

Fuzzy Sets and Fuzzy Logics

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Outline

- 1 The vagueness problem and FST
- 2 Fuzzy systems
- 3 Logics of left-continuous t-norms
- 4 Mathematical Fuzzy Logic

Logic is the science that studies the **relation of logical consequence**:

when a proposition is a logical consequence of (entailed by, follows from, derived from) a set of propositions.

Logic studies **correct reasoning**.

The notion of correct reasoning is not unique, so there is a multiplicity of **logics**.

In Mathematics we typically assume the **Bivalence Principle**.

Bivalence Principle

Every proposition is either true or false.

The usual logic for mathematical reasoning is **classical logic**.

In classical logic every predicate yields a perfect division between those objects it applies to, and those it does not. We call them *crisp*.

Examples: prime number, even number, monotonic function, continuous function, divisible group, ... (any mathematical predicate)

Formulae of classical logic are evaluated at \mathcal{B}_2 (the two-element Boolean algebra).

Correct reasoning in classical logic

Definition

Given $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$ we say that φ is a **logical consequence** of Γ , denoted $\Gamma \models_{\mathcal{B}_2} \varphi$, iff for every \mathcal{B}_2 -evaluation e :
if $e(\gamma) = 1$ for every $\gamma \in \Gamma$, then $e(\varphi) = 1$.

Remark

$$\begin{array}{c} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \\ \hline \varphi \end{array}$$

is a correct reasoning iff **there is no interpretation making the premises true and the conclusion false.**

Example: *modus ponens*

$$\begin{array}{c} \varphi \\ \varphi \rightarrow \psi \\ \hline \psi \end{array}$$

Sorites paradox [Eubulides of Miletus, IV century BC]

A man who has no money is poor. If a poor man earns one dollar, he remains poor. Therefore, a man who has one million dollars is poor.

Formalization:

p_n : A man who has exactly n dollars is poor.

p_0

$p_0 \rightarrow p_1$

$p_1 \rightarrow p_2$

$p_2 \rightarrow p_3$

\vdots

$p_{999999} \rightarrow p_{1000000}$

$p_{1000000}$

- There is no doubt that the premise p_0 is true.
- There is no doubt that the conclusion $p_{1000000}$ is false.
- For each i , the premise $p_i \rightarrow p_{i+1}$ seems to be true.
- The reasoning is logically correct (application of *modus ponens* one million times).
- **We have a paradox!**

The predicates that generate this kind of paradoxes are called *vague*.

Remark

A predicate is vague iff it has **borderline cases**, i.e. there are objects for which we cannot tell whether they fall under the scope of the predicate.

Example: Consider the predicate *tall*. Is a man measuring 1.78 meters tall?

- It is not a problem of **ambiguity**.

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- It cannot be solved by **establishing a crisp definition of the predicate**. The problem is: with the meaning that the predicate *tall* has in the natural language, whatever it might be, is a man measuring 1.78 meters tall?

Many-valued logics to deal with vagueness

- 1949 Sören Halldén in *The Logic of Nonsense* proposes a three-valued logic to model vague predicates.
- 1955 Stephan Körner proposes an alternative three-valued treatment.
- 1965 Lotfi Zadeh proposes **Fuzzy Set Theory (FST)** as a mathematical treatment of vagueness and imprecision. FST becomes an extremely popular paradigm for engineering applications, known also as *Fuzzy Logic*.
- 1969 Goguen shows how to combine Zadeh's fuzzy sets and Łukasiewicz logic to solve the sorites paradox.

Fuzzy sets as a model for vague predicates

Formally a **fuzzy set** is a pair $\langle X, \mu \rangle$ where X is a classical set and $\mu : X \rightarrow [0, 1]$ is a function (called *membership function*) that maps every object $x \in X$ to its membership degree $\mu(x) \in [0, 1]$.

Example: For the predicate *tall* take $X := [0.3, 2.4]$ (containing all possible heights) and

$$\mu(x) = \begin{cases} 0 & \text{if } x \leq 1.2, \\ \frac{5}{3}x - 2 & \text{if } 1.2 \leq x \leq 1.8, \\ 1 & \text{if } x \geq 1.8. \end{cases}$$

If fuzzy sets interpret atomic vague propositions, their set-theoretic operations correspond to logical connectives for vague propositions.

