On Rational Weak Nilpotent Minimum Logics

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Abstract

In this paper we investigate extensions of Gödel and Nilpotent Minimum logics by adding rational truth-values as truth constants in the language and by adding corresponding book-keeping axioms for the truth-constants. We also investigate the rational extensions of some parametric families of Weak Nilpotent Minimum logics, weaker than both Gödel and Nilpotent Minimum logics. Weak and strong standard completeness of these logics are studied in general and in particular when we restrict ourselves to formulas of the kind $\overline{r} \to \varphi$, where r is a rational in [0, 1] and φ is a formula without rational truth-constants.

Keywords: Gödel Logic, Weak Nilpotent Minimum, Rational Pavelka Logic, Rational Gödel logic, Rational Weak Nilpotent Minimum logic.

1 Introduction

As Pavelka pointed out in [11], it seems natural to introduce truth values in the language in order to be able to deal with *partial truth.* With this aim, he obtained a many-valued logical system over Lukasiewicz logic whose language contained as many truth constants as truth values, i.e. a truth constant $\overline{r}$ for each real $r \in [0, 1]$, and a number of additional axioms. Although this Lukasiewicz logic extended with truth-constants, PL, is not strong complete (like Lukasiewicz logic), Pavelka proved that it is complete in a different sense. Indeed, he introduced a weaker notion of strong completeness based on the degrees of provability and truth of a formula φ in an arbitrary theory T. The truth degree of φ in T is defined as

$$\| \varphi \|_T = \inf\{e(\varphi) \mid e \text{ evaluation model of } T\}$$

and the degree of provability of φ in T as

$$| \varphi |_T = \sup\{r \mid T \vdash_{PL} \overline{r} \to \varphi\}.$$

Pavelka proved that these degrees coincide. This kind of completeness, which strongly relies in the continuity of Lukasiewicz logic truth functions, is usually known as Pavelka-style completeness. Moreover he also proved that Pavelka-style completeness is preserved if and only if the language is extended with any connective whose corresponding truth-function on the real unit interval is a continuous (real) function.

Later, Hájek [10] proved that Pavelka's logic PL could be significantly simplified while keeping the completeness results. Namely, Hájek's system is an extension of Lukasiewicz logic by only a countable number of truth-constants, $\overline{r}$ for each *rational* $r \in [0,1]$, and by two additional axiom schemata to deal with the truth-constants, called book-keeping axioms:

$$\overline{r} \& \overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \to \overline{s} \leftrightarrow \overline{r \Rightarrow s}$$

where $*$ and $\Rightarrow$ are the t-norm of Lukasiewicz and its residuum respectively. He denoted this new system as RPL, for Rational Pavelka Logic, and proved the same results that Pavelka proved for his system with continuously many truth-constants. Moreover, in [10] it is proved that RPL is strong complete for finite theories. Remark that the semantics of RPL is kept on the *real* unit interval [0, 1].

Similar *rational* extensions for other popular fuzzy logics can be obviously defined, but Pavelka-style completeness cannot be obtained since Lukasiewicz is the only fuzzy logic with continuous truth-functions in the real unit interval [0, 1]. For instance, in [10] Hájek defines an extension of G_Δ, the extension of Gödel logic with Baaz's Delta operator, with a finite number of rational truth-constants. Later, in [4] the authors define logical systems obtained by adding (rational) truth-constants to $G_\sim$ (Gödel logic with an involutive negation) and to Π (Product logic) and $\Pi_\sim$ (Product logic with an involutive negation). For the first system, $RGL_\sim$, usual strong completeness is proved for finite theories, while for the second systems, $R\Pi L$ and $R\Pi L_\sim$, it is possible to prove Pavelka-style completeness provided an infinitary inference rule is added to overcome the problem that the residuum of the product t-norm is not continuous at the point $(0,0)$. Finally also notice that in [1] standard completeness of Gödel logic with rational truth-constants is stated. Although the result holds true (see Section 3), the proof given there has some flaws.

Another different approach to reasoning with partial degrees of truth is the framework of abstract fuzzy logics developed by Gerla [7] based on the notion of fuzzy consequence or deduction operators over fuzzy sets of formulas, where the membership degree of formulas are interpreted as lower bounds on their truth degrees.

In this paper we investigate the expansions with rational truth-constants, à la Pavelka, of several extensions of the so-called Weak Nilpotent Minimum logic WNM. WNM was introduced in [3] as the axiomatic extension of MTL by the following axiom,

(WNM) $$(\varphi \& \psi \to \overline{0}) \vee (\varphi \wedge \psi \to \varphi \& \psi)$$

and proved to be standard complete with respect to the family of WNM t-norms and their residua. WNM t-norms are left-continuous t-norms defined from a weak negation function n and the minimum t-norm in the following way

$$x *_n y = \begin{cases} 0, & \text{if } x \leq n(y) \\ \min(x,y), & \text{otherwise} \end{cases}$$

Well-known particular cases of WNM t-norms are the minimum t-norm (when n is Gödel negation) and Fodor's nilpotent t-norm (when $n(x) = 1 - x$).

In the first part of the paper we investigate the extensions of Gödel logic G and Nilpotent minimum logic NM with rational truth-constants similar to RPL, that we shall call respectively RG and RNM, and we prove different completeness results, in the usual sense, for RG and for RNM. In the second part of the paper we generalize the results to some particular extensions of the Weak Nilpotent minimum logic WNM.

The paper is structured as follows. In next section we give a general formal account of algebraic semantics for the expansions of usual fuzzy logics with rational truth-constants. This is done by means of Blok and Pigozzi's theory of algebraization of propositional logics. Sections 3 and 4 are devoted to the rational extensions of Gödel and Nilpotent Minimum logics and different completeness results for them. In the first part of Section 5 we consider three different WNM logics and prove their standard completeness. The rest of the section is devoted to several standard completeness results for the rational expansions of these logics. We conclude with some final remarks.

2 Preliminaries

Our general logical framework for this section will be that of MTL and its axiomatic extensions. MTL logic was defined in [3] as a propositional logic in the language $\mathcal{L} = \{\&, \rightarrow, \wedge, \overline{0}\}$. We will denote by $Fm_{\mathcal{L}}$ the set of well-formed formulas built over the language $\mathcal{L}$ and a countable set of propositional variables. Axioms of MTL are:

(A1)	$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(A2)	$(\varphi \& \psi) \rightarrow \varphi$
(A3)	$(\varphi \& \psi) \rightarrow (\psi \& \varphi)$
(A4)	$(\varphi \wedge \psi) \rightarrow \varphi$
(A5)	$(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$
(A6)	$(\varphi \& (\varphi \rightarrow \psi)) \rightarrow (\varphi \wedge \psi)$
(A7a)	$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \& \psi) \rightarrow \chi)$
(A7b)	$((\varphi \& \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(A8)	$((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
(A9)	$\overline{0} \rightarrow \varphi$

The rule of inference of MTL is *modus ponens*.

In the frame of MTL extensions, other usual connectives are definable, in particular $\overline{1}$ is $\varphi \rightarrow \varphi$, $\neg\varphi$ is $\varphi \rightarrow \overline{0}$, $\varphi \vee \psi$ is $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$, and $\varphi \leftrightarrow \psi$ is $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

In [3] an algebraic semantics for MTL was given, based on the notion of MTL-algebras, i.e. bounded integral commutative residuated lattices satisfying the prelinearity equation: $(x \rightarrow y) \vee (y \rightarrow x) \approx 1$. Let $\mathbb{MTL}$ be the variety of all MTL-algebras.

Definition 2.1. *Given $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, we define:*

If $\mathcal{A} \in \mathbb{MTL}$, $\Gamma \models_{\mathcal{A}} \varphi$ iff for all evaluations v in $\mathcal{A}$, we have $v(\varphi) = 1$ whenever $v(\psi) = 1$ for all $\psi \in \Gamma$.

$\Gamma \models \varphi$ iff for all $\mathcal{A} \in \mathbb{MTL}$ we have $\Gamma \models_{\mathcal{A}} \varphi$.

Then, one can prove this theorem of strong completeness for MTL logic:

Theorem 2.2. *If $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, then*

$$\Gamma \models \varphi \text{ iff } \Gamma \vdash_{MTL} \varphi.$$

But this result can be improved by means of the equational consequence:

Definition 2.3. *Let $Eq_{\mathcal{L}}$ be the set of $\mathcal{L}$-equations and let $\Delta \cup \{\varphi \approx \psi\} \subseteq Eq_{\mathcal{L}}$. We define the equational consequence by:*

$\Delta \models_{\mathbb{MTL}} \varphi \approx \psi$ *iff for all* $\mathcal{A} \in \mathbb{MTL}$ *and for all evaluations* v *in* $\mathcal{A}$*, we have* $v(\varphi) = v(\psi)$ *whenever* $v(\alpha) = v(\beta)$ *for all* $\alpha \approx \beta \in \Delta$*.*

Theorem 2.4. *The relation of derivability in the system MTL and the equational consequence in the variety* $\mathbb{MTL}$ *are mutually translatable:*

1. *For every* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*,* $\Gamma \vdash_{MTL} \varphi$ *iff* $\{\psi \approx 1 : \psi \in \Gamma\} \models_{\mathbb{MTL}} \varphi \approx 1$
2. *For every* $\Delta \cup \{\varphi \approx \psi\} \subseteq Eq_{\mathcal{L}}$*,*
   $\Delta \models_{\mathbb{MTL}} \varphi \approx \psi$ *iff* $\{\alpha \leftrightarrow \beta : \alpha \approx \beta \in \Delta\} \vdash_{MTL} \varphi \leftrightarrow \psi$ *.*

In addition, each one of these translations is the inverse of the other, that is:

3. $\varphi \approx \psi \models_{\mathbb{MTL}} \varphi \leftrightarrow \psi \approx 1$ *and* $\varphi \leftrightarrow \psi \approx 1 \models_{\mathbb{MTL}} \varphi \approx \psi$
4. $\varphi \vdash_{MTL} \varphi \leftrightarrow \overline{1}$ *and* $\varphi \leftrightarrow \overline{1} \vdash_{MTL} \varphi$ *.*

Therefore, MTL is an algebraizable logic in the sense of Blok and Pigozzi (see [2]) whose equivalent algebraic semantics is the variety $\mathbb{MTL}$. Thus, using the general theory of [2], all axiomatic extensions of MTL are also algebraizable in this strong sense. Namely, if L denotes the extension of MTL by a given set of axiom schemata Σ, the equivalent algebraic semantics of L is the subvariety of $\mathbb{MTL}$ defined by the translation of the formulas in Σ into equations. We will refer to the algebras of this subvariety as L-algebras.

