.
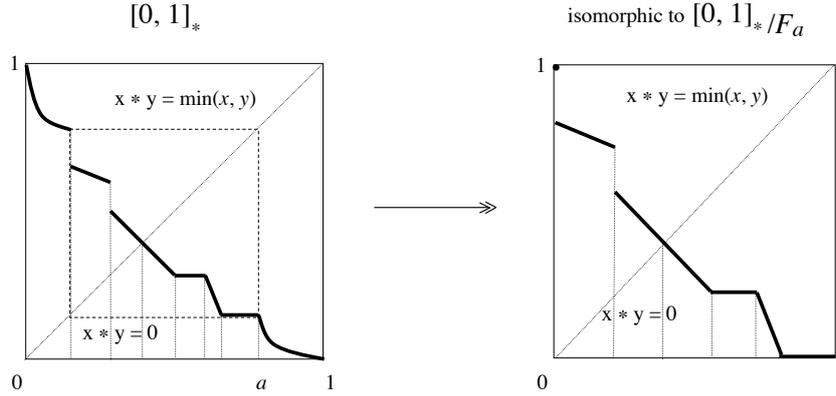

Figure 3: A WNM t-norm with a finite partition such that $I_1 = \{1\}$ (left) and its corresponding t-norm of the quotient algebra $[0,1]_*/F_a$.

4 Expansions of L_* logics with truth-constants

In this section we introduce the basic definitions and first general results regarding the expansions with truth-constants for those extensions of MTL which are the logic of a given left-continuous t-norm.

Definition 17 (Logic $L_*(\mathcal{C})$)**.** *Let* $*$ *be a left-continuous t-norm and let* $\mathcal{C} = \langle C, *, \Rightarrow, \min, \max, 0, 1 \rangle \subseteq [0,1]_*$ *be a* countable *subalgebra. Consider the expanded language* $\mathcal{L}_C = \mathcal{L} \cup \{\overline{r} \mid r \in C \setminus \{0,1\}\}$ *where we introduce a new constant for every element in* $C \setminus \{0,1\}$.

We define $L_*(\mathcal{C})$ *as the expansion of* L_* *in the language* $\mathcal{L}_C$ *obtained by adding the so-called book-keeping axioms:*

$$\overline{r} \& \overline{s} \leftrightarrow \overline{r * s}$$
$$(\overline{r} \to \overline{s}) \leftrightarrow \overline{r \Rightarrow s}$$

for every $r, s \in C$.

Notice that in this definition the book-keeping axioms $\overline{r} \wedge \overline{s} \leftrightarrow \overline{\min\{r, s\}}$ that would correspond to the other primitive connective in MTL, $\wedge$, are not present, since they are easily derivable in $L_*(\mathcal{C})$ as actually defined.

The algebraic counterparts of the $L_*(\mathcal{C})$ logics are defined in the natural way:

Definition 18. *Let* $*$ *be a left-continuous t-norm and let* $\mathcal{C}$ *be a countable subalgebra of* $[0,1]_*$. *An* $L_*(\mathcal{C})$-*algebra is a structure*

$$\mathcal{A} = \langle A, \&^{\mathcal{A}}, \to^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \{\overline{r}^{\mathcal{A}} \mid r \in C\}\rangle$$

such that:
  1. $\langle A, \&^{\mathcal{A}}, \to^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\rangle$ *is an* L_*-*algebra, and*
  2. *for every* $r, s \in C$ *the following identities hold:*
$$\overline{r}^{\mathcal{A}} \&^{\mathcal{A}} \overline{s}^{\mathcal{A}} = \overline{r * s}^{\mathcal{A}}$$
$$\overline{r}^{\mathcal{A}} \to^{\mathcal{A}} \overline{s}^{\mathcal{A}} = \overline{r \Rightarrow s}^{\mathcal{A}}.$$

The canonical standard $L_*(\mathcal{C})$-*chain is the algebra*

$$[0,1]_{L_*(\mathcal{C})} = \langle [0,1], *, \Rightarrow, \min, \max, \{r \mid r \in C\}\rangle,$$

i.e. the $\mathcal{L}_C$-*expansion of* $[0,1]_*$ *where the truth-constants are interpreted on their defining numbers.*

Since the additional symbols added to the language are 0-ary, the condition of algebraizability given in the prelimininaries is trivially fulfilled. Therefore, $L_*(\mathcal{C})$ is also an algebraizable logic and its equivalent algebraic semantics is the variety of $L_*(\mathcal{C})$-algebras, denoted as $\mathbb{L}_*(\mathcal{C})$. In particular this means that the logics $L_*(\mathcal{C})$ are strongly complete with respect to the variety of $L_*(\mathcal{C})$-algebras. Furthermore, reasoning as in the MTL case, we can prove that all $L_*(\mathcal{C})$-algebras are representable as a subdirect product of $L_*(\mathcal{C})$-chains, hence we also have completeness of $L_*(\mathcal{C})$ with respect to $L_*(\mathcal{C})$-chains.

Theorem 19. *For any* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_C}$, $\Gamma \vdash_{L_*(\mathcal{C})} \varphi$ *if, and only if,* $\{\psi \approx \overline{1} \mid \psi \in \Gamma\} \models_{\{L_*(\mathcal{C})-chains\}} \varphi \approx \overline{1}$.

This general completeness with respect to chains, can be refined by using [7, Lemma 3.4.4], where Cintula proves a very general result for expansions of fuzzy logics with rational truth-constants. Adapted to our framework, it reads as follows.

Theorem 20 ([7]). *Let* $*$ *be a left-continuous t-norm such that* L_* *is strongly complete with respect a class* $\mathbb{K}$ *of* L_*-*chains. Then* $L_*(\mathcal{C})$ *is strongly complete with respect to the class of* $L_*(\mathcal{C})$-*chains whose* $\mathcal{L}$-*reducts are in* $\mathbb{K}$.

Notice that when $\mathbb{K}$ is the class of all L_*-chains, then this theorem does not provide anything new other than the result of Theorem 19. If $\mathbb{K}$ is the class of standard L_*-chains, the condition that L_* should be strongly complete with respect to $\mathbb{K}$ (i.e. the $S\mathcal{R}C$ property)

is very demanding. For instance if we restrict ourselves to continuous t-norm based logics, then only Gödel logic G satisfies this condition. On the other hand, if we turn our attention to genuine left-continuous t-norms, the only well-known family with this property are the WNM t-norm logics described in Theorem 16.

Since all the logics $L_*(\mathcal{C})$ are expansions of MTL, sharing *modus ponens* as the only inference rule, they have the same local deduction-detachment theorem as MTL has. In fact, the proof for MTL or BL also applies here.

Theorem 21. *For every* $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}_C}$, $\Gamma, \varphi \vdash_{L_*(\mathcal{C})} \psi$ *if, and only if, there is a natural* $k \geq 1$ *such that* $\Gamma \vdash_{L_*(\mathcal{C})} \varphi^k \to \psi$. *If* $*$ *is a* WNM *t-norm, then one can always take* $k = 2$.

One can also show the following general result about the conservativity of $L_*(\mathcal{C})$ w.r.t. L_*.

Proposition 22. $L_*(\mathcal{C})$ *is a conservative expansion of* L_*.

Proof: Let $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ be arbitrary formulae and suppose that $\Gamma \vdash_{L_*(\mathcal{C})} \varphi$. Then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{L_*(\mathcal{C})} \varphi$. By the above deduction theorem, there exists a natural k such that $\vdash_{L_*(\mathcal{C})} (\Gamma_0)^k \to \varphi$, identifying the set Γ_0 with the strong conjunction of all its formulae. By soundness, this implies that $\models_{[0,1]_{L_*(\mathcal{C})}} (\Gamma_0)^k \to \varphi$. Since the new truth-constants do not occur in $\Gamma_0 \cup \{\varphi\}$, we have $\models_{[0,1]_*} (\Gamma_0)^k \to \varphi$, and by $\mathcal{R}C$ of L_*, $\vdash_{L_*} (\Gamma_0)^k \to \varphi$, and hence $\Gamma \vdash_{L_*} \varphi$ as well. $\qquad\qquad\square$

In the rest of the paper we will study the $\mathcal{R}C$, FS$\mathcal{R}$C and S$\mathcal{R}$C properties for the logics with truth-constants $L_*(\mathcal{C})$ where $*$ is a WNM t-norm, as well as canonical standard completeness properties.

