On completeness results for predicate Łukasiewicz, Product, Gödel, and Nilpotent Minimum logics expanded with truth-constants

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Abstract

In this paper we deal with generic expansions of first-order predicate logics of some left-continuous t-norms with a countable set of truth-constants. Besides already known results for the case of Łukasiewicz logic, we obtain new conservativeness and completenesss results for some other expansions. Namely, we prove that the expansions of predicate Product, Gödel and Nilpotent Minimum logics with truth-constants are conservative, which already implies the failure of standard completeness for the case of Product logic. In contrast, the expansions of predicate Gödel and Nilpotent Minimum logics are proved to be strong standard complete but, when the semantics is restricted to the canonical algebra, they are proved to be complete only for tautologies. Moreover, when the language is restricted to evaluated formulae we prove canonical completeness for deductions from finite sets of premises.

Keywords: Monoidal t-norm based logic, core predicate fuzzy logics, Rational Pavelka Logic, Gödel and Nilpotent Minimum predicate logics, expansions with truth-constants, completeness results.

1 Introduction

As it has been repeatedly stressed, t-norm based fuzzy logics are basically logics of *comparative truth*. In fact, the residuum $\Rightarrow$ of a (left-continuous) t-norm $*$ satisfies the condition $x \Rightarrow y = 1$ if, and only if , $x \leq y$ for all $x, y \in [0, 1]$. This means that a formula $\varphi \to \psi$ is a logical consequence of a theory if the truth degree of ψ is at least as high as the truth degree of φ in any interpretation which is a model of the theory. But in some situations it might be also interesting to explicitly represent and reason with partial degrees of truth. To do so, one convenient and elegant way is introducing truth-constants into the language. As it is well-known, this approach actually goes back to Pavelka [19] who built a propositional many-valued logical system which turned out to be equivalent to the expansion of Łukasiewicz Logic by adding into the language a truth-constant $\overline{r}$ for each *real* $r \in [0, 1]$, together with a number of additional axioms. Pavelka's logic, extended by Novák in [16, 17] for the first-order case and fully elaborated in [18], is based on a infinitary notion of provability which strongly relies on the continuity of the truth functions of Łukasiewicz logic (and hence not applicable to other t-norm based logics), was simplified by Hájek in [8], both for the propositional and first-order cases.

In particular, Hájek defines what he calls *Rational Pavelka Predicate Logic*, RPL$\forall$ for short, as the expansion of Łukasiewicz predicate logic Ł$\forall$ by introducing in the language a truth-constant $\overline{r}$ for each rational r of $[0,1]$ and by adding the well-known book-keeping axioms

$$\overline{r}\&\overline{s} \leftrightarrow \overline{\max(0, r+s-1)}$$

$$(\overline{r} \rightarrow \overline{s}) \leftrightarrow \overline{\min(1, 1-r+s)}$$

Hájek shows that RPL$\forall$ enjoys the so-called *Pavelka style completeness*, which means that for any theory T and formula φ, one has

$$\|\varphi\|_T \; = \; |\varphi|_T,$$

where $\|\varphi\|_T = \inf\{\|\varphi\|_{\mathbf{M}} \mid \mathbf{M} \text{ model of } T\}$ is the truth degree of φ in T and $|\varphi|_T = \sup\{r \mid T \vdash_{\mathrm{RPL}\forall} \overline{r} \rightarrow \varphi\}$ is the provability degree of φ from T.

In contrast to Pavelka-Novák approach, an algebraic analysis of generic expansions of propositional t-norm based fuzzy logics with truth-constantshas been recently used to establish different completeness results (with respect to a finitary notion of deduction) for a number of propositional logics, among them Gödel and Nilpotent Minimum logics [5], Product logic [20], logics of a continuous t-norm [4] and logics of Weak Nilpotent Minimum t-norms [6].[1]

In this paper, following this algebraic approach, we consider the expansions with truth-constants of the corresponding first-order logics, with special attention to the cases of Gödel and Nilpotent Minimum. In fact, to the best of our knowledge, until now only the expansion of Łukasiewicz first-order logic with truth-constants had been considered in the literature. A nice and deep result contained in [12] proves that Rational Pavelka Predicate logic RPL$\forall$ is a conservative expansion of Łukasiewicz first-order logic Ł$\forall$ and, since Ł$\forall$ is not recursively axiomatizable with respect to the standard semantics, so neither is RPL$\forall$. This is a negative result. In this paper we show other negative results, but also some positive new results. Namely, after some preliminary definitions and results in next section, we first prove that the expansions of first-order Product, Gödel, Nilpotent Minimum logics (and more generally any pseudo-complemented t-norm based logic) are conservative expansions of their corresponding first-order logics, which already implies the failure of standard completeness for the case of Product logic. In contrast, the expansions of predicate Gödel and Nilpotent Minimum logics are proved to be strong standard complete but, when the semantics is restricted to the canonical algebra, they are proved to be complete only for tautologies. Moreover, when the language is restricted to evaluated formulae we prove canonical completeness for deductions from finite sets of premises. The paper ends with some conclusions and research open problems.

2 Preliminaries

2.1 Propositional expansions with truth-constants

The basic logic we will consider is the Monoidal t-norm based logic MTL introduced in [3] and proved to be the logic of left-continuous t-norms and their residua in [13]. In this setting, given a left-continuous t-norm $*$ we will denote by $[0,1]_*$ the standard MTL-chain defined by the left-continuous t-norm $*$ and its residuum $\Rightarrow$, i. e. $[0,1]_* = \langle [0,1], *, \Rightarrow, \min, \max, 0, 1\rangle$, and by L_* the axiomatic extension of MTL whose equivalent algebraic semantics is the variety

[1] For a number these logics their complexity issues have been recently studied in [9].

generated by $[0,1]_*$, denoted $\mathbf{V}([0,1]_*)$. Well-known examples of these logics are the cases when $*$ is the minimum t-norm (L_*= G), the Łukasiewicz t-norm ($L_* =$ Ł), the product t-norm ($L_* = \Pi$) or the nilpotent minimum t-norm ($L_* =$ NM).