- Conjunction (intersection): t-norm.
- Disjunction (union): t-conorm.
- Negation (complement): negation function.
- Implication: residuum of the t-norm.

T-norms

Definition

A **t-norm** is a binary function $*$: $[0, 1]^2 \rightarrow [0, 1]$ such that for every $a, b, c \in [0, 1]$:

- $a * (b * c) = (a * b) * c$
- $a * b = b * a$
- if $a \leq b$, then $a * c \leq b * c$
- $a * 1 = a$

T-norms

Example

Three basic continuous t-norms

- Łukasiewicz t-norm: $a *_L b = \max\{a + b - 1, 0\}$
- Product t-norm: $a *_\Pi b = a \cdot b$
- Minimum t-norm: $a *_G b = \min\{a, b\}$

Ordinal sums

Definition

Let $\{[a_i, b_i] \mid i \in I\}$ be a countable family of closed subintervals of $[0, 1]$ such that their interiors are pairwise disjoint. For every $i \in I$, let $*_i$ be a t-norm defined on $[a_i, b_i]^2$. The ordinal sum of this family of t-norms is the operation defined as:

$$x * y = \begin{cases} x *_m y & \text{if } \exists m \in I \text{ such that } x, y \in [a_m, b_m], \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

Theorem

Every continuous t-norm can be decomposed as an ordinal sum of the three basic ones.

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Rule-based systems:

if <premise>, then <conclusion>

What if the involved facts correspond to vague concepts?

Example

Assume we want to control the temperature of a certain machine.

T : desired temperature we want to achieve

t_i : temperature at a given moment i

$$\varepsilon = t_i - T$$

$$\Delta\varepsilon = (t_i - T) - (t_{i-1} - T) = t_i - t_{i-1}$$

Control variable: with positive values heats the machine, with negative values cools it down

Fuzzy rule: if ε is *very positive* and $\Delta\varepsilon$ is *very positive*, then the control variable must be *slightly negative*

Definition (Zadeh, 1975)

A **linguistic variable** is a variable whose values are words or sentences in a natural or artificial language. Formally it is given by a tuple $\langle X, L_X, U_X, S_X \rangle$ where:

- X is the name of the variable (e.g. age, height, temperature)
- L_X is the set of linguistic values X can take (e.g. {hot, warm, cold})
- U_X a domain in which X takes values, usually a set of real numbers (e.g. a range of real numbers measuring possible temperatures)
- S_X a semantical function interpreting the linguistic terms, which maps each value of L_X to a fuzzy set on U_X

Example

- $X = \text{temperature}$
- $L_X = \{\text{hot, warm, cold}\}$
- $U_X = [-50, 50]$
- S_X is the function mapping each element of L_X respectively to:

$$\bullet \mu_{\text{hot}} = \begin{cases} 1 & \text{if } x > 40 \\ (x - 35)/5 & \text{if } 35 \leq x \leq 40 \\ 0 & \text{if } x < 35 \end{cases}$$

$$\bullet \mu_{\text{warm}} = \begin{cases} 0 & \text{if } x > 40 \text{ or } x < 30 \\ (x - 30)/5 & \text{if } 30 \leq x \leq 35 \\ 1 - (x - 35)/5 & \text{if } 35 < x \leq 40 \end{cases}$$

$$\bullet \mu_{\text{cold}} = \begin{cases} 0 & \text{if } x > 35 \\ 1 - (x - 30)/5 & \text{if } 30 \leq x \leq 35 \\ 1 & \text{if } x < 30 \end{cases}$$

Definition

A set of fuzzy sets $\{\mu_1, \dots, \mu_n\}$ over the same domain D is a **fuzzy partition** if for every $x \in D$, $\sum_{i=1}^n \mu_i(x) = 1$.

Example

For every $x \in [-50, 50]$, $\mu_{\text{hot}}(x) + \mu_{\text{warm}}(x) + \mu_{\text{cold}}(x) = 1$.