5   Structure of $L_*(\mathcal{C})$-chains for WNM t-norms $*$

We have seen in Theorem 19 that the logics $L_*(\mathcal{C})$ are complete with respect to the $L_*(\mathcal{C})$-chains. Moreover, Theorem 20 gives strong standard completeness of the logics $L_*(\mathcal{C})$ for those $* \in$ **WNM-fin** which make the logic L_* to enjoy the S$\mathcal{R}$C (see Theorem 16). However, we need to reach a deeper insight into $L_*(\mathcal{C})$-chains to better study which classes of chains give standard completeness or when canonical standard completeness results hold. This is done in this section.

Next we assume $*$ is a left-continuous t-norm and $\mathcal{C}$ is a countable subalgebra of $[0,1]_*$.

Lemma 23. *For any* $L_*(\mathcal{C})$-*chain* $\mathcal{A} = \langle A, \&, \to, \wedge, \vee, \{\bar{r}^{\mathcal{A}} \mid r \in C\}\rangle$, *let* $F_{\mathcal{C}}(\mathcal{A}) = \{r \in C \mid \bar{r}^{\mathcal{A}} = \bar{1}^{\mathcal{A}}\}$ *and* $\overline{F_{\mathcal{C}}(\mathcal{A})} = \{r \in C \mid \neg r \in F_{\mathcal{C}}(\mathcal{A})\}$. *Then:*

(i) $F_{\mathcal{C}}(\mathcal{A})$ *is a filter of* $\mathcal{C}$.

(ii) *The set* $\{\bar{r}^{\mathcal{A}} \mid r \in C\}$ *forms an* L_*-*subalgebra of* $\mathcal{A}$ *isomorphic to* $\mathcal{C}/F_{\mathcal{C}}(\mathcal{A})$, *through the mapping* $\bar{r}^{\mathcal{A}} \mapsto [r]_{\mathcal{A}}$, *in such a way that*

$$[1]_{\mathcal{A}} = F_{\mathcal{C}}(\mathcal{A}) \text{ and } [0]_{\mathcal{A}} = \overline{F_{\mathcal{C}}(\mathcal{A})},$$

where $[r]_{\mathcal{A}}$ *denotes the equivalence class of* $r \in C$ *w.r.t. to the congruence defined by the filter* $F_{\mathcal{C}}(\mathcal{A})$.

Proof: (i) If $r \in F_{\mathcal{C}}(\mathcal{A})$ and $s \in C$ with $s > r$, then $s \in F_{\mathcal{C}}(\mathcal{A})$ because by the book-keeping axioms we have $\overline{s}^{\mathcal{A}} = \overline{\max(r,s)}^{\mathcal{A}} = \overline{r}^{\mathcal{A}} \vee \overline{s}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$. Moreover if $r, s \in F_{\mathcal{C}}(\mathcal{A})$ then $r * s \in F_{\mathcal{C}}(\mathcal{A})$ since $\overline{r * s}^{\mathcal{A}} = \overline{r}^{\mathcal{A}} \& \overline{s}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$. Therefore $F_{\mathcal{C}}(\mathcal{A})$ is a filter.

(ii) An easy computation shows that $\overline{s}^{\mathcal{A}} = \overline{r}^{\mathcal{A}}$ iff $\overline{(r \Rightarrow s) * (s \Rightarrow r)}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$, i.e. elements of the same class have to be interpreted by the same element of A while elements of different classes have to be interpreted by different elements of A. □

In general, the equivalence classes of $\mathcal{C}$ with respect to a filter F, i.e. the elements of $\mathcal{C}/F$, are difficult to describe, but some interesting cases can be indeed fully described, namely when $*$ is a continuous t-norm (see [12]) or when it belongs to **WNM**:

Lemma 24. *Let $* \in$ **WNM** and let $\mathcal{C}$ be a countable subalgebra of $[0,1]_*$. For any $F \in Fi(\mathcal{C})$ and for any $r, s \notin F \cup \overline{F}$, it holds that $[r]_F = [s]_F$ iff $r = s$.*

Proof: The proof is an easy generalization of the proof for NM and some particular WNM t-norm logics given in [14]. □

This lemma shows that the interpretation of the constants over a $L_*(\mathcal{C})$-chain $\mathcal{A}$ depends only on the filter $F_{\mathcal{C}}(\mathcal{A})$. Indeed, if $i : C \to \{\overline{r}^{\mathcal{A}} \mid r \in C\}$ denotes that interpretation, i.e. $i(r) = \overline{r}^{\mathcal{A}}$ for all $r \in C$, then i maps truth-values r to $\overline{1}^{\mathcal{A}}$ or $\overline{0}^{\mathcal{A}}$ depending on whether $r \in F_{\mathcal{C}}(\mathcal{A})$ or $r \in \overline{F_{\mathcal{C}}(\mathcal{A})}$ respectively, and over the rest of the elements of C, i.e. those in $C \setminus (F_{\mathcal{C}}(\mathcal{A}) \cup \overline{F_{\mathcal{C}}(\mathcal{A})})$, i is a one-to-one mapping.

The standard chains of the variety $\mathbb{L}_*(\mathcal{C})$, i.e. the $L_*(\mathcal{C})$-algebras over $[0,1]$, are the key to obtain standard completeness results for the logic $L_*(\mathcal{C})$ when using the technique of partially embedding $L_*(\mathcal{C})$-chains into standard ones. In order to know when such embeddings are possible, it is necessary to study the standard $L_*(\mathcal{C})$-chains in more detail. This question is in fact related to describe the ways the truth-constants from C can be interpreted in $[0,1]$ respecting the book-keeping axioms. We have seen in Lemmas 23 and 24 some necessary conditions showing the preeminent role of the set $Fi(\mathcal{C})$ of proper filters of $\mathcal{C}$ plays in this question. Observe that each proper filter of $\mathcal{C}$ is either of type $F_a = \{x \in C \mid x \geq a\}$ or of type $F_{>a} = \{x \in C \mid x > a\}$ for some positive $a \in C$.

One can wonder whether, given a filter $F \in Fi(\mathcal{C})$, there always exists a standard $L_*(\mathcal{C})$-chain $\mathcal{A}$ such that $F_{\mathcal{C}}(\mathcal{A}) = F$. Obviously, the simplest thing to look at is whether the algebra

$$[0,1]_{L_*(\mathcal{C})}^{F} = \langle [0,1], *, \Rightarrow_*, \min, \max, \{i_F(r) \mid r \in C\} \rangle,$$

where the mapping $i_F : C \to [0,1]$ is defined as

$$i_F(r) = \begin{cases} 1, & \text{if } r \in F \\ 0, & \text{if } r \in \overline{F} \\ r, & \text{otherwise} \end{cases} \tag{1}$$

is always an $L_*(\mathcal{C})$-algebra over $[0,1]_*$, or in other words, whether the mapping i_F is always a proper interpretation of the truth-constants, in the sense of satisfying the book-keeping axioms.

It is easy to check that this is actually the case when $*$ is continuous (see Prop 5.3 in [12]). The case of $L_*(\mathcal{C})$ logics when $* \in$ **WNM-fin** is not so simple. We illustrate the problem with an example. Let $*$ be the WNM t-norm depicted in the left hand side of Figure 3 and

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take $C = \mathbb{Q} \cap [0,1]$. Let a be a positive involutive element such that $I_a \neq \{a\}$ (i.e. such that a is the supremum of a constant interval) and let F_a be the principal filter generated by a. Then the mapping $i_{F_a} : C \to [0,1]$, defined as in expression (1), is not a proper interpretation of the truth-constants since for each $b \in I_a$, $\neg i(b) = \neg b = \neg a$ and $i(\neg b) = i(\neg a) = 0$, i.e. the book-keeping axioms are not satisfied and hence the algebra $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$ is not an $\mathrm{L}_*(\mathcal{C})$-algebra. Thus the mapping (1) used to interpret the truth-constants in the case of continuous t-norms does not always work in the case of a WNM t-norm.