Now, given a left-continuous t-norm $*$ and its corresponding logic L_*, let $\mathcal{C} = \langle C, *, \Rightarrow, \min, \max, 0, 1\rangle$ be a *countable* subalgebra of the standard L_*-algebra $[0,1]_*$. Then, the logic $L_*(\mathcal{C})$ is defined as follows:

(i) the language of $L_*(\mathcal{C})$ is the one of L_* expanded with a new propositional variable $\overline{r}$ for each $r \in C \setminus \{0,1\}$,

(ii) the axioms and rules of $L_*(\mathcal{C})$ are those of L_* plus the book-keeping axioms:

$$\overline{r}\&\overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \to \overline{s} \leftrightarrow \overline{r \Rightarrow_* s}$$

for each $r, s \in C$.

The algebraic counterpart of the $L_*(\mathcal{C})$ logic consists of the class of $L_*(\mathcal{C})$-algebras, defined as structures

$$\mathcal{A} = \langle A, \&, \to, \wedge, \vee, \{\overline{r}^{\mathcal{A}} : r \in C\}\rangle$$

such that:

1. $\langle A, \&, \to, \wedge, \vee, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\rangle$ is an L_*-algebra, and

2. for every $r, s \in C$ the following identities hold:

$$\overline{r}^{\mathcal{A}}\&\overline{s}^{\mathcal{A}} = \overline{r * s}^{\mathcal{A}}$$
$$\overline{r}^{\mathcal{A}} \to \overline{s}^{\mathcal{A}} = \overline{r \Rightarrow s}^{\mathcal{A}}.$$

$L_*(\mathcal{C})$-chains defined over the real unit interval $[0,1]$ are called *standard*. Among them, there is one which reflects the intended semantics, the so-called *canonical standard* $L_*(\mathcal{C})$-chain

$$[0,1]_{L_*(\mathcal{C})} = \langle [0,1], *, \Rightarrow, \min, \max, \{r : r \in C\}\rangle,$$

which is the standard chain where the truth-constants are interpreted by their defining values. It is worth to point out that for a logic $L_*(\mathcal{C})$ there may exist multiple standard chains as soon as there exist different ways of interpreting the truth-constants on $[0,1]$ respecting the book-keeping axioms. For instance, let $C = [0,1] \cap \mathbb{Q}$ and let $*$ be a *pseudo-complemented t-norm*, that is, a left-continuous t-norm $*$ whose definable negation $\neg x = x \Rightarrow 0$ is the so-called Gödel negation ($\neg x = 0$ for all $x \neq 0$ and $\neg 0 = 1$). In such a case, if $*$ is closed on C, it is easy to check that the algebra $\mathcal{A} = \langle [0,1], *, \Rightarrow, \wedge, \vee, \{\overline{r}^{\mathcal{A}} : r \in C\}\rangle$ where

$$\overline{r}^{\mathcal{A}} = \begin{cases} 1, & \text{if } r > 0 \\ 0, & \text{otherwise} \end{cases}$$

is always a standard $L_*(\mathcal{C})$-algebra. This is the case e.g. of minimum and product t-norms. Furthermore, in the particular case of $* = \min$, for any $\alpha > 0$, the algebra $\mathcal{A} = \langle [0,1], \min, \Rightarrow, \wedge, \vee, \{\overline{r}^{\mathcal{A}} : r \in C\}\rangle$ where

$$\overline{r}^{\mathcal{A}} = \begin{cases} 1, & \text{if } r \geq \alpha \\ 0, & \text{otherwise} \end{cases}$$

is also a standard $G_*(\mathcal{C})$-algebra. The structure of $L_*(\mathcal{C})$-chains has been fully described in [4] for the case of $*$ being a continuous t-norm and in [6] for some additional classes of left-continuous t-norms.

Since the additional symbols added to the language are 0-ary, $\mathrm{L}_*(\mathcal{C})$ is also an algebraizable logic and its equivalent algebraic semantics is the variety of $\mathrm{L}_*(\mathcal{C})$-algebras. This, together with the fact that $\mathrm{L}_*(\mathcal{C})$-algebras are representable as a subdirect product of $\mathrm{L}_*(\mathcal{C})$-chains, leads to the following general completeness result of $\mathrm{L}_*(\mathcal{C})$ with respect to the class of $\mathrm{L}_*(\mathcal{C})$-chains. In the following, for any set $\Gamma \cup \{\varphi\}$ of $\mathrm{L}_*(\mathcal{C})$-formulae and any class $\mathbb{K}$ of $\mathrm{L}_*(\mathcal{C})$-chains, we write $\Gamma \models_{\mathbb{K}} \varphi$ to denote that, for each $\mathcal{A} \in \mathbb{K}$, $e(\varphi) = \overline{1}^{\mathcal{A}}$ for all $\mathcal{A}$-evaluation e model of Γ

Theorem 2.1 (Chain completeness)**.** *For any set $\Gamma \cup \{\varphi\}$ of $\mathrm{L}_*(\mathcal{C})$-formulae, it holds that $\Gamma \vdash_{\mathrm{L}_*(\mathcal{C})} \varphi$ if, and only if, $\Gamma \models_{\{\mathrm{L}_*(\mathcal{C})-chains\}} \varphi$.*

The issue of studying when a logic $\mathrm{L}_*(\mathcal{C})$ is also complete with respect to the class of standard $\mathrm{L}_*(\mathcal{C})$-chains (called *standard completeness* property) or with respect to the canonical standard $\mathrm{L}_*(\mathcal{C})$-chain (called *canonical standard completeness* property) has been addressed in the literature for some logics L_*. Hájek already proved in [8] the canonical completeness of the expansion of Łukasiewicz logic with rational truth-constants for finite theories[2]. More recently, the expansions of Gödel (and of some t-norm based logic related to the nilpotent minimum t-norm) and of Product logic with countable sets of truth-constants have been proved to enjoy the canonical standard completeness for theorems in [5] and in [20] respectively. A rather exhaustive description of completeness results for the logics $\mathrm{L}_*(\mathcal{C})$ can be found in [4] and in [6].