Elements of a fuzzy system

- output variable Y
- input variables X_1, \dots, X_n
- components of variables (as linguistic variables)
- fuzzy rules:

if X_1 is t_{i_1} and X_2 is t_{i_2} and \dots and X_n is t_{i_n} , then Y is t_{i_0}

The rules must cover the whole domain of the application.

Example

- input variables: ε (error) and $\Delta\varepsilon$ (increment of error)
- output variable: cv (control variable)
- $L_\varepsilon = \{\text{very positive (VP), positive (P), zero (Z), negative (N), very negative (VN)}\}$
- $L_{\Delta\varepsilon} = \{\text{very positive (VP}_\Delta), \text{quite positive (QP), quasi zero (QZ), quite negative (QN), very negative (VN}_\Delta)\}$
- $L_{cv} = \{\text{almost zero (AZ), slightly negative (SN)}\}$
- $U_\varepsilon = [-5, 5], U_{\Delta\varepsilon} = [-10, 10], U_{cv} = [-15, 15]$

Example

- $\mu_{VP}(x) = \begin{cases} 0 & \text{if } x < 2.5 \\ (x - 2.5)/2.5 & \text{if } x \geq 2.5 \end{cases}$
- $\mu_P(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x/2.5 & \text{if } 0 < x \leq 2.5 \\ 1 - (x - 2.5)/2.5 & \text{if } 2.5 < x \leq 5 \end{cases}$
- $\mu_{VP_\Delta}(x) = \begin{cases} 0 & \text{if } x < 7.5 \\ (x - 7.5)/2.5 & \text{if } x \geq 7.5 \end{cases}$
- $\mu_{QP}(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x/7.5 & \text{if } 0 < x \leq 7.5 \\ 1 - (x - 7.5)/2.5 & \text{if } 7.5 < x \leq 10 \end{cases}$

Example

$$\bullet \mu_{SN}(x) = \begin{cases} 0 & \text{if } x \leq -5 \\ (x+5)/4 & \text{if } -5 < x \leq -1 \\ -x & \text{if } -1 < x \leq 0 \\ 0 & \text{if } 0 \leq x \end{cases}$$

$$\bullet \mu_{AZ}(x) = \begin{cases} 0 & \text{if } x \leq -1 \\ x+1 & \text{if } -1 < x \leq 0 \\ 1-x & \text{if } 0 < x \leq 1 \\ 0 & \text{if } 1 \leq x \end{cases}$$

Example

	VN_{Δ}	QN	QZ	QP	VP_{Δ}
VN					
N					
Z					
P				AZ	SN
VP				SN	SN

If ε is VP and $\Delta\varepsilon$ is VP_{Δ} , then $c\varepsilon$ is SN .

etc.

Mandami's method

Example

- Let us interpret conjunction with the minimum t-norm
- Assume $\varepsilon = 3$ and $\Delta\varepsilon = 8.5$
- $\mu_{VP}(3) = 0.2$ and $\mu_{VP\Delta}(8.5) = 0.4$
- $T(\mu_{VP}(3), \mu_{VP\Delta}(8.5)) = \min\{0.2, 0.4\} = 0.2$
- The membership function of the conclusion of the rule is

$$\mu(x) = \min\{0.2, \mu_{SN}(x)\} = \begin{cases} 0 & \text{if } x \leq -5 \\ (x + 5)/4 & \text{if } -5 < x \leq -4.2 \\ 0.2 & \text{if } -4.2 < x \leq -0.2 \\ -x & \text{if } -0.2 < x \leq 0 \\ 0 & \text{if } 0 \leq x \end{cases}$$

Mandami's method

If we have a set of rules in the system, we apply the same method to each one and obtain a set of fuzzy sets. The conclusion is the union (taking the maximum t-conorm) of all of them, call it μ .

Defuzzification: $(\sum_{x \in D} \mu(x) \cdot x) / \sum_{x \in D} \mu(x)$

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FST solution to sorites paradox

Motto: Truth comes in degrees.