There is another useful kind of completeness result for MTL, completeness with respect to the totally ordered algebras (we will call them 'chains'):

Theorem 2.5. *[3] Each MTL-algebra is isomorphic to a subdirect product of MTL-chains.*

As a consequence, Theorem 2.2 remains valid if the logical consequence $\models$ in Definition 2.1 is restricted to evaluations over MTL-chains. This is also true for every axiomatic extension of MTL.

Now we will consider the algebraization of fuzzy logics with constant symbols for the rationals. The new language we will use is $\mathcal{RL} = \mathcal{L} \cup \{\overline{r} : r \in \mathbb{Q} \cap (0,1)\}$, the expansion of $\mathcal{L}$ with new constant symbols, one for every rational in $(0,1)$.

Definition 2.6. *Let L be MTL or any axiomatic extension of MTL and let* $*$ *be a left-continuous t-norm and* $\Rightarrow$ *its residuum such that* $[0,1]_* = \langle [0,1], *, \Rightarrow, \min, \max, 0, 1\rangle$ *is an L-algebra and such that the set of rational numbers* $\mathbb{Q} \cap [0,1]$ *is closed under* $*$ *and* $\Rightarrow$*. By RL(*$*$*) we will denote the propositional logic in the language* $\mathcal{RL}$ *obtained by adding to L the so-called 'book-keeping axioms':*

$$\overline{r} \& \overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \wedge \overline{s} \leftrightarrow \overline{\min(r,s)}$$
$$(\overline{r} \to \overline{s}) \leftrightarrow \overline{r \Rightarrow s}$$

for every $r, s \in \mathbb{Q} \cap [0,1]$*.*

RL($*$*)-algebras are structures* $\mathcal{A} = \langle A, \&, \to, \wedge, \vee, \{\overline{r}^{\mathcal{A}} : r \in \mathbb{Q} \cap [0,1]\}\rangle$ *such that:*

1. $\langle A, \&, \to, \wedge, \vee, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\rangle$ *is a L-algebra,*
2. *for every* $r, s \in \mathbb{Q} \cap [0,1]$ *:*
   $$\overline{r}^{\mathcal{A}} \& \overline{s}^{\mathcal{A}} = \overline{r * s}^{\mathcal{A}}$$
   $$\overline{r}^{\mathcal{A}} \wedge \overline{s}^{\mathcal{A}} = \overline{\min(r,s)}^{\mathcal{A}}$$
   $$\overline{r}^{\mathcal{A}} \to \overline{s}^{\mathcal{A}} = \overline{r \Rightarrow s}^{\mathcal{A}} \ .$$

Given $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{RL}}$*, we define* $\Gamma \models_{\mathcal{A}} \varphi$ *iff for all evaluations* e *in* $\mathcal{A}$ *(i.e. such that* $e(\overline{r}) = \overline{r}^{\mathcal{A}}$*), we have* $e(\varphi) = \overline{1}^{\mathcal{A}}$ *whenever* $e(\psi) = \overline{1}^{\mathcal{A}}$ *for all* $\psi \in \Gamma$*.*

When $A = [0,1]$ and $\overline{r}^{\mathcal{A}} = r$ for all $r \in \mathbb{Q} \cap [0,1]$, we say that $\mathcal{A}$ is the standard RL($*$)-algebra[1].

Using [2], it is easy to prove that RL($*$) is an algebraizable logic whose equivalent algebraic semantics is the variety of RL($*$)-algebras. Also, using standard techniques, we obtain completeness of RL($*$) with respect to linearly ordered RL($*$)-algebras.

Theorem 2.7. *For any $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{RL}}$, $\Gamma \vdash_{RL(*)} \varphi$ iff $\Gamma \models_{\mathcal{A}} \varphi$ for all RL($*$)-chains $\mathcal{A}$.*

If L_* is an extension of MTL such that is standard complete with respect the left-continuous t-norm $*$, then for simplicity we will write RL_* instead of $RL_*(*)$.

Since these logics are expansions of MTL, sharing modus ponens as the only inference rule, they have the same local deduction-detachment theorem as MTL has. In fact the proof for MTL also applies here.

Theorem 2.8. *Let $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{RL}}$ be such that $\Gamma, \varphi \vdash_{RL(*)} \psi$. Then, there is a natural $k \geq 1$ such that $\Gamma \vdash_{RL(*)} \varphi^k \to \psi$.*

Using this, we can prove the following proposition about conservative extensions:

Proposition 2.9. *If L is weak standard complete w.r.t. $[0,1]_*$, then RL($*$) is a conservative extension of L.*

Proof. Let $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ be such that $\Gamma \vdash_{RL(*)} \varphi$. Then, there is a finite set $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{RL(*)} \varphi$, hence, by the previous theorem, there is $k \geq 1$ such that $\vdash_{RL(*)} (\&\Gamma_0)^k \to \varphi$. Since RL($*$)-algebras are an algebraic semantics, we have $\models_{\mathcal{A}} (\&\Gamma_0)^k \to \varphi$ where $\mathcal{A}$ is the standard L($*$)-algebra. Hence, using that these formulas are written in $\mathcal{L}$, $\models_{[0,1]_*} (\&\Gamma_0)^k \to \varphi$, thus, by weak standard completeness $\vdash_L (\&\Gamma_0)^k \to \varphi$ and, finally, $\Gamma \vdash_L \varphi$.[2] $\square$

In the definition of RL($*$) we could wonder what happens if we consider an isomorphic t-norm $\circ$ instead of $*$. In that case, we would obtain a different logic (actually, the book-keeping axioms of RL($\circ$) are different from those of RL($*$)), but both logics are translatable one to another as the following theorem proves.

Theorem 2.10. *Let L be MTL or an extension of it, and let $*$, $\circ$ be two left-continuous t-norms such that $[0,1]_*$ and $[0,1]_\circ$ are L-algebras and such that the rationals numbers form a subalgebra of both algebras. Suppose that there is an isomorphism[3] from $*$ to $\circ$ such that for every rational number r, $F(r)$ is also rational. For any $\varphi \in Fm_{\mathcal{RL}}$ we write $\varphi(\overline{r_1}, \ldots, \overline{r_n})$ to explicitely denote that the truth-constants $\overline{r_1}, \ldots, \overline{r_n}$ appear in φ, and for every $\varphi(\overline{r_1}, \ldots, \overline{r_n}) \in Fm_{\mathcal{RL}}$ we define $\tau(\varphi(\overline{r_1}, \ldots, \overline{r_n})) := \varphi(\overline{F(r_1)}, \ldots, \overline{F(r_n)})$. Then τ is a* translation *between RL($*$) and RL($\circ$), i. e., for every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{RL}}$, $\Gamma \vdash_{RL(*)} \varphi$ iff $\tau[\Gamma] \vdash_{RL(\circ)} \tau(\varphi)$.*

Proof. Suppose $\langle \varphi_1, \ldots, \varphi_{n-1}, \varphi_n = \varphi \rangle$ is a proof in RL($*$) of φ from Γ. It is sufficient to show that $\langle \tau(\varphi_1), \ldots, \tau(\varphi_{n-1}), \tau(\varphi_n) = \tau(\varphi) \rangle$ is a proof in RL($\circ$) of $\tau(\varphi)$ from $\tau[\Gamma]$. Take any $i \leq n$. If φ_i is an axiom, then $\tau(\varphi_i)$ it is also an axiom. Observe that since F is an isomorphism for the pairs $(*, \circ)$, hence also for $(\Rightarrow_*, \Rightarrow_\circ)$, the translation by τ of a book-keeping axiom of RL($*$) is a book-keeping axiom of RL($\circ$)). It is also clear that if $\varphi_i \in \Gamma$, then $\tau(\varphi_i) \in \tau[\Gamma]$, and if φ_i is obtained by Modus Ponens, then also $\tau(\varphi_i)$ is obtained by Modus Ponens. $\square$

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[1] Notice that in such a case & and $*$ necessarily coincide (hence $\to$ and $\Rightarrow$ as well), since they coincide on the rationals and are left-continuous t-norms.

[2] We thank Petr Cintula for showing us this proof.

[3] That is, an order preserving bijection $F : [0,1] \to [0,1]$ such that $x \circ y = F^{-1}(F(x) * F(y))$ for all $x, y \in [0,1]$.