2.2 Core predicate fuzzy logics

Predicate versions of the propositional t-norm based logics described above have also been defined and studied in the literature. Following [11] we give below a general definition of the first-order logic $\mathrm{L}\forall$ for any (propositional) *core fuzzy logic* L. A finitary logic L in a countable language is a *core fuzzy logic* [10] if:

(i) L expands MTL;

(ii) L satisfies the congruence condition:
$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \chi(\varphi) \leftrightarrow \chi(\psi)$, for every φ, ψ, χ;

(iii) L satisfies the following local deduction theorem:
$\Gamma, \varphi \vdash_{\mathrm{L}} \psi$ iff there a is natural number n such that $\Gamma \vdash_{\mathrm{L}} \varphi \& \overset{n}{\ldots} \& \varphi \to \psi$.

Note that the logics $\mathrm{L}_*(\mathcal{C})$ introduced above are core fuzzy logics, so what follows also applies to them.

Given a core fuzzy logic L, the language $\mathcal{PL}$ of $\mathrm{L}\forall$ is built from the propositional language $\mathcal{L}$ of L by enlarging it with a set of predicates $Pred$, a set of object variables Var and a set of object constants $Const$, together with the two classical quantifiers $\forall$ and $\exists$. The notion of formula trivially generalizes taking into account that now, if φ is a formula and x is an object variable, then $(\forall x)\varphi$ and $(\exists x)\varphi$ are formulae as well.

In first-order fuzzy logics it is usual to restrict the semantics to L-chains only. For each L-chain $\mathcal{A}$, an $\mathcal{A}$-interpretation for a predicate language $\mathcal{PL} = (Pred, Const)$ of $\mathrm{L}\forall$ is a structure

$$\mathbf{M} = (M, (c^{\mathbf{M}})_{c \in Const}, (P^{\mathbf{M}})_{P \in Pred})$$

[2]This result has been recently extended in [1] for the case of expansions with countable sets of truth-constants possibly containing irrational values.

where $M \neq \emptyset$, $c^{\mathbf{M}} \in M$ and $P^{\mathbf{M}} : M^{ar(P)} \to A$ for each $c \in Const$ and $P \in Pred$. For each evaluation of variables $v : Var \to M$, the truth-value $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v}$ of a formula (where $v(x) \in M$ for each variable x) is defined inductively from

$$\|P(x, \cdots, c, \cdots)\|^{\mathcal{A}}_{\mathbf{M},v} = P^{\mathbf{M}}(v(x), \cdots, c^{\mathbf{M}}, \cdots),$$

taking into account that the value commutes with connectives, and defining

$$\|(\forall x)\varphi\|^{\mathcal{A}}_{\mathbf{M},v} = \inf\{\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v'} \mid v(y) = v'(y) \text{ for all variables } y, \text{ except } x\}$$
$$\|(\exists x)\varphi\|^{\mathcal{A}}_{\mathbf{M},v} = \sup\{\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v'} \mid v(y) = v'(y) \text{ for all variables } y, \text{ except } x\}$$

if the infimum and supremum exist in $\mathcal{A}$, otherwise the truth-value(s) remain undefined. A structure $\mathbf{M}$ is called $\mathcal{A}$-safe if all infs and sups needed for the definition of the truth-value of any formula exist in $\mathcal{A}$. Then, the truth-value of a formula φ in a safe $\mathcal{A}$-structure $\mathbf{M}$ is just

$$\|\varphi\|^{\mathcal{A}}_{\mathbf{M}} = \inf\{\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v} \mid v : Var \to M\}.$$

When $\|\varphi\|^{\mathcal{A}}_{\mathbf{M}} = 1$ for a $\mathcal{A}$-safe structure $\mathbf{M}$, the pair $(\mathbf{M}, \mathcal{A})$ is said to be a model for φ, written $(\mathbf{M}, \mathcal{A}) \models \varphi$. Sometimes we will call the pair $(\mathbf{M}, \mathcal{A})$ an L$\forall$-structure.

The axioms for L$\forall$ are the axioms resulting from those of L by substitution of propositional variables with formulae of $\mathcal{PL}$ plus the following axioms on quantifiers (the same used in [8] when defining BL$\forall$):

- ($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$ (t substitutable for x in $\varphi(x)$)
- ($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$ (t substitutable for x in $\varphi(x)$)
- ($\forall$2) $(\forall x)(\nu \to \varphi) \to (\nu \to (\forall x)\varphi)$ (x not free in ν)
- ($\exists$2) $(\forall x)(\varphi \to \nu) \to ((\exists x)\varphi \to \nu)$ (x not free in ν)
- ($\forall$3) $(\forall x)(\varphi \vee \nu) \to ((\forall x)\varphi \vee \nu)$ (x not free in ν)

The rules of inference of L$\forall$ are modus ponens and generalization: from φ infer $(\forall x)\varphi$.

A completeness theorem for first-order BL was proven in [8] and the completeness theorems of other first-order fuzzy logics defined in the literature have been proven in the corresponding papers where the propositional logics are introduced. The following general formulation of completeness for predicate core fuzzy logics is from the paper [11].