- $X = \{0, 1, 2, \dots, 10^6\}$, $\mu : X \rightarrow [0, 1]$.
- The truth value of p_n will be $\mu(n)$.
- $\mu(0) = 1$ and $\mu(10^6) = 0$ (the first premise is **completely true**, the conclusion is **completely false**).
- Take $\varepsilon = 10^{-6}$ and $\mu(n) = 1 - n\varepsilon$.
- Compute the value of $p_n \rightarrow p_{n+1}$ by means of Łukasiewicz implication.
- $\mu(n) \rightarrow \mu(n+1) = 1 - \mu(n) + \mu(n+1) = 1 - (1 - n\varepsilon) + (1 - (n+1)\varepsilon) = 1 - \varepsilon < 1$ (all these premises have the same truth value: **almost completely true**).

Objections

1 Arbitrariness of the interpretation:

- Why should we use Łukasiewicz functions?
- Why should we use this evaluation of atomic propositions?
- Why should we use $[0, 1]$ as the set of truth-values?

Where do the truth values (and their combinations) come from?

2 FST per se, strictly speaking, is not logic.

Requirements from Deductive (Formal, Mathematical) Logic

A logical system must have:

- a formal syntax
- a semantical consequence relation based on some notion of truth
- an (equivalent) notion of proof (completeness theorem)

The formal language (1st requirement)

- Denumerable set of variables X .
- Binary connective $\&$ for **strong conjunction**.
- Binary connective \wedge for **weak conjunction**.
- Binary connective \rightarrow for **implication**.
- Nullary connective $\bar{0}$ for **falsity**.

This gives a set of well-formed formulae $Fm_{\mathcal{L}}$.

Defined connectives:

$$\varphi \vee \psi = ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$$

$$\neg\varphi = \varphi \rightarrow \bar{0}$$

$$\bar{1} = \neg\bar{0}$$

Semantics

- Set of truth-values: $[0, 1]$.
- $\&$ is interpreted by a left-continuous t-norm $*$.
- \wedge is interpreted by the min function.
- \rightarrow is interpreted by the residuum \Rightarrow of $*$.
 $(x \Rightarrow y = \max\{z \in [0, 1] \mid x * z \leq y\})$
- $\bar{0}$ is interpreted by 0.
- \vee is interpreted by the max function.
- \neg is interpreted by the negation function $n(x) = x \Rightarrow 0$.
- $\bar{1}$ is interpreted by 1.

Why should the implication be interpreted as the residuum of a t-norm?

Requirements

- Truth-functionality: the connectives are interpreted by functions (a t-norm $*$ for strong conjunction, a function \Rightarrow for implication).
- Normality: the connectives behave classically in $\{0, 1\}$.
- \Rightarrow captures the order: $x \Rightarrow y = 1$ iff $x \leq y$.
- Truth-preserving *modus ponens*: the truer the premises are, the truer the conclusion is.

Why should the implication be interpreted as the residuum of a t-norm?

Modus ponens is a valid rule in this sense: if $v(\varphi) = 1$ and $v(\varphi) \Rightarrow v(\psi) = 1$, then $v(\psi) = 1$

Why should the implication be interpreted as the residuum of a t-norm?

Modus ponens is a valid rule in this sense: if $v(\varphi) = 1$ and $v(\varphi) \Rightarrow v(\psi) = 1$, then $v(\psi) = 1$

But, we want more:

$$v(\varphi) * [v(\varphi) \Rightarrow v(\psi)] \leq v(\psi).$$

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But, we want more:

$$v(\varphi) * [v(\varphi) \Rightarrow v(\psi)] \leq v(\psi).$$

Take the **residuum**:

$v(\varphi) \Rightarrow v(\psi) = \max\{c \in [0, 1] \mid v(\varphi) * c \leq v(\psi)\}$, i.e. the maximum value that fulfills the required inequality, and so making the *modus ponens* rule as powerful as possible.

- Each continuous t-norm is residuated.
- Actually, in the proof only left-continuity is used.

Proposition

Let $*$ be a t-norm. Then:

$*$ is residuated if, and only if, it is left-continuous.