3 Standard completeness results for RG and RNM

From now on we will consider two particular logics of the type RL($*$), namely for L being Gödel logic G and Nilpotent Minimum logic NM. Gödel logic is the well-known extension of Hájek's BL logic with the contraction axiom:

$$\varphi \to \varphi \& \varphi \tag{G}$$

which forces the equivalence of the connectives & and $\wedge$. G is strong complete w.r.t. the standard G-algebra $[0,1]_G = \langle [0,1], \min, \Rightarrow_G, 0, 1\rangle$, defined by taking $* = \min$. $\Rightarrow_G$ is its residuum, i.e. $x \Rightarrow_G y = 1$ if $x \leq y$, $x \Rightarrow_G y = y$, otherwise. This is actually the only G-algebra on [0, 1]. Moreover G has the usual deduction theorem. Nilpotent Minimum logic NM was defined in [3] as the axiomatic extension of MTL with the following axioms:

$$\neg\neg\varphi \to \varphi \tag{Inv}$$
$$(\varphi\&\psi \to \overline{0}) \vee (\varphi\&\psi \to \varphi \wedge \psi) \tag{WNM}$$

NM is standard complete with respect to any NM-algebra on [0,1] (all are isomorphic), in particular with respect to the NM-algebra $[0,1]_{NM}$ defined by the so-called nilpotent minimum t-norm [6], defined as

$$x *_{NM} y = \begin{cases} min(x,y), & \text{if } x > 1-y \\ 0, & \text{otherwise} \end{cases}$$

and its residuum

$$x \Rightarrow_{NM} y = \begin{cases} 1, & \text{if } x \leq y \\ \max(1-x, y), & \text{otherwise} \end{cases}$$

NM has a weaker form of deduction theorem, namely $\Gamma, \psi \vdash_{NM} \varphi$ iff $\Gamma \vdash_{NM} (\psi\&\psi) \to \varphi$, for any $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$.

In the following we will simplify the notation and we shall write RG for RG(min) and RNM for RNM($*_{NM}$) and we will denote by $[0,1]_{RG}$ and $[0,1]_{RNM}$ their corresponding standard algebras.

Next Theorems 3.3 and 3.4 prove weak standard completeness for RG and RNM logics. But first we need to show how RG-chains and RNM-chains look like.

Lemma 3.1. *For any RG-chain $\mathcal{A} = \langle A, \&, \to, \wedge, \vee, \{\overline{r}^{\mathcal{A}} : r \in \mathbb{Q} \cap [0,1]\}\rangle$ there exists a real $\alpha \in [0,1]$ such that:*
(i) $\overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$ for any rational $r > \alpha$, and
(ii) if $\alpha > 0$, then $\overline{r}^{\mathcal{A}} < \overline{s}^{\mathcal{A}}$ for any rationals $r < s < \alpha$.

Proof. Assume that for two rationals $r < s$ we have that $\overline{r}^{\mathcal{A}} = \overline{s}^{\mathcal{A}}$. Then, on the one hand $\overline{s}^{\mathcal{A}} \to \overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$, but on the other hand, $\overline{s \Rightarrow_G r}^{\mathcal{A}} = \overline{r}^{\mathcal{A}}$, thus by the book-keeping axioms we have $\overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$, and hence $\overline{r'}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$ for each rational $r' > r$ as well. Finally take $\alpha = \inf\{r \mid \overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}\}$. Notice that α can be 0 in the special case that for all rationals $r \neq 0$, $\overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$. □

Lemma 3.2. *For any RNM-chain $\mathcal{A} = \langle A, \&, \to, \wedge, \vee, \{\overline{r}^{\mathcal{A}} : r \in \mathbb{Q} \cap [0,1]\}\rangle$ there exists a real $\alpha \in [\frac{1}{2}, 1]$ such that:*
(i) $\overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$ for any rational $r > \alpha$,
(ii) $\overline{r}^{\mathcal{A}} = \overline{0}^{\mathcal{A}}$ for any rational $r < 1 - \alpha$, and
(iii) if $\alpha > \frac{1}{2}$, then $\overline{r}^{\mathcal{A}} < \overline{s}^{\mathcal{A}}$ for any rationals r and s such that $1 - \alpha < r < s < \alpha$.

Proof. Due to the book-keeping axioms, any RNM-chain has a negation fixpoint, which is $\overline{\frac{1}{2}}^{\mathcal{A}}$. Moreover if for two rationals $\frac{1}{2} < r < s$ we have $\overline{r}^{\mathcal{A}} = \overline{s}^{\mathcal{A}}$, then, on the one hand $\overline{s}^{\mathcal{A}} \to \overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$, and on the other hand, $\overline{s \Rightarrow_{NM} r}^{\mathcal{A}} = \overline{\max(1-s, r)}^{\mathcal{A}} = \overline{r}^{\mathcal{A}}$, which imply, by the book-keeping axioms, that $\overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$, and hence $\overline{r'}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$ for each rational $r' > r$ as well. Then taking $\alpha = \inf\{r \mid \overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}\}$ (i) becomes obvious. (ii) and (iii) easily follow from the involutiveness of the negation. Finally notice that it is possible that $\inf\{r \mid r > \frac{1}{2}, \overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}\} = \frac{1}{2}$. □

Notice that Lemma 3.2 is a natural consequence of previous Lemma 3.1 taking into account that a NM-chain is always isomorphic to a rotation of a Gödel hoop in the sense of Jenei [9].

Theorem 3.3. $\vdash_{RG} \varphi$ *if and only if* $\models_{[0,1]_{RG}} \varphi$

Proof. The soundness part is trivial as usual. To prove completeness, suppose $\nvdash_{RG} \varphi$, then by completeness of RG w.r.t. the RG-chains, there exist a countable RG-chain $\mathcal{C}$ and an evaluation e over $\mathcal{C}$ such that $e(\varphi) <_C \overline{1}^{\mathcal{C}}$. We have to show there is an evaluation e' on the standard algebra $[0,1]_{RG}$ such that $e'(\varphi) < 1$.

Let $X = \{e(\psi) \mid \psi \text{ subformula of } \varphi\} \cup \{\overline{0}^{\mathcal{C}}, \overline{1}^{\mathcal{C}}\}$.

Let $\alpha = \min\{r \mid r = 1 \text{ or } \overline{r} \text{ subformula of } \varphi \text{ with } \overline{r}^{\mathcal{C}} = \overline{1}^{\mathcal{C}}\}$. Clearly $\alpha > 0$. Let g be an order-preserving injection of X into $[0, \alpha]$ satisfying $g(\overline{r}^{\mathcal{C}}) = r$ for $\overline{r}$ a subformula of φ with $r < \alpha$ and, furthermore, $g(\overline{1}^{\mathcal{C}}) = \alpha$.

Then we define an RG-evaluation e' on the standard RG-algebra $[0,1]$ as follows: for all propositional variable p, $e'(p) = g(e(p))$ if p appears in φ and $e'(p) = 1$ otherwise. Then e' is extended to RG-formulas as usual (of course with $e'(\overline{r}) = r$ for each rational r).

Claim. For each ψ subformula of φ,

(1) if $e(\psi) = \overline{1}^{\mathcal{C}}$ then $e'(\psi) \geq \alpha$,

(2) if $e(\psi) < \overline{1}^{\mathcal{C}}$ then $e'(\psi) = g(e(\psi)) < \alpha$.

The claim is clear for variables and for truth-constants $\overline{r}$ subformulas of φ. The induction step for $\wedge$ is trivial. Let us consider the case of $\to$. If $e(\gamma \to \delta) = e(\delta) < \overline{1}^{\mathcal{C}}$ then $e'(\delta) = g(e(\delta)) < \alpha$. Now if $e(\gamma) = \overline{1}^{\mathcal{C}}$ then $e'(\gamma) \geq \alpha$ and $e'(\gamma \to \delta) = e'(\delta) < \alpha$; and if $e(\gamma) < \overline{1}^{\mathcal{C}}$ then $e'(\gamma) = g(e(\gamma)) > g(e(\delta)) = e'(\delta)$, thus again $e'(\gamma \to \delta) = e'(\delta) < \alpha$. On the other hand, assume $e(\gamma \to \delta) = \overline{1}^{\mathcal{C}}$, thus $e(\gamma) \leq e(\delta)$. If $e(\delta) = \overline{1}^{\mathcal{C}}$ then $e'(\gamma \to \delta) \geq e'(\delta) \geq \alpha$. And if $e(\delta) < \overline{1}^{\mathcal{C}}$ then $e'(\gamma) = g(e(\gamma)) \leq g(e(\delta)) = e'(\delta)$ and $e'(\gamma \to \delta) = 1 \geq \alpha$. This proves the claim.

This also finishes the proof of the theorem; indeed, since $e(\varphi) < \overline{1}^{\mathcal{C}}$, then $e'(\varphi) < 1$ as required.

□

Theorem 3.4. $\vdash_{RNM} \varphi$ *if, and only if,* $\models_{[0,1]_{RNM}} \varphi$.

Proof. The proof can be done in a analogous way to the previous theorem for RG, with some necessary changes. Assume $e(\varphi) < \overline{1}^{\mathcal{C}}$ over a countable RNM-chain $\mathcal{C}$ and let $X = \{e(\psi), n_C(e(\psi)) \mid \psi \text{ subformula of } \varphi\} \cup \{\overline{0}^{\mathcal{C}}, \overline{1}^{\mathcal{C}}\}$. Let $\alpha = \min\{r \mid r = 1 \text{ or } \overline{r} \text{ subformula of } \varphi \text{ with } \overline{r}^{\mathcal{C}} = \overline{1}^{\mathcal{C}}\}$. Clearly $\alpha > 0.5$. We define then an order-preserving injection $g : \langle X, \leq_{\mathcal{C}} \rangle \hookrightarrow \langle [1-\alpha, \alpha], \leq \rangle$ which is strictly increasing and negation preserving over the elements of X and such that $g(\overline{r}^{\mathcal{C}}) = r$ for all $\overline{r}^{\mathcal{C}} \in X \setminus \{\overline{0}^{\mathcal{C}}, \overline{1}^{\mathcal{C}}\}$ and, furthermore,

$g(\overline{1}^{\mathcal{C}}) = \alpha$ and $g(\overline{0}^{\mathcal{C}}) = 1-\alpha$. Take into account that the negation is involutive and thus the strict order is preserved by negation. Then, an RNM-evaluation e' over the standard RNM-algebra can be defined by putting for any propositional variable p: $e'(p) = g(e(p))$ if p appears in φ and $e'(p) = 1$ otherwise. Notice that if x is a propositional variable or a truth constant belonging to X, then $e'(x) = g(e(x))$.