Theorem 2.2 ([11])**.** *For any core fuzzy logic* L *over a predicate language* $\mathcal{PL}$*, it holds that*

$$T \vdash_{\mathrm{L}\forall} \varphi \textit{ iff } (\mathbf{M}, \mathcal{A}) \models \varphi \textit{ for each model } (\mathbf{M}, \mathcal{A}) \textit{ of } T,$$

for any set of sentences T *and any formula* φ *of the predicate language* $\mathcal{PL}$*.*

3 Types of completeness properties and their relationships

We will use the following terminology and notation to refer to the usual three notions of completeness for core fuzzy logics.

Definition 3.1. *Let* L *be a core fuzzy and let* $\mathbb{K}$ *be a class of* L*-algebras. We define:*

- L *has the property of* strong $\mathbb{K}$-completeness, S$\mathbb{K}$C *for short, when for every set of* L*-formulae* Γ *and every* L*-formula* φ*,* $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\Gamma \models_{\mathbb{K}} \varphi$*.*

- L *has the property of* finite strong $\mathbb{K}$-completeness*,* FS$\mathbb{K}$C *for short, when for every* finite *set of* L*-formulae* Γ *and every* L*-formula* φ*,* $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\Gamma \models_{\mathbb{K}} \varphi$*.*

- L *has the property of* $\mathbb{K}$-completeness*,* $\mathbb{K}$C *for short, when for every* L*-formula* φ*,* $\vdash_{\mathrm{L}} \varphi$ *iff* $\models_{\mathbb{K}} \varphi$*.*

They are analogously defined for the first-order logics.

Definition 3.2. *Let* L *be a core fuzzy logic. We say that* L$\forall$ *has the* S$\mathbb{K}$C *if for each language* $\mathbf{\Gamma}$*, theory* T*, and formula* φ *the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall} \varphi$*.*

- $(\mathbf{M}, \mathcal{A}) \models \varphi$ *for each* $\mathcal{A} \in \mathbb{K}$ *and each model* $(\mathbf{M}, \mathcal{A})$ *of the theory* T*.*

We say that L$\forall$ *has the* FS$\mathbb{K}$C *if the above condition holds for* finite *theories. Finally, we say that* L$\forall$ *has the* $\mathbb{K}$C *if the above condition holds for the empty theory.*

When $\mathbb{K}$ is the class of standard algebras in the variety of L-algebras, then instead of $\mathbb{K}$-*completeness* properties we talk about *standard completeness* properties and we use the notation $\mathcal{R}$C instead of $\mathbb{K}$C (to stress that it is a completeness with respect to algebras defined of the real unit interval). Moreover, as mentioned above, when the considered core fuzzy logic is of the form $\mathrm{L}_*(\mathcal{C})$ we can think of further restricting the semantics to the canonical standard algebra. Thus, we also consider the three *canonical standard completeness* properties for these logics both in the propositional and in the first-order case.

All these completeness properties, their relationship and algebraic equivalent (or sufficient) conditions have been studied in [2]. In particular, the following results for the S$\mathbb{K}$C have been proved.

Theorem 3.3 ([2])**.** *Let* L *be a core fuzzy logic and let* $\mathbb{K}$ *be a class of* L*-algebras. Then:*

1. L *has the* S$\mathbb{K}$C *if, and only if, every countable* L*-chain can be embedded into some chain from* $\mathbb{K}$*.*

2. L$\forall$ *has the* S$\mathbb{K}$C *if every countable* L*-chain can be* σ*-embedded (i.e. with an embedding which preserves existing suprema and infima) into some chain from* $\mathbb{K}$*.*

Now we recall a relation between completeness of a propositional core fuzzy logic L and completeness of its corresponding first-order logic L$\forall$.

Proposition 3.4 (cf. [2])**.** *If for some family* $\mathbb{K}$ *of* L*-chains* L$\forall$ *enjoys the* $\mathbb{K}$C *(*FS$\mathbb{K}$C*,* S$\mathbb{K}$C *resp.), then* L *enjoys the* $\mathbb{K}$C *(*FS$\mathbb{K}$C*,* S$\mathbb{K}$C *resp.) as well.*

This proposition yields a necessary condition for the completeness properties of first-order fuzzy logics that will be useful to refute some completeness results in the next section. In a similar way we will also use the following proposition relating completeness of two first-order logics when one is a conservative expansion of the other one.

Proposition 3.5. *Let* L *and* L$'$ *be two core fuzzy logics such that* L$'\forall$ *is a conservative expansion of* L$\forall$*. Let* $\mathbb{K}'$ *be a class of* L$'$*-chains and let* $\mathbb{K}$ *be the class of their* L*-reducts. If* L$'\forall$ *enjoys the* $\mathbb{K}'$C *(*FS$\mathbb{K}'$C*,* S$\mathbb{K}'$C *resp.), then* L$\forall$ *enjoys the* $\mathbb{K}$C *(*FS$\mathbb{K}$C*,* S$\mathbb{K}$C *resp.) as well.*

Proof. Assume that L$'\forall$ enjoys the $\mathbb{K}$C and we prove that L$\forall$ also enjoys it. Suppose that $\nvdash_{\mathrm{L}\forall} \varphi$ for some formula φ in language of L$\forall$. Then, since L$'\forall$ is a conservative expansion of L$\forall$ we also have $\nvdash_{\mathrm{L}'\forall} \varphi$, hence there is some structure $(\mathbf{M}, \mathcal{A}')$ with $\mathcal{A}' \in \mathbb{K}'$ such that $(\mathbf{M}, \mathcal{A}') \not\models \varphi$. Let $\mathcal{A}$ be the L-reduct of $\mathcal{A}'$. Since φ is a L$\forall$-formula, it is clear that $(\mathbf{M}, \mathcal{A}) \not\models \varphi$. $\square$

4 Completeness results for some $\mathrm{L}_*\forall(\mathcal{C})$ logics

In the following, given a left-continuous t-norm $*$ and its corresponding logic L_*, and a *countable* subalgebra $\mathcal{C}$ of the standard L_*-algebra $[0,1]_*$, we will denote by $\mathrm{L}_*\forall(\mathcal{C})$ the first-order version of the (core fuzzy) logic $\mathrm{L}_*(\mathcal{C})$ according to the definitions in Section 2.2.