In fact, for the case $* \in$ **WNM-fin**, if we want to associate to each filter $F \in Fi(\mathcal{C})$ a standard chain of $\mathbb{L}_*(\mathcal{C})$ such that $F_{\mathcal{C}}(\mathcal{A}) = F$, we need to proceed in a different way. We will divide the job by cases.

1. If the classes of $\mathcal{C}/F$ satisfy the condition that $\neg[r]_F = [0]_F$ implies $[r]_F = [1]_F$, then the interpretation used in the case of continuous t-norms works well and the chain $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$ is an $\mathrm{L}_*(\mathcal{C})$-chain like in the continuous case.

2. If in $\mathcal{C}/F$ there are classes such that

$$[r]_F \neq [1]_F \text{ (that is, } r \notin F) \text{ and } \neg[r]_F = [0]_F,$$

then the mapping $i_F : C \to [0,1]$ defined by expression (1) is not, in general, an interpretation as the example above proves.

Thus in this case, we consider two further subcases:

(a) If $[0,1]_*$ is such that $I_1^* \neq \{1\}$ (i.e. $\neg x = 0$ for some $x < 1$), then the mapping $i'_F : C \to [0,1]$ defined by,

$$i'_F(r) = \begin{cases} 1, & \text{if } r \in F \\ 0, & \text{if } r \in \overline{F} \\ f(r), & \text{if } \neg r = 0 \text{ and } r \notin (F \cup \overline{F}) \\ r, & \text{otherwise} \end{cases} \qquad (2)$$

where $f : \{r \in C \mid \neg r = 0, r \notin (F \cup \overline{F})\} \to I_1^*$ is an (arbitrary) one-to-one increasing mapping, is an interpretation which satisfies the book-keeping axioms. Then the algebra

$$[0,1]^F_{\mathrm{L}_*(\mathcal{C})} := \langle [0,1], *, \Rightarrow_*, \min, \max, \{i'_F(r) \mid r \in C\} \rangle$$

is an $\mathrm{L}_*(\mathcal{C})$-chain over $[0,1]_*$.

(b) If $[0,1]_*$ is such that $I_1^* = \{1\}$ (i.e. $\neg x = 0$ implies $x = 1$), then the mapping $i'_F : C \to [0,1]$ defined in the previous case does not apply here since having $I_1^* = \{1\}$ makes impossible to define a one-to-one mapping f as required there. In this case we take as initial chain, not the standard chain $[0,1]_*$, but the chain $([0,1]_*)/F_a$ (which still belongs to the variety $\mathbb{L}_*$) where $a \in C$ is the greatest element in the constant intervals of $[0,1]_*$. Notice that $[1]_{F_a} = [a,1]$, $[0]_{F_a} = [0, \neg a]$ and $[r]_{F_a} = \{r\}$ for any $r \in (\neg a, a)$. Hence, $([0,1]_*)/F_a$ is isomorphic to an L_*-chain $[\neg a, a]_{*'}$ by identifying $[1]_{F_a}$ with a, $[0]_{F_a}$ with $\neg a$, and $[r]_{F_a}$ with r for all $r \in (\neg a, a)$, and by taking $*'$ as the obvious adaptation to the interval $[\neg a, a]$ of

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the original $*$. Now it is clear that $[\neg a, a]_{*'}$ is such that $I_1^{*'} \neq \{1\}$ and therefore we can define a mapping $i''_F : C \to [\neg a, a]$ analagously to (2) which makes the algebra

$$\langle [\neg a, a], *', \Rightarrow_{*'}, \min, \max, \{i''_F(r) \mid r \in C\} \rangle$$

an $\mathrm{L}_*(\mathcal{C})$-chain. Finally, by means of an increasing linear transformation $h : [\neg a, a] \to [0, 1]$, it is easy to obtain an isomorphic $\mathrm{L}_*(\mathcal{C})$-chain over $[0, 1]$

$$[0, 1]_{\mathrm{L}_*(\mathcal{C})}^F := \langle [0, 1], \circ, \Rightarrow_\circ, \min, \max, \{j_F(r) \mid r \in C\} \rangle$$

where $x \circ y = h(h^{-1}(x) *' h^{-1}(y))$ and $j_F(r) = h(i''_F(r))$ for all $r \in C$. Notice that $\circ$ needs not coincide with $*$.

Notice that the algebra $[0, 1]_{\mathrm{L}_*(\mathcal{C})}^F$ built in case (a) and in case (b) is not univocally defined since its definition depends on the choice of some mappings, but all possible choices would yield isomorphic algebras.

Thus, we have the following corollary:

Corollary 25. *Let* $* \in$ **WNM-fin** *and let* $\mathcal{C}$ *be a countable subalgebra of* $[0, 1]_*$. *Then for any filter* $F \in Fi(\mathcal{C})$, *there exists a standard* $\mathrm{L}_*(\mathcal{C})$-*chain* $\mathcal{A}$ *such that* $F_{\mathcal{C}}(\mathcal{A}) = F$, *namely* $\mathcal{A} = [0, 1]_{\mathrm{L}_*(\mathcal{C})}^F$.

Any standard $\mathrm{L}_*(\mathcal{C})$-chain $\mathcal{A}$ such that $F_{\mathcal{C}}(\mathcal{A}) = F$ will be called from now on *standard* $\mathrm{L}_*(\mathcal{C})$-*chain of type* F.

6   Completeness results

In this section we will prove completeness results for the logics $\mathrm{L}_*(\mathcal{C})$ in the following particular case:

- when $* \in$ **WNM-fin** and $\mathcal{C}$ is a countable subalgebra of $[0, 1]_*$ such that has elements in the interior of each interval of the partition.[8]

Thus, from now on we will assume that the algebra $\mathcal{C}$ satisfies these conditions.

In the following subsection we will focus on strong and finite strong standard completeness results while in the second subsection we will focus on the issue of canonical standard completeness.

6.1   About $S\mathcal{R}C$ and $FS\mathcal{R}C$ results

We start with a general result on strong standard completeness when $* \in$ **WNM-fin** which is a consequence of Cintula's Theorem 20 and the $S\mathcal{R}C$ results given in statements 2 and 3 of Theorem 16.

Theorem 26. *For every* $* \in$ **WNM-fin** *and every suitable* $\mathcal{C}$, *the logic* $\mathrm{L}_*(\mathcal{C})$ *enjoys the* $S\mathcal{R}C$ *restricted to the family* $\{[0, 1]_{\mathrm{L}_*(\mathcal{C})}^F \mid F \in Fi(\mathcal{C})\}$.

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[8]This condition is analogous to that of [12] where $* \in$ **CONT-fin** and $\mathcal{C}$ is required to have elements in the interior of each component of the decomposition of $*$ as an ordinal sum of basic components.

	G($\mathcal{C}$)	NM($\mathcal{C}$)	L$_*$($\mathcal{C}$), for other $* \in$ **WNM-fin**
$\mathcal{RC}$	Yes	Yes	Yes
FS$\mathcal{RC}$	Yes	Yes	Yes
S$\mathcal{RC}$	Yes	Yes	Yes
Canonical FS$\mathcal{RC}$	No	No	No
Canonical S$\mathcal{RC}$	No	No	No

Table 3: (Finite) strong standard completeness results for logics of a t-norm from **WNM-fin**.

As particular cases of the above theorem we obtain that the logics G($\mathcal{C}$) and NM($\mathcal{C}$) enjoy the S$\mathcal{RC}$ restricted to the corresponding family $\{[0,1]^F_{\mathrm{L}_*(\mathcal{C})} \mid F \in Fi(\mathcal{C})\}$. If differs radically from the situation in continuous t-norm based logics L$_*$($\mathcal{C}$) where the S$\mathcal{RC}$ fails for each continuous $* \neq \min$ (see [12]).