Claim. For each ψ which is either a subformula of φ or the negation of a subformula of φ,
(1) if $e(\psi) = \overline{1}^{\mathcal{C}}$, then $e'(\psi) \geq \alpha$,
(2) if $\overline{0}^{\mathcal{C}} < e(\psi) < \overline{1}^{\mathcal{C}}$, then $1-\alpha < e'(\psi) = g(e(\psi)) < \alpha$,
(3) if $e(\psi) = \overline{0}^{\mathcal{C}}$, then $e'(\psi) \leq 1-\alpha$. Therefore, in particular, $e'(\varphi) < 1$.

This is proved again by induction on the complexity of ψ. It is clear for variables and for truth-constants. The induction step for $\neg$ is also straightforward. Suppose $\psi = \gamma \& \delta$.
If $e(\psi) = \overline{1}^{\mathcal{C}}$, then $e(\gamma) = e(\delta) = \overline{1}^{\mathcal{C}}$, hence $e'(\gamma), e'(\delta) \geq \alpha$, so $e'(\psi) \geq \alpha$.
If $\overline{0}^{\mathcal{C}} < e(\psi) < \overline{1}^{\mathcal{C}}$, then $e(\psi) = e(\gamma) \wedge e(\delta)$ and $e(\gamma) > \neg e(\delta)$. Suppose, for instance, $e(\gamma) \leq e(\delta)$. So, $e(\psi) = e(\gamma)$ and $e'(\gamma) = g(e(\gamma)) \in (1-\alpha, \alpha)$. We distinguish two cases:
(a) if $e(\delta) = \overline{1}^{\mathcal{C}}$, then $e'(\delta) \geq \alpha$. Therefore, $e'(\psi) = e'(\gamma) \& e'(\delta) = e'(\gamma) \wedge e'(\delta) = e'(\gamma) = g(e(\gamma)) = g(e(\psi)) \in (1-\alpha, \alpha)$.
(b) if $e(\delta) < \overline{1}^{\mathcal{C}}$, then $e'(\delta) = g(e(\delta)) \in (1-\alpha, \alpha)$. Since g is increasing, $e'(\gamma) \leq e'(\delta)$. Moreover, since $\overline{1}^{\mathcal{C}} > e(\gamma) > e(\neg\delta) > \overline{0}^{\mathcal{C}}$, we obtain $e'(\gamma) > e'(\neg\delta)$ also applying the monotonocity of g. Thus $e'(\psi) = e'(\gamma) \& e'(\delta) = e'(\gamma) \wedge e'(\delta) = e'(\gamma) = g(e(\gamma)) = g(e(\psi)) \in (1-\alpha, \alpha)$.
If $e(\psi) = \overline{0}^{\mathcal{C}}$, then $e(\gamma) \leq \neg e(\delta)$. Now we distinguish three cases:
(a) if $e(\gamma) = \overline{0}^{\mathcal{C}}$, then $e'(\gamma) \leq 1-\alpha$, so $e'(\psi) \leq 1-\alpha$.
(b) if $\overline{0}^{\mathcal{C}} < e(\gamma) \leq \neg e(\delta) < \overline{1}^{\mathcal{C}}$, then $e'(\gamma) \leq \neg e'(\delta)$, so $e'(\psi) = 0 \leq 1-\alpha$.
(c) $\overline{0}^{\mathcal{C}} < e(\gamma) \leq \neg e(\delta) = \overline{1}^{\mathcal{C}}$, then $e(\delta) = \overline{0}^{\mathcal{C}}$, so $e'(\delta) \leq 1-\alpha$ and $e'(\psi) \leq 1-\alpha$. □

4 On finite strong standard completeness for RG and RNM

G and NM are strongly standard complete[4] for arbitrary theories, hence, by Proposition 2.9, RG and RNM are conservative extensions of G an NM respectively. On the other hand RG and RNM are not strong standard complete for arbitrary theories, even for finite theories. Namely, for any rational $0 < r < 1$ and any propositional variable p, $\overline{r} \nvdash_{RG} p$ but it trivially holds that $\overline{r} \models_{[0,1]_{RG}} p$ since there is no evaluation which is a model of $\overline{r}$. The same is also true for RNM. Looking at this example, one could think that the reason of failure is that the theory used, $T = \{\overline{r}\}$, is somewhat special, in the sense that it is not satisfiable. So we could try to check whether strong standard completeness holds restricted to satisfiable theories. Unfortunately, being satisfiable is not a sufficient condition either for strong standard completeness, even for Pavelka-style completeness, as the following example shows.

Example 4.1. Let $T = \{\overline{r} \vee p\}$, where $0 < r < 1$ and p is a propositional variable. It is clear that T is satisfiable for any evaluation e such that $e(p) = 1$, and that $T \models_{[0,1]_{RG}} p$. But again $T \nvdash_{RG} p$ since if so, by the deduction and weak standard completeness theorems for RG, it should also be true that $\models_{[0,1]_{RG}} (\overline{r} \vee p) \to p$, which is false for any evaluation with $e(p) < r$. Moreover Pavelka-style completeness also fails. Namely, it is clear that,

$$\| p \|_T = \inf\{e(p) \mid e(\overline{r} \vee p) = 1\} = 1$$

[4]In [10] it is proved for G and an analogous proof shows the completeness of NM.

but

$$| p |_T = \sup\{s \mid T \vdash_{RG} \overline{s} \to p\} = 0.$$

To prove this last equality take into account that again by the deduction and weak standard completeness theorems for RG, $T \vdash_{RG} \overline{s} \to p$ iff $\models_{[0,1]_{RG}} (\overline{r} \vee p) \to (\overline{s} \to p)$ and this only holds true for $s = 0$. Indeed, if $s \neq 0$, take an evaluation such that $e(p) = 0$, then $e((\overline{r} \vee p) \to (\overline{s} \to p)) = r \Rightarrow_G 0 = 0$. [5]

The same example is also valid for RNM. If $T \vdash_{[0,1]_{RNM}} p$ were true, then we would have $\models_{[0,1]_{RNM}} ((\overline{r} \vee p)\&(\overline{r} \vee p)) \to p$, but this is false (take e such that $e(p) = 0$ and $r > 0.5$). For Pavelka-style completeness notice that similar arguments to the case of RG lead to $|| p ||_T = 1$ and $| p |_T \leq 1 - (r *_{NM} r)$. □

Finally we will prove that RG is strongly standard complete if we restrict ourselves to formulas of the type $\overline{r} \to \varphi$, where φ is a formula without rational truth-constants (a formula of G), and expressing that φ is true at least to the degree r. This type of formulas are also commonly denoted in the fuzzy logic setting as pairs (φ, r). We shall also adopt this notation from now on. Notice that Gerla's fuzzy sets of formulas [7] exactly correspond to sets of this kind of formulas.

We want to show that the following equivalence holds true:

$$\{(\psi_i, r_i) \mid i = 1, 2, .., n\} \vdash_{RG} (\varphi, s)$$
$$\text{if and only if}$$
$$\{(\psi_i, r_i) \mid i = 1, 2, .., n\} \models_{[0,1]_{RG}} (\varphi, s)$$

To prove this result we need some previous results and lemmas. Actually, due to the fact that RG enjoys the (syntactical) deduction theorem and to the weak standard completeness, one can easily notice that proving the above restricted (finite) strong standard completeness for RG amounts to prove the following semantical version of the deduction theorem:

$$\{(\psi_i, r_i) \mid i = 1, 2, .., n\} \models_{[0,1]_{RG}} (\varphi, s)$$
$$\text{if, and only if,}$$
$$\models_{[0,1]_{RG}} (\textstyle\bigwedge_{i=1,2,..,n} (\psi_i, r_i)) \to (\varphi, s).$$

Accordingly, in what follows we prove this.

Lemma 4.2. *Let $a \in (0, 1]$ and define a mapping $f_a : [0, 1] \to [0, 1]$ as follows:*

$$f_a(x) = \begin{cases} 1, & \text{if } x \geq a \\ x, & \text{otherwise} \end{cases}$$

Then f_a is a morphism with respect to the standard Gödel truth functions. Therefore, if e is a G-evaluation of formulas, then $e_a = f_a \circ e$ is another G-evaluation.

Proof. We have to prove: (i) $f_a(0) = 0$, (ii) $f_a(\min(x, y)) = \min(f_a(x), f_a(y))$, and (iii) $f_a(x \Rightarrow_G y) = f_a(x) \Rightarrow_G f_a(y)$. (i) is obvious and (ii) is also easy immediate since f_a is a non-decreasing function. So let us prove (iii). We consider two cases:

[5] Notice that these negative results are also valid for any logic standard complete with respect to a continuous t-norm defining a SBL-algebra, because in such a logic the generalized deduction theorem (like in BL) holds and the negation is Gödel negation.

Case A : $x \leq y$, $x \Rightarrow_G y = 1$. In this case, $f_a(x) \leq f_a(y)$ as well, hence $f_a(x \Rightarrow_G y) = f(1) = 1 = f_a(x) \Rightarrow_G f_a(y)$.