Remark about the notation used. In the way we have just defined $\mathrm{L}_*\forall(\mathcal{C})$, we should have rather used the notation $\mathrm{L}_*(\mathcal{C})\forall$, since we have started by the expanded logic $\mathrm{L}_*(\mathcal{C})$ and then we have defined the first-order variant over it. But in fact, starting with the $\mathrm{L}_*\forall$ logic and then expanding it with the truth constants from $\mathcal{C}$ leads exactly to the same first-order logic and thus we will keep on using $\mathrm{L}_*\forall(\mathcal{C})$. Moreover, we will also denote as usual by G$\forall$, Ł$\forall$, $\Pi\forall$ and NM$\forall$ the logics $\mathrm{L}_*\forall$ logic when the t-norm $*$ is the minimum, Łukasiewicz, product and nilpotent minimum t-norm respectively.

In the case of expansions of $\mathrm{L}_*\forall$ logics with truth-constants, it was already proved by Hájek *et al.* in [12] that RPL$\forall$ (Rational Pavelka predicate logic[3]) is a conservative expansion of Ł$\forall$. Next theorem proves that an analogous result also holds for some other logics.

Proposition 4.1. *If $*$ is a pseudo-complemented t-norm or the nilpotent minimum t-norm, then $\mathrm{L}_*\forall(\mathcal{C})$ is a conservative expansion of $\mathrm{L}_*\forall$.*

Proof. Let φ be an $\mathrm{L}_*\forall$-formula such that $\nvdash_{\mathrm{L}_*\forall} \varphi$. We must show that $\nvdash_{\mathrm{L}_*\forall(\mathcal{C})} \varphi$. By hypothesis, there is some $\mathrm{L}_*\forall$-structure $(\mathbf{M},\mathcal{A})$ such that $(\mathbf{M},\mathcal{A}) \not\models \varphi$. It is enough to show that $\mathcal{A}$ can be expanded to an $\mathrm{L}_*(\mathcal{C})$-chain. If $*$ is a pseudo-complemented t-norm we can define the interpretation of every truth-constant $\overline{r}$ in $\mathcal{A}$ by putting $\overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$ for $r \neq 0$ (see Section 2.1). Assume now that $*$ is the nilpotent minimum t-norm. If $\mathcal{C}$ has no negation fixpoint, we define the interpretation in $\mathcal{A}$ of a truth-constant $\overline{r}$ as $\overline{1}^{\mathcal{A}}$ when $\neg r < r$, and we define it as $\overline{0}^{\mathcal{A}}$ when $\neg r > r$. If $\mathcal{C}$ has the negation fixpoint $\frac{1}{2}$, we can suppose that $\mathcal{A}$ also has a negation fixpoint a (otherwise it could be added as shown in [15]), and then we interpret $\overline{\frac{1}{2}}$ and the rest of the constants as in the previous case. $\square$

This result, together with the one above mentioned by Hájek *et al.*, shows that when $*$ is one of the three basic continuous t-norms (Łukasiewicz, product and minimum), $\mathrm{L}_*\forall(\mathcal{C})$ is a conservative expansion of $\mathrm{L}_*\forall$ for every subalgebra $\mathcal{C}$ of truth-constants, except for the case of Łukasiewicz t-norm where the result has only been proved for $C = [0,1] \cap \mathbb{Q}$.

Now we are prepared to deal with the standard completeness properties of first-order logics with truth-constants. It is well known that Product and Łukasiewicz first-order logics do not enjoy standard completeness. Therefore, by Propositions 3.5 and 4.1, Ł$\forall(\mathcal{C})$ and $\Pi\forall(\mathcal{C})$ do not have the $\mathbb{K}$C when $\mathbb{K}$ is the class of all standard Ł$\forall(\mathcal{C})$-chains and the class of all standard $\Pi\forall(\mathcal{C})$-chains, respectively; hence these logics do not enjoy canonical standard completeness neither. The same reasoning would also hold for every logic based on a pseudo-complemented t-norm $*$ for which we know that $\mathrm{L}_*\forall$ fails to enjoy the standard completeness.[4]

This is not the case for Gödel and Nilpotent Minimum first-order logics which, in fact, are strongly standard complete. Next we show that in these two particular cases, their completeness properties extend to their expansions with truth-constants.

Theorem 4.2. *The logics G$\forall(\mathcal{C})$ and NM$\forall(\mathcal{C})$ enjoy the S$\mathcal{R}$C.*

[3] In our notation RPL$\forall$ corresponds to Ł$\forall(\mathcal{C})$ when $C = [0,1] \cap \mathbb{Q}$.

[4] This is the case of all pseudo-complemented continuous t-norms except for $* = \min$.