Historical remarks: two converging lines

- 1930 Łukasiewicz and Tarski: $[0, 1]$ -valued logic that uses a particular continuous t-norm to interpret (strong) conjunction.
- 1942 Menger: Statistical metrics. Triangular norms (t-norms) as functions to deal with triangular inequality.
- 1957 Mostert, Shields: Representation of continuous t-norms as ordinal sums of three basic components.
- 1959 Dummett resumes Gödel's work from 1932 and proposes a $[0, 1]$ -valued logic that uses \min function (a continuous t-norm) to interpret conjunction.
- 1960 Schweizer and Sklar: Development of statistical metric spaces.

- 1965 Zadeh: Fuzzy Set Theory (FST). He proposes min function for the intersection of fuzzy sets.
- 1969 Goguen: combined use of Zadeh's fuzzy sets and Łukasiewicz functions to solve the sorites paradox.
- late 70s Several researchers in FST propose the usage of t-norms to interpret intersection of fuzzy sets and conjunction in the logics. All known residuated t-norms are continuous.
- 1995 Fodor: Nilpotent Minimum, a left-continuous non-continuous t-norm.
- 1996 Esteva, Godo, Hájek: Product logic as a $[0, 1]$ -valued logic that uses product t-norm to interpret (strong) conjunction.

- 1998 Hájek introduces the logic BL and conjectures that it is the logic of all continuous t-norms. He shows that Ł, G, and II are axiomatic extensions of BL.
- 2000 Hájek, Cignoli, Esteva, Godo and Torrens: BL is the logic of all continuous t-norms.
- 2001 Esteva and Godo introduce the logic MTL and conjecture that it is the logic of all left-continuous t-norms. They show that the logic NM of Fodor's t-norm and BL are axiomatic extensions of MTL.
- 2002 Jenei and Montagna: MTL is the logic of all left-continuous t-norms.

Hilbert-style calculus for MTL

$$(MTL1) \quad (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$$

$$(MTL2) \quad \varphi \&\psi \rightarrow \varphi$$

$$(MTL3) \quad \varphi \&\psi \rightarrow \psi \&\varphi$$

$$(MTL4) \quad \varphi \wedge \psi \rightarrow \varphi$$

$$(MTL5) \quad \varphi \wedge \psi \rightarrow \psi \wedge \varphi$$

$$(MTL6) \quad \varphi \&(\varphi \rightarrow \psi) \rightarrow \varphi \wedge \psi$$

$$(MTL7a) \quad (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \&\psi \rightarrow \chi)$$

$$(MTL7b) \quad (\varphi \&\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$$

$$(MTL8) \quad (((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi))$$

$$(MTL9) \quad \bar{0} \rightarrow \varphi$$

Inference rule: *modus ponens*

$\Gamma \vdash_{\text{MTL}} \varphi$ means that there is a proof of φ from Γ in this system.

The algebraic semantics

Definition

An **MTL-algebra** $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$ is a **residuated lattice satisfying the prelinearity equation:**

$$(x \rightarrow y) \vee (y \rightarrow x) \approx \bar{1}.$$

If the lattice order is linear, we call it *MTL-chain*.

Let $*$ be a left-continuous t-norm and \Rightarrow_* its residuum. Then $[0, 1]_* = \langle [0, 1], *, \Rightarrow_*, \min, \max, 0, 1 \rangle$ is an MTL-chain.

All MTL-chains over $[0, 1]$ are of this form and are called *standard chains*.

Theorem

The class of all MTL-algebras, MTL , is a variety.

Important subvarieties of MTL :

- 1 BL : $x \& (x \rightarrow y) \approx x \wedge y$ (**divisibility equation**).
- 2 MV : **divisibility** + $\neg \neg x \approx x$ (**involution equation**).
- 3 P : **divisibility** + $\neg x \vee ((x \rightarrow x \& y) \rightarrow y) \approx \bar{1}$ (**cancellativity equation**).
- 4 G : $x \approx x \& x$ (**contraction equation**).
- 5 BA : $x \vee \neg x \approx \bar{1}$ (**excluded middle**).
- 6 ...