Case B : $x > y$, $x \Rightarrow_G y = y$. Now we distinguish the following three sub-cases:

B.1 : $a \leq y < x$, $f_a(x \Rightarrow_G y) = 1$. In this case $f_a(x) = f_a(y) = 1$ and hence $f_a(x) \Rightarrow_G f_a(y) = 1$;

B.2 : $y < a \leq x$, $f_a(x \Rightarrow_G y) = y$. In this case $f_a(x) = 1, f_a(y) = y$ and hence $f_a(x) \Rightarrow_G f_a(y) = y$;

B.3 : $y < x < a$, $f_a(x \Rightarrow_G y) = y$. In this case $f_a(y) = y, f_a(x) = x$, and hence $f_a(x) \Rightarrow_G f_a(y) = y$.

So, in any of the subcases, $f_a(x \Rightarrow_G y) = f_a(x) \Rightarrow_G f_a(y)$.

This ends the proof. □

Theorem 4.3.
$\{(\varphi_1, \alpha_1), \ldots, (\varphi_n, \alpha_n)\} \models_{[0,1]_{RG}} (\psi, \beta)$ *iff* $\models_{[0,1]_{RG}} (\bigwedge_{i=1}^n (\varphi_i, \alpha_i)) \to (\psi, \beta)$.

Proof. One direction is easy. As for the difficult one, it is enough to prove that if there is an evaluation e which is not a model of $(\bigwedge_{i=1}^n (\varphi_i, \alpha_i)) \to (\psi, \beta)$, then we can find another evaluation e' which is model of $\{(\varphi_1, \alpha_1), \ldots, (\varphi_n, \alpha_n)\}$ and not of (ψ, β).

So let e be such that $e((\bigwedge_{i=1}^n (\varphi_i, \alpha_i)) \to (\psi, \beta)) < 1$. If e is a model of every (φ_i, α_i) for $i = 1, ..., n$, then we can take $e' = e$ and the problem is solved. Otherwise, there exists some $1 \leq j \leq n$ for which $\alpha_j > e(\varphi_j)$ and thus $e((\varphi_j, \alpha_j)) = e(\varphi_j) < 1$. Let $J = \{j \mid \alpha_j > e(\varphi_j)\}$ and let $a = e(\bigwedge_{i=1}^n (\varphi_i, \alpha_i)) = \min\{e(\varphi_j) \mid j \in J\}$. Then the RG-evaluation e' such that $e' = e_a$ over the propositional variables does the job. Namely, by Lemma 4.2, over Gödel formulas we have $e' = e_a \geq e$, so e' is still model of those (φ_i, α_i)'s for $i \in \{1, \ldots, n\} \setminus J$. But now, $e'(\varphi_j) = 1$ for every $j \in J$, so e' is also a model of $\{(\varphi_1, \alpha_1), \ldots, (\varphi_n, \alpha_n)\}$. On the other hand, since $e((\bigwedge_{i=1}^n (\varphi_i, \alpha_i)) \to (\psi, \beta)) < 1$, it must be $\beta > e(\psi)$ and $a = e(\bigwedge_{i=1}^n (\varphi_i, \alpha_i)) > e(\psi)$. Now, by Lemma 4.2, $e'(\psi) = e_a(\psi) = e(\psi)$, hence $e'(\psi, \beta)) = e(\psi, \beta) < 1$. Therefore we have proved the theorem. □

After this last theorem, the announced result of finite strong completeness of RG when restricted to formulas of the kind (φ, α) comes as an easy corollary.

Theorem 4.4. $\{(\psi_i, r_i) \mid i = 1, 2, .., n\} \vdash_{RG} (\varphi, s)$ *if, and only if,* $\{(\psi_i, r_i) \mid i = 1, 2, .., n\} \models_{[0,1]_{RG}} (\varphi, s)$.

Remark 4.5. Last theorem is valid for the restricted language of formulas of the kind $\overline{r} \to \varphi$. The validity of these type of formulas expresses that r is a lower bound for the truth value of φ. We might wonder whether an analogous theorem could be also valid for formulas of the type $\varphi \to \overline{r}$, whose validity expresses that r is an upper bound for the truth value of φ. Unfortunately this is not true as we can see with the following simple example. It is easy to check that

$$\neg\neg p \to \overline{0.3} \models_{[0,1]_{RG}} p \to \overline{0}$$

since the premise is only true if $e(p) = 0$, while

$$\not\models_{[0,1]_{RG}} (\neg\neg p \to \overline{0.3}) \to (p \to \overline{0})$$

since if $e(p) = c$ for $c > 0.3$ an easy computation shows that $e((\neg\neg p \to \overline{0.3}) \to (p \to \overline{0})) = 0$.

Finally, we show that a similar, although a bit weaker, complenetess result holds for RNM.

Lemma 4.6. *Let $a \in (\frac{1}{2}, 1]$ and define a mapping $f^a : [0,1] \to [0,1]$ as follows:*

$$f^a(x) = \begin{cases} 1, & \text{if } x \geq a \\ 0, & \text{if } x \leq 1-a \\ x, & \text{otherwise} \end{cases}$$

Then f^a is a morphism with respect to the standard Nilpotent Minimum logic truth functions. Therefore, if e is a NM-evaluation of formulas, then $e^a = f^a \circ e$ is another NM-evaluation.

Proof. Since $\Rightarrow_{NM}$ is definable from $*_{NM}$ and the standard negation $n(x) = 1-x$, it is enough to prove: (i) $f^a(0) = 0$, (ii) $f^a(x *_{NM} y) = f^a(x) *_{NM} f^a(y)$, and (iii) $f^a(1-x) = 1 - f^a(x)$. (i) is obvious and (iii) is easy. As for (ii) we consider the following cases. For $x, y > \frac{1}{2}$, $x *_{NM} y = \min(x,y)$ and f^a is non-decreasing, so (ii) easily holds true. For $x, y \leq \frac{1}{2}$, $x *_{NM} y = 0$ and f^a is non-increasing, so $f^a(x *_{NM} y) = f^a(x) *_{NM} f^a(y) = 0$. Finally, assume $x > \frac{1}{2} \geq y$. In this case a careful check shows that

$$f^a(x *_{NM} y) = f^a(x) *_{NM} f^a(y) = \begin{cases} y, & \text{if } y > \max(1-x, 1-a) \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the lemma is proved. □

Theorem 4.7. *Let $\alpha_1, \ldots, \alpha_n \in (\frac{1}{2}, 1]$. Then:*
$\{(\varphi_1, \alpha_1), \ldots, (\varphi_n, \alpha_n)\} \models_{[0,1]_{RNM}} (\psi, \beta)$ *iff* $\models_{[0,1]_{RNM}} (\&_{i=1}^n (\varphi_i, \alpha_i))^2 \to (\psi, \beta)$.

Proof. One direction is easy. As for the difficult one, it is enough to prove that if there is an evaluation e which is not a model of $(\&_{i=1}^n (\varphi_i, \alpha_i))^2 \to (\psi, \beta)$, then we can find another evaluation e' which is model of $\{(\varphi_1, \alpha_1), \ldots, (\varphi_n, \alpha_n)\}$ and not of (ψ, β).

So let e be such that $e((\&_{i=1}^n (\varphi_i, \alpha_i))^2 \to (\psi, \beta)) < 1$, i.e. $e((\&_{i=1}^n (\varphi_i, \alpha_i))^2) > e((\psi, \beta))$. This means that:
(i) $e((\&_{i=1}^n (\varphi_i, \alpha_i))^2) > 0$, hence for all i we have $e((\varphi_i, \alpha_i)^2) > 0$, hence for all i we have $e((\varphi_i, \alpha_i)^2) = e((\varphi_i, \alpha_i)) = \max(1-\alpha_i, e(\varphi_i)) = e(\varphi_i) > \frac{1}{2}$, and also $e((\&_{i=1}^n (\varphi_i, \alpha_i))^2) = \min_{i=1}^n e((\varphi_i, \alpha_i))$;
(ii) $e((\psi, \beta)) < 1$, hence $\beta > e(\psi)$ and $e((\psi, \beta)) = \max(1-\beta, e(\psi))$.
Therefore we are assuming an evaluation e such that $\min_{i=1}^n e((\varphi_i, \alpha_i)) > \max(1-\beta, e(\psi))$, with $\beta > e(\psi)$.

If e is a model of every (φ_i, α_i) for $i = 1, ..., n$, then we can take $e' = e$ and the problem is solved. Otherwise, there exists some $1 \leq j \leq n$ for which $\alpha_j > e(\varphi_j)$ and thus $e((\varphi_j, \alpha_j)) = \max(1-\alpha_j, e(\varphi_j)) = e(\varphi_j) < 1$, the last equality due to the fact that we are assuming $\max(1-\alpha_j, e(\varphi_j)) > \frac{1}{2}$ and $\alpha_j > \frac{1}{2}$.

Let $J = \{j \mid \alpha_j > e(\varphi_j)\}$ and let $a = e((\&_{i=1}^n (\varphi_i, \alpha_i))^2) = \min\{e(\varphi_j) \mid j \in J\}$. Then the RNM-evaluation e' such that $e' = e^a$ over the propositional variables does the job. Namely, by Lemma 4.6, over NM-formulas χ such that $e(\chi) \geq 1-a$, we have $e'(\chi) = e^a(\chi) \geq e(\chi)$, so e' is still model of those (φ_i, α_i)'s for $i \in \{1, \ldots, n\} \setminus J$. Moreover now, $e'(\varphi_j) = 1$ for every $j \in J$, so e' is also a model of $\{(\varphi_1, \alpha_1), \ldots, (\varphi_n, \alpha_n)\}$.