Proof. As stated in the preliminaries, the strong standard completeness follows from the property of σ-embeddability. Since the S$\mathcal{R}$C for first-order Gödel logic was proved in this way, we know that any countable G-chain is σ-embeddable into $[0,1]_{\mathrm{G}}$, thus every countable G($\mathcal{C}$)-chain is also σ-embeddable into a standard G($\mathcal{C}$)-chain. Indeed, given a countable G($\mathcal{C}$)-chain $\mathcal{A}$ let f be the σ-embedding of its G-reduct into $[0,1]_{\mathrm{G}}$. Then $\mathcal{A}$, as G($\mathcal{C}$)-chain is also σ-embeddable into the standard G($\mathcal{C}$)-chain defined over $[0,1]_{\mathrm{G}}$ interpreting each truth-constant $\overline{r}$ as $f(\overline{r}^{\mathcal{A}})$. The proof for NM$\forall$($\mathcal{C}$) is completely analogous. □

A natural question here is whether these completeness results can be improved by restricting the semantics to the canonical standard algebra. As a matter of fact, the canonical FS$\mathcal{R}$C fails for the logics G($\mathcal{C}$) and NM($\mathcal{C}$), as shown in [4, 6]. Therefore, by Proposition 3.4, this completeness property also fails for their first-order versions. Nevertheless, we can still prove the canonical standard completeness for these logics.

Theorem 4.3. *The logics* G$\forall$($\mathcal{C}$) *and* NM$\forall$($\mathcal{C}$) *enjoy the canonical* $\mathcal{R}$C*, i.e. the provable formulae coincide with the 1-tautologies of the canonical standard chain* $[0,1]_{\mathrm{G}(\mathcal{C})}$ *and of* $[0,1]_{\mathrm{NM}(\mathcal{C})}$ *respectively.*

Proof. We only prove it for G$\forall$($\mathcal{C}$) (the proof for NM$\forall$($\mathcal{C}$) is analogous with the obvious changes). Soudness is obvious as usual. For the converse direction we will argue by contraposition, i.e. we will prove that if $\nvdash_{\mathrm{G}\forall(\mathcal{C})} \varphi$ for some formula φ, then there is a G$\forall$($\mathcal{C}$)-structure $(\mathbf{M}, [0,1]_{\mathrm{G}(\mathcal{C})})$ such that $(\mathbf{M}, [0,1]_{\mathrm{G}(\mathcal{C})}) \not\models \varphi$.

If $\nvdash_{\mathrm{G}\forall(\mathcal{C})} \varphi$, then there exists a G$\forall$($\mathcal{C}$)-structure $(\mathbf{M}, \mathcal{A})$ over a countable G-chain $\mathcal{A}$ and an evaluation v such that $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v} < \overline{1}^{\mathcal{A}}$. Take $s = \min(\{r \in C \mid \overline{r}^{A} = \overline{1}^{\mathcal{A}}, r \text{ appears in } \varphi\} \cup \{1\})$ and define $g : A \to [0,1]$ by taking $g(\overline{1}^{\mathcal{A}}) = 1$ and $g\mid_{A\setminus\{\overline{1}^{\mathcal{A}}\}}$ a bijection of $A \setminus \{\overline{1}^{\mathcal{A}}\}$ into $[0,s)$ preserving existing suprema and infima, and such that $g(\overline{r}^{\mathcal{A}}) = r$ for every truth-constant appearing in φ such that $\overline{r}^{\mathcal{A}} \neq \overline{1}^{\mathcal{A}}$. If $\mathbf{M} = (M, (c^{\mathbf{M}})_{c\in Const}, (P^{\mathbf{M}})_{P\in Pred})$, using the mapping g we can produce a structure $(\mathbf{M}', [0,1]_{\mathrm{G}(\mathcal{C})})$, where $\mathbf{M}' = (M, (c^{\mathbf{M}})_{c\in Const}, (P^{\mathbf{M}'})_{P\in Pred})$, where $P^{\mathbf{M}'} : M^{ar(P)} \to [0,1]$ is defined as $P^{\mathbf{M}'} = g \circ P^{\mathbf{M}}$, and hence

for every evaluation of variables e on M one has

$$\|P(t_1, t_2, \ldots, t_n)\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e} = g(\|P(t_1, t_2, \ldots, t_n)\|^{\mathcal{A}}_{\mathbf{M},e})$$

for each predicate symbol P and terms $t_1, t_2, \ldots, t_n$. Now we can prove that the following statements hold for any subformula ψ of φ and every evaluation of variables e on M:

a) If $\|\psi\|^{\mathcal{A}}_{\mathbf{M},e} = \overline{1}^{\mathcal{A}}$, then $\|\psi\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e} \geq s$,

b) If $\|\psi\|^{\mathcal{A}}_{\mathbf{M},e} \neq \overline{1}^{\mathcal{A}}$, then $\|\psi\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e} = g(\|\psi\|^{\mathcal{A}}_{\mathbf{M},e})$.

The proof is by induction on the structure of ψ and we provide some details next:

- If $\psi = \overline{r}$, then $\|\psi\|^{\mathcal{A}}_{\mathbf{M},e} = \overline{r}^{\mathcal{A}}$. Then either $\overline{r}^{A} = \overline{1}^{\mathcal{A}}$ and hence $r \geq s$, or $\overline{r}^{A} \neq \overline{1}^{\mathcal{A}}$, and hence $g(\overline{r}^{A}) = r = \|\overline{r}\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e}$.
- If $\psi = P(t_1, t_2, \ldots, t_n)$, then it holds by definition.

- Suppose $\psi = \alpha\&\beta$. If $\|\alpha\&\beta\|^{\mathcal{A}}_{\mathbf{M},e} = \overline{1}^{\mathcal{A}}$, then $\|\alpha\|^{\mathcal{A}}_{\mathbf{M},e} = \|\beta\|^{\mathcal{A}}_{\mathbf{M},e} = \overline{1}^{\mathcal{A}}$ and thus by induction hypothesis $\|\alpha\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e}, \|\beta\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e} \geq s$, and hence $\|\alpha\&\beta\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e} \geq s$. If $\|\alpha\&\beta\|^{\mathcal{A}}_{\mathbf{M},e} \neq \overline{1}^{\mathcal{A}}$, then $\|\alpha\|^{\mathcal{A}}_{\mathbf{M},e} \neq \overline{1}^{\mathcal{A}}$ or $\|\beta\|^{\mathcal{A}}_{\mathbf{M},e} \neq \overline{1}^{\mathcal{A}}$, and by using the induction hypothesis the result easily follows.