Notice that these results can never be improved to canonical FS$\mathcal{RC}$, as the following example shows.

Example 1. *For every non-trivial filter F (that exists in all these cases) and every $r \in F\backslash\{1\}$, the derivation*
$$(p \rightarrow q) \rightarrow \overline{r} \models q \rightarrow p$$
is valid in $[0,1]_{\mathrm{L}_*(\mathcal{C})}$ but not in $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$.

These results are collected in Table 3.

6.2 About canonical standard completeness

From the results of the last sections, we already know that all the considered logics enjoy the $\mathcal{RC}$ restricted to the family of standard chains associated to proper filters of $\mathcal{C}$, i.e, their theorems are exactly the common tautologies of the chains of the family $\{[0,1]^F_{\mathrm{L}_*(\mathcal{C})} \mid F \in Fi(\mathcal{C})\}$. But, although the logics considered in the last sections do not enjoy the canonical FS$\mathcal{RC}$ (even in the continuous case only L($\mathcal{C}$) enjoys it when $C \subseteq \mathbb{Q} \cap [0,1]$), some of them still have the canonical $\mathcal{RC}$, i.e. their theorems are exactly the tautologies of their corresponding canonical standard algebra. In order to prove it, we need to show that tautologies of the canonical standard chain are a subset of the tautologies of each one of the standard chains associated to each proper filter of $\mathcal{C}$.

In [14] it is proved that the expansions of Gödel logic, NM logic and the logics corresponding to the t-norms $\otimes_c$ and $\star_c$ depicted in Figure 2 enjoy the canonical $\mathcal{RC}$.[9] Here we give a new unified and simpler proof.

Theorem 27. *If $* \in$ **WNM-fin** such that its negation on the set of positive elements is either both involutive and continuous, or is identically 0, then L$_*$($\mathcal{C}$) enjoys the canonical $\mathcal{RC}$.*

Proof: Suppose φ is a tautology with respect to $[0,1]_{\mathrm{L}_*(\mathcal{C})}$. We will prove that φ is also a tautology with respect to $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$ for each $F \in Fi(\mathcal{C})$, which implies that $\vdash_{\mathrm{L}_*(\mathcal{C})} \varphi$. Let e be an interpretation over the chain $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$. Suppose that $\mathcal{A}$ is the finite algebra

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[9]In [14] it is wrongly claimed that the expansions L$_*$($\mathcal{C}$) for $* = \odot_c$ (see Figure 2) were also canonical standard complete, in Example 2 we provide a counter-example.

generated by $\{e(\psi) \mid \psi \text{ subformula of } \varphi\}$ and $\alpha = \min\{r \in F \mid \bar{r} \text{ occurs in } \varphi\}$. Suppose that $f : (\neg\alpha, \alpha) \to (0, 1)$ is a bijection such that $f(r) = r$ for all $r \notin F \cup \overline{F}$ such that $\bar{r}$ in φ and f is a homomorphism from $\mathcal{A}$ to the canonical standard chain. Then define an evaluation e' on the canonical standard chain defined by $e'(p) = f^{-1}(e(p))$ if p is a propositional variable that appears in φ and $e'(p) = 1$ otherwise. Since φ is a tautology for the canonical standard chain, $e'(\varphi) = 1$. Take the algebra $[0, 1]_*/F_\alpha$ where F_α is the principal filter generated by α. By hypothesis this algebra is isomorphic to $[0, 1]_*$. Define the evaluation e'' on the quotient algebra obtained from e' and it obviously satisfies $e''(\varphi) = [1]_{F_\alpha}$. But a simple computation shows that the algebra $\mathcal{B}$ generated by $\{e''(\psi) \mid \psi \text{ subformula of } \varphi\}$ is isomorphic to $\mathcal{A}$ and $e''(\varphi)$ over the quotient algebra corresponds to $e(\varphi)$ over the chain $[0, 1]^F_{\mathrm{L}_*(\mathcal{C})}$ and thus $e(\varphi) = 1$. $\qquad\square$

Actually, the only expansions of logics L_* with $* \in \mathbf{WNM\text{-}fin}$ that enjoy the canonical $\mathcal{R}C$ are those falling under the hypotheses of last theorem. This is proved below by showing counterexamples for the remaining cases, where $p(x)$ and $n(x)$ denote the terms $x \vee \neg x$ and $x \wedge \neg x$ respectively.

Example 2. *Let $* \in \mathbf{WNM\text{-}fin}$ not falling under the hypotheses of last theorem. We distinguish the following three cases:*

- *Suppose the negation is continuous on the set of positive elements and the only constant interval formed by positive elements is I_1. In such a case, there is an interval I of involutive positive elements, followed by I_1. Take a truth-constant b in the interior of I. Then the formula*

$$(\neg\neg p(x) \to p(x)) \vee (\bar{b} \to p(x))$$

  is a tautology for $[0, 1]_{\mathrm{L}_*(\mathcal{C})}$ and it is not a tautology for $[0, 1]^F_{\mathrm{L}_*(\mathcal{C})}$ for any F containing b. Take into account that in $[0, 1]_{\mathrm{L}_*(\mathcal{C})}$ a positive element is either involutive or greater than b.

- *Suppose the negation is continuous on the set of positive elements and there is some constant interval formed by positive elements different from I_1 (this is the case of the family of t-norms $\odot_c$ in Figure 2). Let b be the minimum involutive positive element with a non-trivial associated interval. Then the formula*

$$(\neg\neg p(x) \to p(x)) \vee (\neg p(x) \to \neg\bar{b})$$

  is a tautology for $[0, 1]_{\mathrm{L}_*(\mathcal{C})}$ and it is not a tautology for $[0, 1]^F_{\mathrm{L}_*(\mathcal{C})}$ for any F containing b. Notice that in this case $[0, 1]^F_{\mathrm{L}_*(\mathcal{C})}$ is such that either a positive element is involutive or its negation is not greater than $\neg b$.

- *Suppose the negation is not continuous on the set of positive elements. Let b be the minimum discontinuity point of the negation function in the set of positive elements. Then $I_{\neg b}$ is the greatest constant interval in the negative part with biggest element $\neg b$ and not containing the fixpoint. Then the formula*

$$(\neg\neg n(x) \to n(x)) \vee (\neg n(x) \to \neg\neg n(x)) \vee (n(x) \to \neg\bar{b})$$

  is a tautology for $[0, 1]_{\mathrm{L}_*(\mathcal{C})}$ and it is not a tautology for $[0, 1]^F_{\mathrm{L}_*(\mathcal{C})}$ for any F containing b. Notice that in $[0, 1]_{\mathrm{L}_*(\mathcal{C})}$ a negative element is either involutive or belongs to a constant interval whose greatest element is the fixpoint (if it exists) or it is less or equal than $\neg b$.

All the results on canonical $\mathcal{R}C$ are gathered in Table 4.

$[0,1]_*$	Canonical $\mathcal{RC}$ for $L_*(\mathcal{C})$
$[0,1]_{\mathrm{NM}}$	Yes
$[0,1]_{\otimes_c}$	Yes
$[0,1]_{\star_c}$	Yes
$[0,1]_*$, for other $* \in$ **WNM-fin**	No

Table 4: Canonical standard completeness results for logics $L_*(\mathcal{C})$ when $* \in$ **WNM-fin**. Recall that $\otimes_c$, and $\star_c$ are those WNM t-norms depicted in Figure 2.

7 Completeness results for evaluated formulae

This section deals with completeness results when we restrict to what we call *evaluated formulae*, formulae of type $\overline{r} \to \varphi$, where φ is a formula without new truth-constants (different from $\overline{0}$ and $\overline{1}$). These formulae can be seen as a special kind of Novák's *evaluated formulae*, which are expressions a/A where a is a truth value (as in our definition) but A is a formula that may contain truth-constants again, and whose interpretation is that the truth-value of A is at least a. Hence our formulae $\overline{r} \to \varphi$ would be expressed as r/φ in Novák's evaluated syntax. On the other hand, formulae $\overline{r} \to \varphi$ when φ is a Horn-like rule of the form $b_1 \& \ldots \& b_n \to h$ also correspond to typical fuzzy logic programming rules $(b_1 \& \ldots \& b_n \to h, r)$, where r specifies a lower bound for the validity of the rule.