On the other hand, since $\beta > e(\psi)$ and $e(\psi) < a$, it turns out, due to Lemma 4.6, that $e'(\psi) = e^a(\psi) \leq e(\psi)$ and therefore $\beta > e'(\psi)$ as well. So, $e'((\psi, \beta)) < 1$. This ends the proof. □

As in the case of RG, from this last result it follows the next restricted form of finite strong standard completeness for RNM.

Theorem 4.8. *Let* $\alpha_1, \ldots, \alpha_n \in (\frac{1}{2}, 1]$. *Then:*
$\{(\varphi_1, \alpha_1), \ldots, (\varphi_n, \alpha_n)\} \vdash_{RNM} (\psi, \beta)$ *iff* $\{(\varphi_1, \alpha_1), \ldots, (\varphi_n, \alpha_n)\} \models_{[0,1]_{RNM}} (\psi, \beta)$.

One could ask whether the conditions $\alpha_1, \ldots, \alpha_n \in (\frac{1}{2}, 1]$ in the above theorem are actually necessary. In fact this is so, as the following example shows. It is easy to check that

$$\{(p \vee q, 0.7), (\neg p, 0.35)\} \models_{[0,1]_{RNM}} (q, 0.7)$$

since for any evaluation e in $[0,1]_{RNM}$ such that $\max(e(p), e(q)) \geq 0.7$ and $e(p) \leq 0.65$ necessarily it must be $e(q) \geq 0.7$. On the other hand, it is also not difficult to check that

$$\not\models_{[0,1]_{RNM}} [(p \vee q, 0.7)^2 \& (\neg p, 0.35)^2] \to (q, 0.7).$$

It is enough to take an RNM-evaluation e such that $e(p) = 0.7$ and $e(q) = 0.6$: the left-hand side of the implication is evaluated to 0.65 while the right-hand side is evaluated to 0.6.

Finally notice in this case the fact that negation is involutive implies the equivalence between formulas $\varphi \to \overline{r}$ and $\neg\overline{r} \to \neg\varphi$, which in turn implies that, in contrast to what happens with RG, Theorem 4.7, and thus the finite strong standard completeness as well, is also valid for formulas of type $\varphi \to \overline{r}$ with $r \in [0, \frac{1}{2})$.

5 Three families of rational extensions of the Weak Nilpotent logic

Weak Nilpotent Minimum logic was introduced in [3] as the axiomatic extension of MTL by the following axiom,

(WNM) $$(\varphi \& \psi \to \overline{0}) \vee (\varphi \wedge \psi \to \varphi \& \psi)$$

and proved to be standard complete with respect to the family of WNM t-norms and their residua. WNM t-norms are left-continuous t-norms defined from a weak negation function[6] n and the minimum t-norm in the following way

$$x *_n y = \begin{cases} 0, & \text{if } x \leq n(y) \\ \min(x, y), & \text{otherwise} \end{cases}$$

Well-known particular cases of WNM t-norms are the minimum t-norm (when n is Gödel negation) and Fodor's nilpotent t-norm (when $n(x) = 1 - x$).

WNM logic enjoys the same type of deduction theorem as NM logic, that is, it holds that $T \cup \{\varphi\} \vdash_{WNM} \psi$ iff $T \vdash_{WNM} \varphi \& \varphi \to \psi$.

The whole structure of the variety of WNM-algebras is still not known and there is not a general result giving, for each WNM t-norm $*$, the axiomatic characterization of the logic complete with respect to $*$ and its residuum, denoted WNM_*. The lack of this axiomatics makes impossible to define in general its corresponding rational logic $RWNM_*$ (using the notation introduced in the preliminaries), with the exception of Gödel logic (G) and Nilpotent

[6] A weak negation function is a mapping $n : [0,1] \to [0,1]$ such that n is decreasing, $n(0) = 1$, $n(1) = 0$ and $n(n(x)) \geq x$ for all $x \in [0,1]$.

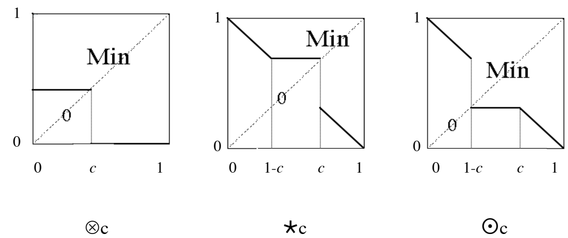

Figure 1: WNM t-norms $*_c$, $\star_c$ and $\circ_c$, respectively.

Minimum logic (NM), both extensions of WNM logic, whose rational expansions have been studied in the previous sections.

In this section we first introduce three new axiomatic extensions of WNM, two weaker than G and two weaker than NM, whose corresponding varieties are proved to be generated by a WNM t-norm and hence they are suitable for defining their rational expansions. We then prove the (weak) standard completeness of these rational expansions and finally, analogously to the cases of RG and RNM, we prove they are strongly finite standard complete when the language is restricted to formulas of type (φ, r).

5.1 Characterization of three subvarieties of WNM

We now introduce three classes of WNM t-norms. Let $*_c$ for any $c \in [0, 1)$, $\star_c$ for any $c \in [\frac{1}{2}, 1)$ and $\circ_c$ for any $c \in [\frac{1}{2}, 1]$ be the WNM t-norms defined respectively by the negation functions on [0, 1] n_{*_c}, $n_{\star_c}$ and $n_{\circ_c}$ given in Figure 1 and defined by,

$$n_{*_c}(x) = \begin{cases} 1, & \text{if } x = 0 \\ c, & \text{if } x \leq c \\ 0, & \text{otherwise} \end{cases}$$

$$n_{\star_c}(x) = \begin{cases} 1 - x, & \text{if } x \in [0, 1-c] \cup (c, 1] \\ c, & \text{otherwise} \end{cases}$$

$$n_{\circ_c}(x) = \begin{cases} 1 - x, & \text{if } x \in [0, 1-c] \cup [c, 1] \\ 1 - c, & \text{otherwise} \end{cases}$$

Remark 5.1. It is the case that for any $c_1, c_2 > 0$ the corresponding t-norms $*_{c_1}$ and $*_{c_2}$ are isomorphic, while for any $1 > c_1, c_2 > 1/2$ the pairs of t-norms $\star_{c_1}$ and $\star_{c_2}$, and $\circ_{c_1}$, and $\circ_{c_2}$ are also isomorphic[7]. However the limiting cases of these families behave different. Namely, when $c = 0$, $*_0 = \min$ and it is not isomorphic to any $*_c$ for $c > 0$. Similarly, for $c = 1/2$, we have that $\star_{1/2} = \circ_{1/2} = *_{NM}$ (the nilpotent minimum t-norm), which is isomorphic neither to $\star_c$ nor $\circ_c$ for $c > 1/2$. Finally, for $c = 1$, $\circ_1 = \min$, which is obvioulsy not isomorphic to $\circ_c$ for any $c < 1$.

[7] Actually, the two families of t-norms $\{\star_c : c > 1/2\}$ and $\{\circ_c : c > 1/2\}$ are also respectively isomorphic to those families of weak nilpotent minimums obtained by distorting any involutive negation function n (instead of the standard negation $\neg x = 1 - x$) with a constant segment of value $n(c)$ or c respectively in the interval $[n(c), c]$.

Next we define three corresponding axiomatic extensions of WNM logic. WNM_* is the axiomatic extension of WNM by adding the axiom

$$(\varphi \to \varphi^2) \to ((\varphi \wedge \psi) \to (\varphi \& \psi)),$$

$WNM_\star$ is the axiomatic extension of WNM by adding the axiom

$$(\neg\neg\varphi \to \varphi) \vee (\neg\neg\varphi \leftrightarrow \neg\varphi),$$

and $WNM_\circ$ is the axiomatic extension of WNM by adding the axioms

$$(\neg\neg\varphi \to \varphi) \vee (\neg\neg\psi \to \psi) \vee ((\neg\varphi \leftrightarrow \neg\psi) \wedge (\neg\varphi \to \varphi))$$
$$(\neg\neg p(\varphi) \to p(\varphi)) \vee ((\neg\neg p(\psi) \to p(\psi)) \to (p(\varphi) \to p(\psi)))$$

where $p(\chi)$ denotes $\chi \vee \neg\chi$. Actually, Gödel logic is an axiomatic extension of both WNM_* and $WNM_\circ$, while Nilpotent Minimum logic NM is an axiomatic extension of both $WNM_\star$ and $WNM_\circ$.

Next we will prove that, for any $0 < c < 1$, WNM_* is the logic of the t-norm $*_c$, and for any $1/2 < c < 1$, $WNM_\star$ and $WNM_\circ$ are the logics of $\star_c$ and $\circ_c$ respectively.

Lemma 5.2. *Let $\mathcal{A}$ be a WNM-chain . Then:*

(i) $\mathcal{A}$ is a WNM_*-chain iff there exists an element $c \in A$ such that the negation in $\mathcal{A}$ is like n_{*_c}, i.e. $\neg x = c$ for $\overline{0}^{\mathcal{A}} < x \leq c$ and $\neg x = \overline{0}^{\mathcal{A}}$ for $x > c$.

(ii) $\mathcal{A}$ is a $WNM_\star$-chain iff it has a fixpoint c and it is such that the negation is involutive except for a segment ending in c in which $\neg$ is constantly c.

(iii) $\mathcal{A}$ is a $WNM_\circ$-chain iff there exists an element d such that $d \leq \neg d$ and that $\neg$ is involutive in the segments $[\overline{0}^{\mathcal{A}}, d]$ and in $[\neg d, \overline{1}^{\mathcal{A}}]$ and $\neg x = d$ for $x \in (d, \neg d]$.