Semantical notion of logical consequence (2nd requirement)

Definition

Given $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$, we define the consequence relation $\Gamma \models_{\text{MTL}} \varphi$ iff for every $\mathcal{A} \in \text{MTL}$ and every \mathcal{A} -evaluation e : if $e[\Gamma] \subseteq \{\bar{1}^{\mathcal{A}}\}$, then $e(\varphi) = \bar{1}^{\mathcal{A}}$. (Preservation of complete truth)

Completeness (3rd requirement)

Theorem

For every $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\text{MTL}} \varphi$ iff $\Gamma \models_{\text{MTL}} \varphi$.

MTL is the **equivalent algebraic semantics** of MTL.

The rôle of chains

Theorem

Every MTL-algebra is representable as a subdirect product of MTL-chains.

Corollary

For every $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\text{MTL}} \varphi$ iff $\Gamma \models_{\{\text{MTL-chains}\}} \varphi$.

Remark

These properties remain valid for axiomatic extensions as well.

Some axiomatic extensions of MTL

BL	=	MTL	+	$\varphi \wedge \psi \rightarrow \varphi \& (\varphi \rightarrow \psi)$	(Div)
\mathcal{L}	=	BL	+	$\neg\neg\varphi \rightarrow \varphi$	(Inv)
G	=	BL	+	$\varphi \rightarrow \varphi \& \varphi$	(C)
Π	=	BL	+	$\neg\psi \vee ((\psi \rightarrow \varphi \& \psi) \rightarrow \varphi)$	(Can)
SBL	=	BL	+	$\varphi \wedge \neg\varphi \rightarrow \bar{0}$	(PC)
IMTL	=	MTL	+	$\neg\neg\varphi \rightarrow \varphi$	(Inv)
SMTL	=	MTL	+	$\varphi \wedge \neg\varphi \rightarrow \bar{0}$	(PC)
Π IMTL	=	MTL	+	$\neg\psi \vee ((\psi \rightarrow \varphi \& \psi) \rightarrow \varphi)$	(Can)
WCMTL	=	MTL	+	$\neg(\varphi \& \psi) \vee ((\psi \rightarrow \varphi \& \psi) \rightarrow \varphi)$	(WCan)
WNM	=	MTL	+	$(\varphi \& \psi \rightarrow \bar{0}) \vee (\varphi \wedge \psi \rightarrow \varphi \& \psi)$	(WNM)
NM	=	IMTL	+	$(\varphi \& \psi \rightarrow \bar{0}) \vee (\varphi \wedge \psi \rightarrow \varphi \& \psi)$	(WNM)
C_n MTL	=	MTL	+	$\varphi^{n-1} \rightarrow \varphi^n$	(C_n)
C_n IMTL	=	IMTL	+	$\varphi^{n-1} \rightarrow \varphi^n$	(C_n)
CPC	=	MTL	+	$\varphi \vee \neg\varphi$	(EM)

...

Strong standard completeness of MTL

Theorem

For every $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$,

$\Gamma \vdash_{\text{MTL}} \varphi$ iff $\Gamma \models_{\{[0,1]_{*} \mid * \text{ left-continuous t-norm}\}} \varphi$.

Solution to the objection of arbitrariness

The logical consequence does not depend on particular arbitrary choices, but on *all possible interpretations*.

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Mathematical Fuzzy Logic is the study of systems of fuzzy logic with the methodology of **Mathematical Logic**

Other systems of fuzzy logic

- **Removing bottom:** Logics based on hoops on the real interval (proponents: Esteva, Godo, Hájek, Montagna 2002)
- **Removing commutativity:** Logics based non-commutative t-norms (proponents: Hájek 2002)
- **Adding expressive power:** Logics based on expanded standard algebras (with Δ, \sim , truth-constants, hedges, etc.)
- **Leaving the real interval behind:** Logics complete w.r.t. rational, finite, hyperreal, and in general (expansions of) linearly ordered MTL-algebras.
- **General classes of fuzzy logics:** Core and Δ -core fuzzy logics (proponents: Hájek and Cintula 2006)