From the previous sections we know that the FS$\mathcal{RC}$ is true for the expansion of L_* with a suitable subalgebra of truth-constants (not only for evaluated formulae). But restricting the language to evaluated formulae these results can be improved.

For the case of **WNM-fin** t-norms, the only available results are those from [14] for evaluated formulae of the kind $\overline{r} \to \varphi$ where r is a positive constant (i.e. $r > \neg r$), that we will call *positively evaluated formulae*. Indeed, as shown in [15, Prop. 3], canonical FS$\mathcal{RC}$ can only be achieved with these kind of formulae.

Theorem 28 ([14])**.** *If $*$ is either $\otimes_c$ or $\star_c$ in Fig. 2, then $L_*(\mathcal{C})$ has the following canonical FS$\mathcal{RC}$ if we restrict the language to positively evaluated formulae:*

$$\{\overline{r}_i \to \varphi_i\}_{i \in I} \vdash_{L_*(\mathcal{C})} \overline{s} \to \psi \text{ iff } \{\overline{r}_i \to \varphi_i\}_{i \in I} \models_{[0,1]_{L_*(\mathcal{C})}} \overline{s} \to \psi.$$

where I is a finite index set, $\psi, \varphi_i \in Fm_{\mathcal{L}}$ and $r_i \in (c, 1]$.

For $* \in$ **WNM-fin** other than $\otimes_c, \star_c$ the canonical FS$\mathcal{RC}$ restricted to positively evaluated formulae does not hold as the following counterexamples show.

Example 3. *Let $* = \odot_c$ in Fig. 2 with $c > 1/2$. Let $r \in C$ such that $1 - c < r \le c$. Then the semantical deduction*

$$\neg\neg p(x) \to p(x) \models \overline{r} \to p(x)$$

is valid in $[0,1]_{L_*(\mathcal{C})}$ but not in $[0,1]_{L_*(\mathcal{C})}^F$ for any F containing r. Obviously, in $[0,1]_{L_*(\mathcal{C})}$ any involutive and positive element is greater than r.

Example 4. *Let $* \in$ **WNM-fin** be such that the first interval I of the partition associated to $*$ formed by positive elements is involutive and there is a constant interval on the right of it. In such a case, take a truth-constant r in the interior of I. Then the semantical deduction,*

$$(\neg\neg p(x) \to p(x)) \to p(x) \models \overline{r} \to p(x)$$

is valid in $[0,1]_{\mathrm{L}_*(\mathcal{C})}$ but not in $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$ for any F containing r. Observe that in $[0,1]_{\mathrm{L}_*(\mathcal{C})}$ the premise is true if, and only if, $p(x)$ is not involutive or 1, and for these cases $p(x)$ is greater than r.

Example 5. *Let $* \in$ **WNM-fin** such that the first interval of the partition associated to $*$ formed by positive elements is a constant interval with respect to the negation (I_c being c the biggest element of the interval). Additionally suppose that there is another interval of positive elements that is also a constant interval with respect to the negation. In such a case, take a truth-constant $r \in I_c$. Then the formula,*

$$\overline{r} \to \neg\neg p(x)$$

is a tautology for $[0,1]_{\mathrm{L}_*(\mathcal{C})}$ and it is not a tautology for $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$ for any F containing r. Obviously in $[0,1]_{\mathrm{L}_*(\mathcal{C})}$ any involutive and positive element is greater than r.

Example 6. *Let $* \in$ **WNM-fin** be such that there is a positive element which is a discontinuity point of the negation function. Then, due to symmetry of negation functions, there is a constant interval whose elements are negative and whose greatest element is not the fixpoint. Denote by I the greatest constant interval formed by negative elements whose greatest element is different from the fixpoint and take r as the greatest element of I, i.e. $I = I_r$. Then the semantical deduction,*

$$\left\{ \begin{array}{l} \neg\neg n(x) \to \neg(\neg\neg n(x) \to n(x)), \\ \neg n(x) \to \neg(\neg n(x) \to \neg\neg n(x)) \end{array} \right\} \models \neg\overline{r} \to \neg n(x)$$

is valid deduction in $[0,1]_{\mathrm{L}_*(\mathcal{C})}$ but it is not in $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$ for any F containing r. Observe that the first premise is true if, and only if, $n(x)$ is either not involutive or $n(x) = 0$ and the second premise is true if and only if $n(x)$ does not belong to a constant interval whose greatest element is the fixpoint. Thus, if x satisfies the premises, it is clear that $n(x)$ belongs to a constant interval which does not contain the fixpoint, thus it is less or equal to r, and hence the conclusion is also satisfied.

These four examples, as in the case of general canonical $\mathcal{R}C$ studied in the last section, prove that a rather large family of expansions of the logic of a WNM t-norm with truth-constants do not enjoy the canonical FS$\mathcal{R}C$ even when we restrict the language to positively evaluated formulae. Indeed only four logics logics are not covered for the previous examples, namely $\mathrm{G}(\mathcal{C})$, $\mathrm{NM}(\mathcal{C})$ and the logics $\mathrm{L}_*(\mathcal{C})$, for$*$ being either $\otimes_c$ with $c > 0$ or $\star_c$ with $c > 1/2$ (see Figure 2). Thus only these four logics with truth-constants enjoy the canonical FS$\mathcal{R}C$ when restricted to positively evaluated formulae.

As for canonical $\mathcal{R}C$, the same examples above indeed prove that canonical $\mathcal{R}C$ does not hold except for the previous four logics. The proof is based in the use, when applicable, of the deduction theorem for WNM and their extensions, which reads as follows: $\varphi \vdash_{\mathrm{L}} \psi$ iff $\vdash_{\mathrm{L}} \varphi^2 \to \psi$, for any extension L of WNM. The result of applying the deduction theorem to the premises in the the above Examples 3, 4 and 6 yield non evaluated formulae, but they are actually equivalent to an evaluated formula using the exchange property of the implication. Indeed, the formulae obtained then in each of these 3 examples are respectively:

$$\overline{r} \to ((\neg\neg p(x) \to p(x))^2 \to p(x)),$$

$$\overline{r} \to (((\neg\neg p(x) \to p(x)) \to p(x))^2 \to p(x)),$$

	Restricted to pos. evaluated formulae of $L_*(\mathcal{C})$	
$[0,1]_*$	Canonical $\mathcal{R}C$,	Canonical FS$\mathcal{R}C$
$[0,1]_{\mathrm{NM}}$	Yes	
$[0,1]_{\otimes_c}$	Yes	
$[0,1]_{\star_c}$	Yes	
$[0,1]_*$, for other $* \in$ **WNM-fin**	No	

Table 5: Canonical $\mathcal{R}C$ and FS$\mathcal{R}C$ results restricted to positively evaluated formulae for logics $L_*(\mathcal{C})$ when $* \in$ **CONT-fin** $\cup$ **WNM-fin**.

and

$$\neg \overline{r} \to [((\neg\neg n(x) \to \neg(\neg\neg n(x) \to n(x))) \& (\neg n(x) \to \neg(\neg n(x) \to \neg\neg n(x)))^2 \to \neg n(x)].$$

Obviously, these three formulae and the one in Example 5, $\overline{r} \to \neg\neg p(x)$, where the truth-constant $\overline{r}$ is suitably chosen as in each corresponding example above, are evaluated to 1 for each canonical evaluation but not when $\overline{r}$ is evaluated as 1.

A summary of these completeness results is shown in Table 5. Furthermore, comparing this table with Table 4 we realise that for a logic $L_*(\mathcal{C})$ where $* \in$ **WNM-fin** the canonical $\mathcal{R}C$ turns out to be equivalent to the canonical $\mathcal{R}C$ (and to the canonical FS$\mathcal{R}C$) restricted to positively evaluated formulae.