Proof. (i) If $\neg x = \overline{0}^{\mathcal{A}}$ for all x different from $\overline{0}^{\mathcal{A}}$, then $\neg$ is Gödel negation and $\& = \min$. Otherwise assume there exists $x \in A$ different from $\overline{0}^{\mathcal{A}}$ such that $\neg x > \overline{0}^{\mathcal{A}}$, and let $c = \neg x$. Let us show that $c = \max\{x \in A \mid x\&x = \overline{0}^{\mathcal{A}}\}$. Indeed, if $c\&c = n(x)\&n(x) > 0$ then, by the axiom of WNM_*, $n(x)\&x = x \wedge n(x) > 0$, contradiction. Hence $c\&c = \overline{0}^{\mathcal{A}}$. On the other hand, if $y\&y > 0$ then it must be $y > n(x) = c$ (otherwise, if $y \leq n(x)$, then $y\&y \leq n(x)\&n(x) = 0$).

Moreover, by the axiom of WNM_*, if $y > c$, then $y\&x = x \wedge y$ for all x. Altogether leads to have, for all $x, y \in A$:

$$x\&y = \begin{cases} x \wedge y, & \text{if } x > c \,\text{or}\, y > c \\ 0, & \text{otherwise} \end{cases}$$

This proves (i).

(ii) If $x = \neg\neg x$ for all x then $\neg$ is involutive and $\mathcal{A}$ is a NM-chain. Otherwise assume there exists $x \in A$ such that $x < \neg\neg x$, and let $c = \neg x$. Observe that $\neg c = c$ since, by the axiom of $\mathrm{WNM}_\star$, $x < \neg\neg x$ implies $\neg c = \neg\neg x = \neg x = c$. Hence also $x \leq \neg\neg x = c$. Let us show:

(1) If $y < \neg\neg y$ then $\neg y = c$.

proof: if $y < \neg\neg y$ then $\neg y = \neg\neg y$, hence $\neg y$ is a fix point of $\neg$ and thus necessarily $\neg y = c$ (since c is a fix point and at most there is one).

(2) If $y > c$ then $y = \neg\neg y$.

proof: if $y > c$ then $\neg y \leq c$. If $\neg y < c$ then by (1) $y = \neg\neg y$. If $\neg y = c$ assume $y < \neg\neg y$, hence $c = \neg y = \neg\neg y > y$, contradiction.

(3) If $y < \neg y$ and $\neg y \neq c$ then $y = \neg\neg y$.

proof: otherwise, by the axiom of WNM$_\star$, it would be $\neg\neg y = \neg y$, and hence $\neg y = c$, contradiction.

Therefore, $\neg$ is such that it is involutive on the set $B = \{x, \neg x \mid x > c\}$ and $\neg x = c$ for $x \in A \setminus B$. This proves (ii).

(iii) If $x = \neg\neg x$ for all x then $\neg$ is involutive and $\mathcal{A}$ is a NM-chain. Otherwise assume there exists $x \in A$ such that $x < \neg\neg x$, and let $d = \neg x$ and $c = \neg d = \neg\neg c$. Observe the following:

(1) By the first axiom of WNM$_\circ$, if y and z are such that $y < \neg\neg y$ and $z < \neg\neg z$ then $\neg y = \neg z = d$. Moreover, if $y < \neg\neg y$ then $\neg y \leq y$ and reciprocally if $\neg y > y$ then $y = \neg\neg y$. If also follows that $d < c$. Otherwise, if $d \geq c$ then $\neg x = d \geq c = \neg\neg x > x$, contradiction.

(2) If $y > c$ then $y = \neg\neg y$.

proof: indeed, if $y < \neg\neg y$ then $\neg y = d$, hence $y < \neg\neg y = c$, contradiction.

(3) If $y = \neg\neg y > \neg y$ and $z < \neg\neg z$ then $z > y$.

proof: this follows from the second axiom of WNM$_\circ$.

(4) If $y > d$ then $y > \neg y$.

proof: If $y > d$ then $\neg y \leq \neg d = c$. Now assume $y \leq \neg y$, then by (3) $\neg y \geq c$, hence $\neg y = c$, hence $\neg\neg y = \neg c = d < y$, contradiction.

All these properties lead to have $\neg$ defined as follows: if $d < x \leq c$ then $\neg x = d$, otherwise it is such that $\neg\neg x = x$. □

Corollary 5.3. *Let $\mathcal{A}$ be a WNM-chain on [0, 1] and let $\neg$ be its negation. Then:*

- *$\mathcal{A}$ is a WNM_*-chain iff there exists $c < 1$ such that $\neg = n_{*_c}$.*
- *$\mathcal{A}$ is a $WNM_\star$-chain iff there exists $c \in [1/2, 1)$ such that $\neg$ is isomorphic[8] to $n_{\star_c}$.*
- *$\mathcal{A}$ is a $WNM_\circ$-chain iff there exists $c \in [1/2, 1]$ such that $\neg$ is isomorphic to $n_{\circ_c}$.*

Theorem 5.4. *If $+$ denotes $*$, $\star$ or $\circ$, the logic WNM_+ is complete with respect to each one of the following sets of chains:*

(1) the linearly ordered WNM_+-chains.

(2) the WNM_+-chains over [0,1].

(3) the WNM_+-chain over [0,1] defined by a t-norm $+_c$, where $0 < c < 1$ for $+$ being $*$ and $1/2 < c < 1$ for $+$ being $\star$ or $\circ$.

Proof. We sketch the proof for WNM_*, the proofs for $WNM_\star$ and $WNM_\circ$ are similar. The first result (1) is a particular case of the general result about all axiomatic extensions of MTL (see preliminaries). The proof of (2) is actually completely analogous with the obvious changes to the proof of standard completeness of WNM given in [3]. Finally, from the last corollary, the only WNM_*-chains on [0, 1] are those defined by the WNM t-norms $*_c$, for $c \in [0, 1)$. But it is easy to prove that all t-norms $*_c$ with $c > 0$ are isomorphic and that $*_0$ (= min) is isomorphic to the subalgebra of any of them defined on the subset $\{0\} \cup (c, 1]$. Thus any WNM_*-chain over [0,1] defined by a t-norm $*_c$ with $c > 0$ generates the whole variety of WNM_*-algebras. Hence (3) is proved. □

[8] In the sense of Trillas [12].

Notice that, according to Footnote 7, in (3) of the above theorem we could also consider other isomorphic t-norms for the cases of $+$ being $\star$ and $\circ$.

5.2 Standard completeness of logics $RWNM_*$, $RWNM_\star$ and $RWNM_\circ$

Now we will consider rational expansions of the logics WNM_*, $WNM_\star$ and $WNM_\circ$. To do so, as previously done, we add to the language as many truth-constants as rationals in (0,1), and we add to the each one of these logics a set of book-keeping axioms corresponding to one t-norm whose induced standard algebra belongs to the corresponding variety of the logic. Actually, given any of the three logics, we obtain a different rational expansion for each particular t-norm used to define the book-keeping axioms, since these axioms are obviously different. However, for each one of these three logics, only two of their rational expansions are really different, in the sense of Theorem 2.10 of not being translatable. Namely, according to Remark 5.1 and Corollary 5.3, it will suffice for WNM_* to consider the rational expansions $RWNM_*(*_c)$ only for two t-norms, $*_0 = \min$ and one $*_c$ for some $c > 0$, while for the cases of $WNM_\star$ and $WNM_\circ$ it will suffice to consider the rational expansions $RWNM_\star(\star_c)$ and $RWNM_\circ(\circ_c)$ only for the t-norm $\star_{1/2} = \circ_{1/2}$ and for one pair of t-norms $\star_c$ and $\circ_c$ respectively for some $c > 1/2$.

In order to prove standard completeness for these rational expansions, we start with the following general lemma that describes how rational constants are distributed in the linearly ordered algebras of these logics.

Lemma 5.5. *Let $+$ denote $*$, $\star$ or $\circ$, and let c be any suitable parameter defining the t-norms $*_c$, $\star_c$ or $\circ_c$. For any $RWNM_+(+_c)$-chain $\mathcal{A} = \langle A, \&, \to, \wedge, \vee, \{\overline{r}^{\mathcal{A}} : r \in \mathbb{Q} \cap [0,1]\}\rangle$ (hence satisfying the book-keeping axioms of $+_c$) the following conditions hold:*

(i) The set $R_1 = \{r \in \mathbb{Q} \mid \overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}\}$ is a right-closed interval with 1 as upper bound.

(ii) The set $R_0 = \{r \in \mathbb{Q} \mid \overline{r}^{\mathcal{A}} = \overline{0}^{\mathcal{A}}\}$ coincides with $n_{+_c}(R_1)$.

(iii) $\overline{r}^{\mathcal{A}} < \overline{s}^{\mathcal{A}}$ for any rationals $r < s$ not belonging to $R_1 \cup R_0$.

(iv) If $r \in R_1$ then $n_{+_c}(r) < r$. If $r \in R_0$ then $n_{+_c}(r) \geq r$.

(v) If $r < n_{+_c}(r)$ then $\overline{r}^{\mathcal{A}} < \neg\overline{r}^{\mathcal{A}}$.

(vi) If $r = n_{+_c}(r)$ then $\overline{r}^{\mathcal{A}} = \neg\overline{r}^{\mathcal{A}}$.

(vii) If $r > n_{+_c}(r)$ then $\overline{r}^{\mathcal{A}} > \neg\overline{r}^{\mathcal{A}}$.

(v) If $n_{+_c}(r) = n_{+_c}(s)$ then $\neg\overline{r}^{\mathcal{A}} = \neg\overline{s}^{\mathcal{A}}$.

(v) If $r = n_{+_c}(n_{+_c}(r))$ then $\overline{r}^{\mathcal{A}} = \neg\neg\overline{r}^{\mathcal{A}}$.