- If $\psi = \alpha \to \beta$, to check the result is again a matter of routinary proof by cases and usage of the induction hypothesis.

- Suppose $\psi = (\forall x)\alpha$, and let $V(e)$ denote the set of evaluations of variables v such that $e(y) = v(y)$ for all variables y, except x. Recall that $\|(\forall x)\alpha\|^{\mathcal{A}}_{\mathbf{M},e} = \inf\{\|\alpha\|^{\mathcal{A}}_{\mathbf{M},v} \mid v \in V(e)\}$.

  If $\|(\forall x)\alpha\|^{\mathcal{A}}_{\mathbf{M},e} = \overline{1}^{\mathcal{A}}$, then for every such $v \in V(e)$ we have $\|\alpha\|^{\mathcal{A}}_{\mathbf{M},v} = \overline{1}^{\mathcal{A}}$, and hence $\|\alpha\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',v} \geq s$, which implies that $\|(\forall x)\alpha\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e} \geq s$.

  If $\|(\forall x)\alpha\|^{\mathcal{A}}_{\mathbf{M},e} \neq \overline{1}^{\mathcal{A}}$, then it is enough to consider the infimum over the subset $V^+(e) \subseteq V(e)$ of those evaluations v such that $\|\alpha\|^{\mathcal{A}}_{\mathbf{M},v} \neq \overline{1}^{\mathcal{A}}$, i.e. $\|(\forall x)\alpha\|^{\mathcal{A}}_{\mathbf{M},e} = \inf\{\|\alpha\|^{\mathcal{A}}_{\mathbf{M},v} \mid v \in V^+(e)\} \neq \overline{1}^{\mathcal{A}}$. Then, since g preserves the existing infima, we have the following equalities: $g(\|(\forall x)\alpha\|^{\mathcal{A}}_{\mathbf{M},e}) = g(\inf\{\|\alpha\|^{\mathcal{A}}_{\mathbf{M},v} \mid v \in V^+(e)\}) = \inf\{g(\|\alpha\|^{\mathcal{A}}_{\mathbf{M},v}) \mid v \in V^+(e)\} = \inf\{\|\alpha\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',v} \mid v \in V^+(e)\} = \inf\{\|\alpha\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',v} \mid v \in V(e)\} = \|(\forall x)\alpha\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e}$.

- If $\psi = (\exists x)\alpha$, the reasoning is similar to the previous one (now it uses that g preserves existing suprema).

Finally, from the above statements the theorem easily follows since $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v} < \overline{1}^{\mathcal{A}}$, and thus $\|\varphi\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',v} = g(\|\varphi\|^{\mathcal{A}}_{\mathbf{M}',v}) < s < 1$. □

As regards to canonical finite strong standard completeness, we have already argued why it fails for all the logics considered in this paper. However, for those logics enjoying canonical standard completeness, G∀($\mathcal{C}$) and NM∀($\mathcal{C}$) according to the last theorem, we may still wonder whether this result can be extended to a canonical FS$\mathcal{R}$C for evaluated formulae as it has been done in previous works for propositional fuzzy logics. Recall that an *evaluated formula* is a formula of type $\overline{r} \to \varphi$, where φ is a formula without new truth-constants (i.e. different from $\overline{0}$ and $\overline{1}$); it is called *positively evaluated formula* if $r > \neg r$.

Theorem 4.4. G∀($\mathcal{C}$) *enjoys the canonical* FS$\mathcal{R}$C *restricted to evaluated formulae.* NM∀($\mathcal{C}$) *enjoys the canonical* FS$\mathcal{R}$C *restricted to positively evaluated formulae.*

Proof. Again we write the proof for the case of Gödel logic (for Nilpotent Minimum logic the same kind of reasoning with a few changes would prove the result). We have to show that for every $\varphi_1, \ldots, \varphi_k, \psi$ formulae in the language of G∀ and constants $\overline{r_1}, \ldots, \overline{r_k}, \overline{s}$:

$$\{\overline{r_i} \to \varphi_i \mid i = 1, \ldots, k\} \vdash_{\mathrm{G}(\mathcal{C})\forall} \overline{s} \to \psi \text{ if, and only if, } \{\overline{r_i} \to \varphi_i \mid i = 1, \ldots, k\} \models_{[0,1]_{\mathrm{G}(\mathcal{C})}} \overline{s} \to \psi$$

By the deduction theorem and the canonical standard completeness for G∀($\mathcal{C}$), a finite deduction of type $\{\overline{r_i} \to \varphi_i \mid i = 1, \ldots, k\} \vdash_{\mathrm{G}\forall(\mathcal{C})} \overline{s} \to \psi$ is equivalent to $\models_{[0,1]_{\mathrm{G}(\mathcal{C})}} \bigwedge_{i=1,\ldots,k}(\overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)$. Thus what we need to prove is the semantical version of the deduction theorem for G∀($\mathcal{C}$), i.e. the equivalence between $\{\overline{r_i} \to \varphi_i \mid i = 1, \ldots, k\} \models_{[0,1]_{\mathrm{G}(\mathcal{C})}} \overline{s} \to \psi$ and $\models_{[0,1]_{\mathrm{G}(\mathcal{C})}} \bigwedge_{i=1,\ldots,k}(\overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)$.