A short remark about the canonical $S\mathcal{R}C$ property. Very recently, and thus after the original manuscript of this paper was written, the same authors have published [15] where the rational completeness properties of expansions of t-norm based logics with truth-constants. In particular, in that paper the authors solve the question whether these expansions of WNM t-norm logics enjoy the *canonical strong rational completeness*, denoted CanS$\mathcal{Q}$C in [15], for positively evaluated formulae. It turns out that an easy checking reveals that the same proofs apply for real semantics as well. For the readers' convenience, we show in Table 6 the adapted canonical $S\mathcal{R}C$ results, where C^+ denotes the set of positive elements of the algebra $\mathcal{C}$, and $\mathcal{P}_{\sup-acc}([0,1])$ denotes the set of subsets of $[0,1]$ containing at least one *sup-accessible* point. By sup-accessible point it is meant an accumulation point $r \in C$ which is the supremum of a strictly increasing sequence $\langle r_i \rangle_{i \in \mathbb{N}}$ of points of C.

8 Adding truth-constants to expansions with Δ connective

For every left-continuous t-norm $*$, consider the logic $L_{*\Delta}$, the expansion of the logic L_* by adding to the language the unary connective Δ as introduced in Section 2.

Since there is a one-to-one correspondence between L_*-chains and $L_{*\Delta}$-chains, Theorem 11 leads to the next statement about the $S\mathcal{R}C$ and FS$\mathcal{R}C$ of logics $L_{*\Delta}$.

Theorem 29. *For any left-continuous t-norm $*$, L_* has the $S\mathcal{R}C$ (resp. FS$\mathcal{R}C$) with respect to a class of standard L_*-chains $\mathbb{K}$ if, and only if, $L_{*\Delta}$ has the $S\mathcal{R}C$ (resp. FS$\mathcal{R}C$) with respect to the class of standard $L_{*\Delta}$-chains $\mathbb{K}_\Delta$, where $\mathbb{K}_\Delta$ denotes the class of Δ-expansions of chains in $\mathbb{K}$.*

Logic	Canonical $\mathcal{SRC}$
$\mathrm{G}(\mathcal{C})$, $C^+ \in \mathcal{P}_{\sup-acc}([0,1])$	No
$\mathrm{G}(\mathcal{C})$, $C^+ \notin \mathcal{P}_{\sup-acc}([0,1])$	Yes
$\mathrm{NM}(\mathcal{C})$, $C^+ \in \mathcal{P}_{\sup-acc}([0,1])$	No
$\mathrm{NM}(\mathcal{C})$, $C^+ \notin \mathcal{P}_{\sup-acc}([0,1])$	Yes
$\mathrm{L}_{\otimes_c}(\mathcal{C})$, $C^+ \in \mathcal{P}_{\sup-acc}([0,1])$	No
$\mathrm{L}_{\otimes_c}(\mathcal{C})$, $C^+ \notin \mathcal{P}_{\sup-acc}([0,1])$	Yes
$\mathrm{L}_{\star_c}(\mathcal{C})$, $C^+ \in \mathcal{P}_{\sup-acc}([0,1])$	No
$\mathrm{L}_{\star_c}(\mathcal{C})$, $C^+ \notin \mathcal{P}_{\sup-acc}([0,1])$	Yes
$\mathrm{L}_*(\mathcal{C})$, for other $* \in$ **WNM-fin**	No

Table 6: Canonical $\mathcal{SRC}$ properties for propositional WNM fuzzy logics with truth-constants restricted to positively evaluated formulae.

Now we will consider expansions with truth-constants for these logics with Δ. Given a left-continuous t-norm $*$ and a countable subalgebra $\mathcal{C} \subseteq [0,1]_*$, we define the logic $\mathrm{L}_{*\Delta}(\mathcal{C})$ as the expansion of $\mathrm{L}_{*\Delta}$ in the language $\mathcal{L}_C$ obtained by adding the following book-keeping axioms:

$$\overline{r} \& \overline{s} \leftrightarrow \overline{r * s}$$
$$(\overline{r} \rightarrow \overline{s}) \leftrightarrow \overline{r \Rightarrow s}$$
$$\Delta \overline{r} \leftrightarrow \overline{\Delta r}$$

for every $r, s \in C$.

Again, using the general facts mentioned in the preliminaries we know that $\mathrm{L}_{*\Delta}(\mathcal{C})$ is an algebraizable logic and we can axiomatize its equivalent algebraic semantics, the variety of $\mathrm{L}_{*\Delta}(\mathcal{C})$-algebras. Moreover, it can be easily checked that $\mathrm{L}_{*\Delta}(\mathcal{C})$-algebras are representable as subdirect product of chains.

Proposition 30. *For every left-continuous t-norm $*$ and every countable subalgebra $\mathcal{C} \subseteq [0,1]_*$, the logic $\mathrm{L}_{*\Delta}(\mathcal{C})$ is a conservative expansion of $\mathrm{L}_{*\Delta}$, whenever $\mathrm{L}_{*\Delta}$ has the FSRC.*

Proof: Let us denote by $\mathbb{S}$ is the class of standard $\mathrm{L}_{*\Delta}$-chains and by $\mathbb{S}(\mathcal{C})$ is the class of standard $\mathrm{L}_{*\Delta}(\mathcal{C})$-chains. Let $\Gamma \cup \{\varphi\}$ be arbitrary formulae of $\mathrm{L}_{*\Delta}$ and suppose that $\Gamma \vdash_{\mathrm{L}_{*\Delta}(\mathcal{C})} \varphi$. Then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathrm{L}_{*\Delta}(\mathcal{C})} \varphi$, and this implies that $\Gamma_0 \models_{\mathbb{S}(\mathcal{C})} \varphi$. Since the new truth-constants do not occur in $\Gamma_0 \cup \{\varphi\}$, we have $\Gamma_0 \models_{\mathbb{S}} \varphi$, and by FSRC of $\mathrm{L}_{*\Delta}$, $\Gamma_0 \vdash_{\mathrm{L}_{*\Delta}} \varphi$, and hence $\Gamma \vdash_{\mathrm{L}_{*\Delta}} \varphi$. $\square$

Hence, for each $* \in$ **WNM**, $\mathrm{L}_{*\Delta}(\mathcal{C})$ is a conservative expansion of $\mathrm{L}_{*\Delta}$. Since $\mathrm{L}_{*\Delta}$-chains are simple, adding Δ to $\mathrm{L}_*(\mathcal{C})$-chains simplifies significantly their structure as next lemma shows.

Lemma 31. *Let $\mathcal{A}$ be a non-trivial $\mathrm{L}_{*\Delta}(\mathcal{C})$-chain, $*$ be a left-continuous t-norm and $\mathcal{C} \subseteq [0,1]_*$ be a countable subalgebra. Then, for every $r, s \in C$ such that $r < s$, we have $\overline{r}^{\mathcal{A}} < \overline{s}^{\mathcal{A}}$.*

Proof: Suppose $\overline{r}^{\mathcal{A}} = \overline{s}^{\mathcal{A}}$. Then $\overline{1}^{\mathcal{A}} = \Delta \overline{1}^{\mathcal{A}} = \Delta \overline{s \rightarrow r}^{\mathcal{A}} = \overline{\Delta(s \rightarrow t)}^{\mathcal{A}} = \overline{0}^{\mathcal{A}}$; a contradiction. $\square$

Therefore, in the variety of $L_{*\Delta}(\mathcal{C})$-algebras all chains $\mathcal{A}$ are such that $F_{\mathcal{C}}(\mathcal{A}) = \{1\}$, among them the canonical standard chain that we denote by $[0,1]_{L_{*\Delta}(\mathcal{C})}$. Furthermore, observe that the condition that $\mathcal{C}$ has elements in the interior of each interval of the partition, together with the previous lemma and the book-keeping axioms, implies that all $L_{*\Delta}(\mathcal{C})$-chains must have the same kind of partition (the same sequence of constant and involutive intervals). In particular, this implies that $[0,1]_{L_{*\Delta}(\mathcal{C})}$ is the only standard $L_{*\Delta}(\mathcal{C})$-chain up to isomorphism.