Other systems of fuzzy logic

- **Removing commutativity:** PseudoBL, PseudoMTL (proponents: Hájek, Jenei, Montagna, 2002)
- **Removing integrality:** Uninorm logics (proponents: Metcalfe, Montagna, 2007)
- **Removing associativity:** SL^ℓ (proponents: Cintula, Horčík, Noguera, 2012)

Relation to other families of logics

$$(lin) \quad (\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$$

$$(lin_1) \quad ((\varphi \rightarrow \psi) \wedge \bar{1}) \vee ((\psi \rightarrow \varphi) \wedge \bar{1})$$

- G is the axiomatic extension of **intuitionistic logic** obtained by adding (lin)
- MTL is the axiomatic extension of **FL_{ew} (monoidal logic)** obtained by adding (lin)
- UL is the axiomatic extension of **FL_e (linear logic)** obtained by adding (lin_1)
- ...

Fuzzy logics are a subfamily of **substructural logics**

Taking intermediate truth degrees seriously

- 1 **Truth degrees as syntactical objects:** add a truth-constant \bar{r} for each truth value r (Pavelka logic, Hájek RPL, Novák's evaluated syntax, algebraic approach by Esteva et al).
- 2 **Logics preserving truth degrees:** $\Gamma \models_{\mathcal{A}} \varphi$ iff for every \mathcal{A} -evaluation e : if $\min e[\Gamma] \leq e(\varphi)$. (**Preservation of partial truth**)

The essence of Mathematical Fuzzy Logic

- 1 Completeness w.r.t. a semantics based on chains (Hájek, Pavelka, Novák, Esteva, Godo, Montagna ...)
- 2 **Fuzzy logics are the logics of chains** (Běhounek-Cintula 2006)
- 3 Mathematical definition: semilinear logics (Cintula-Noguera 2010)

Agenda of Mathematical Fuzzy Logic

- **Semantics:** algebraic and relational semantics, algebraic study of logical properties, ...
- **Model theory:** study of first-order models
- **Set theory:** axiomatic set theory over fuzzy logics
- **Proof theory:** sequent systems, hypersequents systems, ...
- **Recursion theory:** computational complexity, arithmetical complexity

Agenda of Mathematical Fuzzy Logic

- **Game theory:** dialogical games, Rényi-Ulam games, evaluation games
- **Philosophy of Logic:** liar paradox, vagueness paradoxes
- **Applied logical calculi:** logic programming, possibilistic and probabilistic reasoning, description logics, similarity-based reasoning, ...

International working group (www.mathfuzzlog.org)

Series of international conferences: *Logic, Algebra and Truth Degrees*

Handbook of Mathematical Fuzzy Logic

Edited by Petr Cintula, Petr Hájek, and Carles Noguera.
Studies in Logic, Mathematical Logic and Foundations, vols. 37
and 38, College Publications, 2011.

- 1 Introduction to Mathematical Fuzzy Logic (Behounek, Cintula, Hájek).
- 2 A general Framework for Mathematical Fuzzy Logic (Cintula, Noguera)
- 3 Algebraic Semantics (Horcík)
- 4 Proof Theory (Metcalf)
- 5 Łukasiewicz Logic and MV-Algebras (Di Nola, Leustean)

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- 6 Gödel-Dummett Logics (Baaz, Preining)
- 7 Basic Fuzzy Logic and BL-Algebras (Busaniche, Montagna)
- 8 Logics with Enriched Language (Esteva, Godo, Marchioni)
- 9 Free Algebras and Functional Representation (Aguzzoli, Bova, Gerla)
- 10 Computational Complexity (Hanikova)
- 11 Arithmetical Complexity (Hájek, Montagna, Noguera)

Conclusions

- FST offers a mathematical treatment of vagueness with a great flexibility for (engineering) applications.
- MFL is the study of fuzzy logics arising from FST.
- Fuzzy logics allow a mathematical treatment of *some aspects* of vagueness, those corresponding to **gradual properties**.
- Fuzzy logics are logics of **graduality**.
- MFL is a very active branch of **Mathematical Logic** with many challenging open problems.

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