In one direction the implication is obvious. For the other one we do it by contraposition. If $\not\models_{[0,1]_{\mathrm{G}(\mathcal{C})}} \bigwedge_{i=1,\dots,k}(\overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)$ there must exist a G$\forall(\mathcal{C})$-structure $(\mathbf{M}, [0,1]_{\mathrm{G}(\mathcal{C})})$ and an evaluation e such that

$$\| \bigwedge_{i=1,\dots,k} (\overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M},e} < 1$$

We have to build a G$\forall(\mathcal{C})$-structure $(\mathbf{M}', [0,1]_{\mathrm{G}(\mathcal{C})})$ and an evaluation of variables e' such that $\| \bigwedge_{i=1,\dots,k}(\overline{r_i} \to \varphi_i)\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e'} = 1$ and $\|\overline{s} \to \psi\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e'} < 1$. Observe first that the previous inequality implies that $\| \bigwedge_{i=1,\dots,k}(\overline{r_i} \to \varphi_i)\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M},e} > \|\overline{s} \to \psi\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M},e}$ and thus $\|\overline{s} \to \psi\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M},e} = \|\psi\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M},e} < 1$. We follow the proof by cases:

(i) If $\|\overline{r_i} \to \varphi_i\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M},e} = 1$ for every $i \in \{1,\dots,k\}$, then we just take $\mathbf{M}' = \mathbf{M}$ and $e' = e$.

(ii) Suppose there exists a non-empty set of indexes $J \subseteq \{1,\dots,k\}$ such that for all $j \in J$, $\|\overline{r_i} \to \varphi_i\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M},e} = \|\varphi_i\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M},e} < 1$. Let

$$b = \min\{\|\varphi_i\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M},e} \mid j \in J\} \text{ and } c = \|\psi\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M},e}.$$

Define f as the automorphism of $[0,1]_{\mathrm{G}(\mathcal{C})}$ given by $f(x) = 1$ for every $x \geq b$, an ordered bijection between $[c,b)$ and $[c,1)$ such that $x \leq f(x)$ for every x (e.g. $f(x) = (x-c)(1-c)/(b-c)+c$), and $f(x) = x$ for every $x < c$. It is obvious that this mapping is an ordered bijection that preserves existing suprema and infima. Now we consider a structure $\mathbf{M}'$ over the same domain as $\mathbf{M}$ with the same interpretation of the object constants, with the same evaluation of variables $e' = e$, and we will just change the interpretation of the predicate symbols. Indeed, for every n-ary predicate P and arbitrary elements of the domain $m_1,\dots,m_n$, we define $P^{\mathbf{M}'}(m_1,\dots,m_n) = f(P^{\mathbf{M}}(m_1,\dots,m_n))$. Then, since f is a homomorphism that preserves existing suprema and infima, it is obvious that for every G$\forall$-formula φ we have $\|\varphi\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e'} = f(\|\varphi\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M},e})$. An easy computation shows that $\| \bigwedge_{i=1,\dots,k}(\overline{r_i} \to \varphi_i)\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e'} = 1$ (observe that $\|\overline{r_i} \to \varphi\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M},e} = 1$ for all $i \notin J$ by assumption, and for all $i \in J$ we have $\|\varphi_i\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e'} = 1$), while $\|\overline{s} \to \psi\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e'} < 1$ (since the value of ψ has not changed).

The proof for NM$\forall(\mathcal{C})$ is very similar, it uses the fact that if all the r_i's are positive, i.e. $r_i > \neg r_i$, then c, as defined above in (ii), is also positive, and this allows one to define the mapping f as $f(x) = 1$ if $x \geq b$, $f(x) = 0$ if $x \leq \neg b$ and as an ordered bijection between $(\neg b, b)$ and $(0,1)$ such that $f(x) = x$ for $x \in (\neg c, c)$. □

5 Conclusions and open problems

In this paper we have considered the (canonical) standard completeness properties for several prominent first-order fuzzy logics enriched with constants for intermediate truth-values. Some of these properties have been shown to fail because they already fail in the corresponding propositional logics. In some other cases the standard completeness properties have been refuted by showing that the logic is a conservative expansion of the corresponding logic without additional truth-constants and for which standard completeness already fails. In the remaining cases dealt with in the paper the answer has turned out to be positive by some ad hoc proofs. The following table collects the results.

Logic	$\mathcal{R}$C	FS$\mathcal{R}$C	S$\mathcal{R}$C	Can. $\mathcal{R}$C	Can. FS$\mathcal{R}$C	Can. S$\mathcal{R}$C
Ł$\forall(\mathcal{C})$	No	No	No	No	No	No
$\Pi\forall(\mathcal{C})$	No	No	No	No	No	No
G$\forall(\mathcal{C})$	Yes	Yes	Yes	Yes	No	No
NM$\forall(\mathcal{C})$	Yes	Yes	Yes	Yes	No	No

Finally, we have also obtained the canonical FS$\mathcal{R}$C restricted to (positively) evaluated formulae for G$\forall(\mathcal{C})$ and NM$\forall(\mathcal{C})$. Of course, there are a number of problems left open and which we plan to address in future research. Among them, we can mention the following ones:

1. Investigate for which left-continuous t-norms $*$ and algebras of truth constants $\mathcal{C}$ is $\mathrm{L}_*\forall(\mathcal{C})$ a conservative extension of $\mathrm{L}_*\forall$. In particular, is Ł$\forall(\mathcal{C})$ is a conservative expansion of Ł$\forall$ when $\mathcal{C}$ contains irrational values?

2. Investigate completeness results for the expansions of the logics $\mathrm{L}_*\forall(\mathcal{C})$ with the projection connective Δ.

3. Investigate completeness results with respect to the semantics defined over the rational unit interval.

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