Theorem 32. *Let $* \in$ **WNM-fin** and let $\mathcal{C} \subseteq [0,1]_*$ be a suitable countable subalgebra. Then:*

1. $L_{*\Delta}(\mathcal{C})$ *has the canonical* $\mathcal{SRC}$.

2. $L_{*\Delta}(\mathcal{C})$ *is not a conservative expansion of* $L_*(\mathcal{C})$.

Proof: The fact that all $L_{*\Delta}(\mathcal{C})$-chains have the same kind of partition gives that all the countable ones are embeddable into $[0,1]_{L_{*\Delta}(\mathcal{C})}$. Thus Theorem 11 ensures the canonical $\mathcal{SRC}$. Let us now prove the second statement. Recall the example in Section 6.1 that showed $L_*(\mathcal{C})$ does not enjoy the canonical $\mathcal{FSRC}$: $(p \to q) \to \overline{r} \models_{[0,1]_{L_*(\mathcal{C})}} q \to p$ and $(p \to q) \to \overline{r} \not\vdash_{L_*(\mathcal{C})} q \to p$. But then, $(p \to q) \to \overline{r} \models_{[0,1]_{L_{*\Delta}(\mathcal{C})}} q \to p$ and hence $(p \to q) \to \overline{r} \vdash_{L_{*\Delta}(\mathcal{C})} q \to p$, by the canonical $\mathcal{SRC}$ of $L_{*\Delta}(\mathcal{C})$. Therefore, $L_{*\Delta}(\mathcal{C})$ is not a conservative expansion of $L_*(\mathcal{C})$. $\square$

9 Complexity results

In the paper [22], Hájek has studied the computational complexity of relevant subsets of formulae with rational truth-constants, i.e. formulae of the language $\mathcal{L}_C$ where $C = \mathbb{Q} \cap [0,1]$. Here we follow his approach to determine the computational complexity of some logics $L_*(\mathcal{C})$ with $* \in$ **WNM-fin**.

In the following we will use $[0,1]^{\mathbb{Q}}$ to denote $\mathbb{Q} \cap [0,1]$. A left-continuous t-norm $*$ is called *r-admissible* when both $*$ and its residuum $\Rightarrow$ are closed operations on $[0,1]^{\mathbb{Q}}$. Notice that if $*$ is r-admissible, then $\mathcal{Q}_* = \langle [0,1]^{\mathbb{Q}}, *, \Rightarrow, \min, \max, 0, 1 \rangle$ is a countable subalgebra of the standard algebra $[0,1]_*$, and hence it is meaningful to consider the logic $L_*(\mathcal{Q}_*)$ and the canonical standard algebra $[0,1]_{L_*(\mathcal{Q}_*)}$. To simplify a bit the notation we will denote the latter as $[0,1]_{L_*(\mathcal{Q})}$.

We introduce the following three sets of formulae, namely the set of tautologies, the set of satisfiable formulae and the set of pairs of formulae in the semantical consequence relation, everything with respect to the canonical standard chain $[0,1]_{L_*(\mathcal{Q})}$:

$$RTAUT(*) = \{\varphi \mid [0,1]_{L_*(\mathcal{Q})} \models \varphi \approx \overline{1}\}$$

$$RSAT(*) = \{\varphi \mid [0,1]_{L_*(\mathcal{Q})} \not\models \neg\varphi \approx \overline{1}\}$$

$$RSECON(*) = \{\langle \varphi, \psi \rangle \mid \varphi \approx \overline{1} \models_{[0,1]_{L_*(\mathcal{Q})}} \psi \approx \overline{1}\}$$

Hájek's results in [22] can be summarized as follows. An r-admissible t-norm $* \in$ **CONT-fin** is called *strong r-admissible* when each Ł-component and Π-component is isomorphic to $[0,1]_{\text{Ł}}$ and $[0,1]_{\Pi}$ respectively via a bijection f mapping rationals into rationals such that both f and f^{-1} restricted to rationals are deterministically polynomially computable. Then for a strong r-admissible t-norm $* \in$ **CONT-fin** with rational endpoints in all its basic components:

(i) when $[0,1]_*$ has no Π-component, $RTAUT(*)$ and $RSECON(*)$ are coNP-complete and $RSAT(*)$ is NP-complete;

(ii) otherwise, $RTAUT(*)$, $RSECON(*)$ and $RSAT(*)$ are in PSPACE.

Now, let us consider the set of theorems of $\mathrm{L}_*(\mathcal{Q}_*)$, the set of consistent formulae in $\mathrm{L}_*(\mathcal{Q}_*)$ and the set of pairs of formulae such that the second is derivable from the first in $\mathrm{L}_*(\mathcal{Q}_*)$:

$$RTHEO(*) = \{\varphi \mid \mathrm{L}_*(\mathcal{Q}) \vdash \varphi\}$$

$$RCONS(*) = \{\varphi \mid \varphi \nvdash_{\mathrm{L}_*(\mathcal{Q})} \overline{0}\}$$

$$RSYCON(*) = \{\langle \varphi, \psi \rangle \mid \varphi \vdash_{\mathrm{L}_*(\mathcal{Q})} \psi\}$$

Theorem 33. *Let $* \in$ **WNM-fin** be r-admissible such that all the endpoints of its partition are rational. Then $RTAUT(*)$ and $RSECON(*)$ are coNP-complete and $RSAT(*)$ is NP-complete.*

Proof: The proof is a generalization of the one for $* = \min$ in [22, Theorem 2]. Given a formula φ, let $R(\varphi)$ be the universe of the WNM-subalgebra of $\mathcal{Q}_*$ generated by the set of truth-constants appearing in φ. It is clear that $R(\varphi)$ is finite. Let $Part(*)$ be the set $0 < s_1 < \ldots < s_m < 1$ of the endpoints of the partition associated to $*$ (recall that it includes the negation fixpoint if it exists) and let $X = R(\varphi) \cup Part(*) = \{0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = 1\}$ which forms another finite WNM-subalgebra of $\mathcal{Q}_*$. If φ contains n propositional variables, then we choose n rational elements $a_{i1}, \ldots, a_{in}$ in each open interval (t_i, t_{i+1}) such that $X \cup \bigcup_{i=0}^{k-1}\{a_{i1}, \ldots, a_{in}\}$ forms a WNM-subalgebra $\mathcal{A}_\varphi$ of $\mathcal{Q}_*$. Now, one can prove the following:

Claim: $\varphi \in RSAT(*)$ if and only if there exists an evaluation e on $\mathcal{A}_\varphi$ such that $e(\varphi) = 1$.
Proof: Let v be an evaluation on $[0,1]_{\mathrm{L}_*(\mathcal{Q})}$ such that $v(\varphi) = 1$, and let $\mathcal{B}$ the $\mathrm{L}_*(R(\varphi))$-algebra generated by the set $\{v(q) \mid q$ propositional variable in $\varphi\}$. Then one can check that $\mathcal{B}$ can be embedded in $\mathcal{A}_\varphi$. $\qquad\square$

Therefore $\varphi \in RSAT(*)$ if and only if one can guess such an evaluation.

Analogously, one can prove that $\varphi \notin TAUT(*)$ iff one can guess an evaluation e on $\mathcal{A}_\varphi$ such that $e(\varphi) < 1$.

Finally, the case of checking $\langle \varphi, \psi \rangle \in RSECON(*)$ is reduced, due to the deduction theorem for WNM, to checking $\varphi^2 \to \psi \in TAUT(*)$. This ends the proof. $\qquad\square$

Finally, taking into account the canonical standard completeness results for expansions of WNM logics (see Section 6.2), we can state the computational complexity of the following logics with rational truth-constants.