Proof. Items (i), and (ii) and (iii) are generalizations of Lemmas 3.1 and 3.2 and are proved very similarly. The rest of items are quite straightforward. □

Let $RWNM_{*_c}$, $RWNM_{\star_c}$ and $RWNM_{\circ_c}$ be these logics respectively, where c is any suitable parameter defining the t-norms $*_c$, $\star_c$ or $\circ_c$. Next result establishes the (weak) standard completeness of the logics with respect to any WNM t-norm of their corresponding families.

Theorem 5.6. *If $+$ denotes $*$, $\star$ or $\circ$, and c any suitable parameter defining the t-norms $*_c$, $\star_c$ or $\circ_c$, then $\vdash_{RWNM_{+_c}} \varphi$ if, and only if, $\models_{[0,1]_{+_c}} \varphi$.*

Proof. The proof for $RWNM_{*_c}$ is very similar to that of Theorem 3.3 for RG and the proof for $RWNM_{\star_c}$ is very similar to that of Theorem 3.4 for RNM. We will not repeat them at full detail here since the contructions are essentially the same. Given that $\nvdash_{RWNM_+} \varphi$, there exists a countable $RWNM_+$-chain $\mathcal{C}$ and an evaluation over C such that $e(\varphi) < \overline{1}^{\mathcal{C}}$, and the task is to define another evaluation e' over a $RWNM_+$-chain on [0, 1] such that $e'(\varphi) < 1$. Let again $X = \{e(\psi), n_C(e(\psi)), n_C(n_C(e(\psi))) \mid \psi \text{ subformula of } \varphi\} \cup \{\overline{0}^{\mathcal{C}}, \overline{1}^{\mathcal{C}}\}$ and define $\alpha = \min\{r \mid r = 1 \text{ or } \overline{r} \text{ subformula of } \varphi \text{ with } \overline{r}^{\mathcal{C}} = \overline{1}^{\mathcal{C}}\}$. Note that $\alpha > c$.

The previous Lemma 5.5 allows to define a strictly increasing injection

$$g : \langle X, \leq_C \rangle \hookrightarrow \langle [n_{+_c}(\alpha), \alpha], \leq \rangle$$

which also preserves the negation in X and such that $g(\overline{r}^{\mathcal{C}}) = r$ for all $\overline{r}^{\mathcal{C}} \in X \setminus \{\overline{0}^{\mathcal{C}}, \overline{1}^{\mathcal{C}}\}$ and, furthermore, $g(\overline{1}^{\mathcal{C}}) = \alpha$ and $g(\overline{0}^{\mathcal{C}}) = n_{+_c}(\alpha)$. Then the evaluation e' we are looking for is defined by putting for each propositional variable p, $e'(p) = g(e(p))$ if $p \in X$ and $e'(p) = 1$ otherwise, together with $e'(\overline{r}) = r$ for each rational r. The proof for $RWNM_{\circ_c}$ is a slight modification of the previous. □

5.3 On finite strong standard completeness for $RWNM_*$, $RWNM_\star$ and $RWNM_\circ$

Regarding the issue of strong (standard) completeness the situation is again very similar to that for RG and RNM. Actually, the same examples given in Section 4 also work, with slight adaptations, to show that the rational logics $RWNM_*$, $RWNM_\star$ and $RWNM_\circ$ have neither (finite) strong standard completeness nor Pavelka-style completeness. Nevertheless, again as in the cases of RG and RNM, these logics are (finite) strong standard complete if we restrict ourselves to formulas of type $\overline{r} \to \varphi$. And again, due to the (syntactical) deduction theorem for WNM, to show these results it will be enough to prove the following semantical counterpart of the deduction theorem for our logics.

Lemma 5.7. *Let $a \in (c, 1]$ and define a mapping $f^a : [0,1] \to [0,1]$ as follows:*

$$f^a(x) = \begin{cases} 1, & \text{if } x \geq a \\ 0, & \text{if } x \leq n_{+_c}(a) \\ x, & \text{otherwise} \end{cases}$$

Then f^a is a morphism with respect to the operations of the algebra $[0,1]_{+_c}$. Therefore, if e is a WNM_{+_c}-evaluation of formulas, then $e^a = f^a \circ e$ is another WNM_{+_c}-evaluation.

Theorem 5.8. *Let $r_1, \ldots, r_n \in (c, 1]$ and $s \in [0, 1]$. Then:*
$\{(\varphi_1, r_1), \ldots, (\varphi_n, r_n)\} \models_{[0,1]_{+_c}} (\psi, s)$ iff $\models_{[0,1]_{+_c}} (\&_{i=1}^n (\varphi_i, r_i))^2 \to (\psi, s)$.

Proof. One direction is easy. As for the difficult one, it is enough to prove that if there is an evaluation e which is not a model of $(\&_{i=1}^n (\varphi_i, r_i))^2 \to (\psi, s)$, then we can find another evaluation e' which is model of $\{(\varphi_1, r_1), \ldots, (\varphi_n, r_n)\}$ and not of (ψ, s).

Let e be an evaluation such that $e((\&_{i=1}^n (\varphi_i, r_i))^2 \to (\psi, s)) < 1$, i.e. $e((\&_{i=1}^n (\varphi_i, r_i))^2) > e((\psi, s))$. This means that:

(i) $e((\&_{i=1}^{n}(\varphi_i, r_i))^2) > 0$, hence this is also valid for each i and thus $e((\varphi_i, r_i)^2) = e((\varphi_i, r_i)) > c$ and $e((\&_{i=1}^{n}(\varphi_i, r_i))^2) = \min_{i=1}^{n} e((\varphi_i, r_i)) > c$;
(ii) $e((\psi, s)) < 1$, hence $s > e(\psi)$ and $e((\psi, s)) = \max(1 - s, e(\psi))$.
Therefore we are assuming an evaluation e such that $\min_{i=1}^{n} e((\varphi_i, r_i)) > \max(1 - s, e(\psi))$, with $s > e(\psi)$.

If e is a model of every (φ_i, r_i) for $i = 1, ..., n$, then we can take $e' = e$ and the problem is solved. Otherwise, there exists some $1 \leq j \leq n$ for which $r_j > e(\varphi_j)$ and thus $e((\varphi_j, r_j)) = \max(1 - r_j, e(\varphi_j)) = e(\varphi_j) < 1$, the last equality due to the fact that we are assuming $\max(1 - r_j, e(\varphi_j)) > c$ and $r_j > c$.

Let $J = \{j \mid r_j > e(\varphi_j)\}$ and let $a = e((\&_{i=1}^{n}(\varphi_i, r_i))^2) = \min\{e(\varphi_j) \mid j \in J\}$. Then the $RWNM_+$-evaluation e' such that $e' = e^a$ over the propositional variables does the job. Namely, by the corresponding transformation of Lemma 5.7, over $RWNM_+$-formulas, e' is a model of (φ_i, r_i)'s for all i.

On the other hand, since $s > e(\psi)$ and $e(\psi) < a$, it turns out, due to the corresponding translation of Lemma 5.7, that $e'(\psi) = e^a(\psi) \leq e(\psi)$ and therefore $s > e'(\psi)$ as well. So, $e'((\psi, s)) < 1$ and the proof is completed. □

Finally, finite strong standard completeness results for $RWNM_*$, $RWNM_\star$ and $RWNM_\circ$, when restricted to formulas of the kind (φ, r), come as an easy consequence as the last theorem. Next corollary summarizes these three results (for $+$ being any one of the three symbols $*$, $\star$ or $\circ$) with the restriction for the values r_i as in the corresponding theorem.

Corollary 5.9. *$\{(\psi_i, r_i) \mid i = 1, 2, .., n\} \vdash_{RWNM_{+c}} (\varphi, s)$ if, and only if, $\{(\psi_i, r_i) \mid i = 1, 2, .., n\} \models_{[0,1]_{+c}} (\varphi, s)$.*

6 Conclusions and further work

Book-keeping axioms can be defined only when we have chosen a t-norm and its residuum. Thus the process to create a new logic by adding truth constants has only sense if the initial logic is "the" logic of a left continuous t-norm and its residuum. RPL, RG (rational Gödel), RNM (rational Nilpotent Minimum) and the examples of RWNM studied in the paper, have sense since Ł, G and NM and the consisered axiomatic extensions of WNM are "the" logics of Lukasiewicz t-norm, of minimum t-norm, nilpotent minimum t-norm and of the t-norms $*_c$, $\star_c$ and $\circ_c$ (and their residua) respectively. For example, RMTL has no sense since MTL, the logic of all left-continuous t-norms and their residua, is the logic of a family of t-norms and thus it is not determined what t-norm can be used in the book-keeping axioms.

In [5], for each continuous t-norm $*$, the extension of BL, noted BL($*$), which is standard complete with respect to the BL-algebra in $[0, 1]$ defined by the continuous t-norm $*$ and its residuum, is defined. Moreover an algorithm to obtain a finite axiomatization of them is also given. This group of logics are suitable to be extended with rational truth-constants by adding the corresponding book-keeping axioms. In this way an interesting matter of future research could be the definition and study the rational extensions of the logics BL($*$).

As for logics of left-continuous (but not continuous) t-norms, in this paper we have also studied rational extensions of some of them. However, so far, few logics standard complete with respect to a (non-continuous) left-continuous t-norm and its residuum are known (see [13] for an interesting example of a logic, called NMŁ, for a family of involutive non-continuous t-norms not belonging to the weak nilpotent minimum family). It will be an interesting future task to study the rational expansions of other left-continuous t-norm logics.

Acknowledgments Authors are indebted to Petr Hájek for pointing out and solving some technical problems in the original proof of standard completeness of RG. Authors acknowledge partial support of the Spanish project MULOG TIN2004-07933-C03-01.

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