Theorem 34. *Let $* \in$ **WNM-fin** be r-admissible such that all the endpoints of its partition are rational. If L_* is G, NM, $\mathrm{L}_{\otimes_c}$ or $\mathrm{L}_{\star_c}$, we have that $RTHEO(*)$ and $RSYCON(*)$ are coNP-complete and $RCONS(*)$ is NP-complete.*

10 Conclusions

In this paper we have been concerned with expansions of logics of WNM t-norms with countable sets C of truth-constants when (i) the t-norm is a WNM t-norm with an associated finite partition, and (ii) the set of truth-constants *covers* all the unit interval in the sense that the interior of each interval of the partition contains at least one value of C. In particular we have considered different kinds of standard completeness properties and have identified which logics satisfy them. Some modest results on computational complexity for the expanded logics are also presented.

This paper is a natural follow-up of [12], where the authors studied the expansions with truth-constants of logics of continuous t-norms. However, as already mentioned in a footnote at the beginning of the paper, by some reasons this paper comes when further developments have been already or are near to be published, namely the study of completeness properties for rational-valued semantics (as opposed to real-based semantics in this paper) in [15] and an exhaustive study of expansions with truth-constants for predicate t-norm logics in [16].

References

[1] M. Baaz. Infinite-valued Gödel logics with $0 - 1$ projections and relativizations. In GÖDEL '96 - Logical foundations of mathematics, computer science and physics. *Lecture Notes in Logic* 6 (1996), P. Hájek (Ed.), Springer Verlag, 23–33.

[2] W.J. Blok and D. Pigozzi. Algebraizable logics, *Memoirs of the American Mathematical Society* 396, vol 77, 1989.

[3] F. Bou, F. Esteva, A. Gil, J. Gispert, L. Godo, J.M. Font, A. Torrens and V. Verdú. Logics preserving degrees of truth from varieties of residuated lattices. To appear in *Journal of Logic and Computation*. A short version, with the title "T-norm based fuzzy logics preserving degrees of truth", appears in the Proceedings of IPMU 2008 conference, (L. Magdalena et al. eds.), Málaga, June 2008, pp. 1053-1060.

[4] S. Burris and H.P. Sankappanavar. *A course in Universal Algebra*, Springer Verlag, New York, 1981.

[5] R. Cignoli, F. Esteva and L. Godo. On Łukasiewicz logic with truth constants. In *Theoretical Advances and Applications of Fuzzy Logic and Soft Computing*, ASC 42: Springer-Verlag, pp. 869–875, 2007.

[6] R. Cignoli, F. Esteva, L. Godo and A. Torrens. Basic Fuzzy Logic is the logic of continuous t-norms and their residua, *Soft Computing* 4 (2000) 106–112.

[7] P. Cintula. *From Fuzzy Logic to Fuzzy Mathematics*. Ph.D. dissertation, Czech Technical University, Prague (Czech Republic), 2005.

[8] P. Cintula, F. Esteva, J. Gispert, L. Godo, F. Montagna and C. Noguera. Distinguished algebraic semantics for t-norm based fuzzy logics: methods and algebraic equivalencies. To appear in *Annals of Pure and Applied Logic*, 2009.

[9] M. Dummett. A propositional calculus with denumerable matrix, *The Journal of Symbolic Logic* 24 (1959) 97–106.

[10] F. Esteva, X. Domingo. Sobre negaciones fuertes y débiles en $[0,1]$, *Stochastica* 4 (1980) 141–166.

[11] F. Esteva, J. Gispert, L. Godo and F. Montagna. On the standard and Rational Completeness of some axiomatic extensions of Monoidal t-norm Based Logic, *Studia Logica* 71 (2002) 199–226.

[12] F. Esteva, J. Gispert, L. Godo and C. Noguera. Adding truth-constants to logics of continuous t-norms: axiomatization and completeness results, *Fuzzy Sets and Systems* 158 (2007) 597–618.

[13] F. Esteva and L. Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms, *Fuzzy Sets and Systems* 124 (2001) 271–288.

[14] F. Esteva, L. Godo and C. Noguera. On rational weak nilpotent minimum logics, *Journal of Multiple-valued Logic & Soft Computing* 12 (2006) 9–32.

[15] F. Esteva, L. Godo and C. Noguera. Expanding the propositional logic of a t-norm with truth-constants: completeness results for rational semantics, *Soft Computing*, to appear (DOI 10.1007/s00500-009-0402-8).

[16] F. Esteva, L. Godo and C. Noguera. First-order t-norm based fuzzy logics with truth-constants: distinguished semantics and completeness properties, *Annals of Pure and Applied Logic*, to appear.

[17] J. Fodor. Nilpotent minimum and related connectives for fuzzy logic, *Proceedings of FUZZ–IEEE'95*, 1995, pp. 2077–2082.

[18] J.M. Font, À. Gil, A. Torrens and V. Verdú. An infinite-valued Łukasiewicz logic that preserves degrees of truth, *Archive for Mathematical Logic* 45 (2006) 839–868.

[19] G. Gerla. *Fuzzy Logic: Mathematical Tools for Approximate Reasoning*, Trends in Logic, vol. 11, Kluwer, Dordrecht, 2001.

[20] J. Gispert. Axiomatic extensions of the nilpotent minimum logic, *Reports on Mathematical Logic* 37 (2003) 113–123.

[21] P. Hájek. *Metamathematics of fuzzy logic*, Trends in Logic, vol. 4, Kluwer, Dordrecht, 1998.

[22] P. Hájek. Computational complexity of t-norm based propositional fuzzy logics with rational truth-constants, *Fuzzy Sets and Systems* 157 (2006) 677–682.

[23] P. Hájek, L. Godo and F. Esteva. A complete many-valued logic with product-conjunction, *Archive for Mathematical Logic* 35 (1996) 191–208.

[24] U. Höhle. Commutative, residuated l-monoids. In Höhle, U. and Klement. E.P. eds., *Non-Classical Logics and Their Applications to Fuzzy Subsets*, Kluwer Acad. Publ., Dordrecht, 1995, pp. 55–106.

[25] S. Jenei and F. Montagna. A proof of standard completeness for Esteva and Godo's logic MTL, *Studia Logica* 70 (2002) 183–192.

[26] T. Lukasiewicz and U. Straccia. Managing Uncertainty and Vagueness in Description Logics for the Semantic Web. *Journal of Web Semantics* 6 (2008), pp. 291308.

[27] C. Noguera, F. Esteva and J. Gispert. On some varieties of MTL-algebras, *Logic Journal of the IGPL* 13 (2005) 443–466.

[28] C. Noguera, F. Esteva and J. Gispert. On triangular norm based axiomatic extensions of the Weak Nilpotent Minimum logic, *Mathematical Logic Quarterly* 54 (2008) 387–409.

[29] C. Noguera. *Algebraic study of axiomatic extensions of triangular norm based fuzzy logics*, Monografies de l'Institut d'Investigació en Intel·ligència Artificial vol. 27, Barcelona, 2007.

[30] V. Novák. On the syntactico-semantical completeness of first-order fuzzy logic. Part I. Syntax and Semantics, *Kybernetika* 26 (1990) 47–66.

[31] V. Novák. On the syntactico-semantical completeness of first-order fuzzy logic. Part II. Main Results, *Kybernetika* 26 (1990) 134–154.

[32] V. Novák, I. Perfilieva, J. Močkoř. *Mathematical Principles of Fuzzy Logic*. Kluwer Academic Pub., 1999.

[33] J. Pavelka. On Fuzzy Logic I, II, III. *Zeitschrift fur Math. Logik und Grundlagen der Math.* 25 (1979) 45–52, 119–134, 447–464.

[34] P. Savický, R. Cignoli, F. Esteva, L. Godo and C. Noguera. On product logic with truth-constants, *Journal of Logic and Computation* 16 (2006) 205–225.

[35] E. Trillas. Sobre funciones de negación en la teoría de conjuntos difusos, *Stochastica* 3 (1979) 47–60. English translation in: S. Barro, A. Bugarin and A. Sobrino eds., *Advances in Fuzzy logic*, Public. Univ. Santiago de Compostela, Spain, pp. 31–45.

[36] P. Vojtáš. Fuzzy logic programming, *Fuzzy Sets and Systems* 124 (2001) 361–370.