General Theories of Logical Systems
1st lesson

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Logic is the science that studies correct reasoning.

It is studied as part of Philosophy, Mathematics, and Computer Science.

From XIXth century, it has become a formal science that studies symbolic abstractions capturing the formal aspects of inference: symbolic logic or mathematical logic.

What is a correct reasoning?

Example 1.1

“If God exists, He must be good and ommipotent. If God was good and omnipotent, He would not allow human suffering. But, there is human suffering. Therefore, God does not exist."

Is this a correct reasoning?

What is a correct reasoning?

Formalization

Atomic parts:

- p: God exists
- q: God is good
- r: God is ommipotent
- s: There is human suffering

The form of the reasoning:

$$\begin{array}{l} p \to q \wedge r \\ \neg(q \wedge r \wedge s) \\ s \\ \hline \neg p \end{array}$$

Is this a correct reasoning?

Classical logic

Syntax:

Formulae $\boldsymbol{Fm}_{\mathcal{L}}$ built from atoms combined by connectives $\mathcal{L} = \{\neg, \wedge, \vee, \to\}$.

Semantics:

Bivalence Principle

Every proposition is either true or false.

Definition 1.2

The Boolean algebra of two elements, $\mathbf{2}$, is defined over the universe $\{0, 1\}$ with the following operations:

$\neg^{\mathbf{2}}$	
0	1
1	0

$\wedge^{\mathbf{2}}$	0	1
0	0	0
1	0	1

$\vee^{\mathbf{2}}$	0	1
0	0	1
1	1	1

$\to^{\mathbf{2}}$	0	1
0	1	1
1	0	1

$\mathbf{2} = \langle \{0, 1\}, \neg^{\mathbf{2}}, \wedge^{\mathbf{2}}, \vee^{\mathbf{2}}, \to^{\mathbf{2}} \rangle$

Correct reasoning in classical logic

Definition 1.3

Given $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$ we say that φ is a logical consequence of Γ, denoted $\Gamma \models_{\mathbf{2}} \varphi$, iff for every $\mathbf{2}$-evaluation e such that $e(\gamma) = 1$ for every $\gamma \in \Gamma$, we have $e(\varphi) = 1$.

Correct reasoning = logical consequence

Definition 1.4

Given $\psi_1, \ldots, \psi_n, \varphi \in \boldsymbol{Fm}_{\mathcal{L}}$ we say that $\langle \psi_1, \ldots, \psi_n, \varphi \rangle$ is a correct reasoning if $\{\psi_1, \ldots, \psi_n\} \models_{\mathbf{2}} \varphi$. In this case, $\psi_1, \ldots, \psi_n$ are the premises of the reasoning and φ is the conclusion.

Correct reasoning in classical logic

Remark

$$\frac{\begin{array}{c}\psi_1\\ \psi_2\\ \vdots\\ \psi_n\end{array}}{\varphi}$$

is a correct reasoning iff there is no interpretation making the premises true and the conclusion false.

Correct reasoning in classical logic

Example 1.5

Modus ponens:

$$\frac{\begin{array}{l} p \to q \\ p \end{array}}{q}$$

It is a correct reasoning (if $e(p \to q) = e(p) = 1$, then $e(q) = 1$).

Example 1.6

Abduction:

$$\frac{\begin{array}{l} p \to q \\ q \end{array}}{p}$$

It is NOT a correct reasoning (take: $e(p) = 0$ and $e(q) = 1$).

Correct reasoning in classical logic

Example 1.7

$p \to q \wedge r$
$\neg(q \wedge r \wedge s)$
s
___
$\neg p$

Assume $e(p \to q \wedge r) = e(\neg(q \wedge r \wedge s)) = e(s) = 1$. Then $e(q \wedge r \wedge s) = 0$, so $e(q \wedge r) = 0$. But, since $e(p \to q \wedge r) = 1$, we must have $e(p) = 0$, and therefore: $e(\neg p) = 1$.

It is a correct reasoning!

BUT, is this really a proof that God does not exist?

NO. We only know that if the premisses were true, then the conclusion would be true as well.

Logic(s)

Logic studies the notion of logical consequence. There are many kinds of logical consequence, i.e. many different logics:

1. Classical logic
2. Non-classical logics:
   - Modal logics
   - Intuitionistic logic
   - Superintuitionistic logics
   - Linear logics
   - Fuzzy logics
   - Relevance logics
   - Substructural logics
   - Paraconsistent logics
   - Dynamic logics
   - Non-monotonic logics

   ⋮

Algebraic Logic is the subdiscipline of Mathematical Logic which studies logical systems (classical and non-classical) by using tools from Universal Algebra.

Logic	Algebraic counterpart
Classical logic	Boolean algebras
Modal logics	Modal algebras
Intuitionistic logic	Heyting algebras
Linear logics	Commutative residuated lattices
Fuzzy logics	Semilinear residuated lattices
Relevance logics	Commutative contractive residuated lattices
⋮	⋮

Universal Algebra is the field of Mathematics which studies algebraic structures.

Abstract Algebraic Logic

AAL is the evolution of Algebraic Logic that wants to:

- understand the several ways by which a logic can be given an algebraic semantics
- build a general and abstract theory of non-classical logics based on their relation to algebras
- understand the rôle of connectives in (non-)classical logics.
- classify non-classical logics
- find general results connecting logical and algebraic properties (bridge theorems)
- generalize properties from syntax to semantics (transfer theorems)
- advance the study of particular (families of) non-classical logics by using the abstract notions and results

It works best, by far, when restricted to propositional logics.

A little history of (Abstract) Algebraic Logic – 1

1847	George Boole, *Mathematical Analysis of Logic.* Augustus De Morgan, *Formal Logic*.
1854	George Boole, *The Laws of Thought*.
1880	Charles Sanders Peirce, *On the Algebra of Logic*.
1890	Ernst Schröder, *Algebra der Logik* (in three volumes).
1920	Jan Łukasiewicz, *O logice trojwartosciowej*. Three-valued logic.
1930	Jan Łukasiewicz, Alfred Tarski, *Untersuchungen über den Aussagenkalkül*. Infinitely-valued logic.
1930	Alfred Tarski, *Über einige fundamentale Begriffe der Metamathematik*. Consequence operators.
1931	Alfred Tarski, *Grundzüge der Systemenkalküls*. Precise connection between classical logic and Boolean algebras. Lindenbaum–Tarski method.

A little history of (Abstract) Algebraic Logic – 2

1935 Garrett Birkhoff, *On the Structure of Abstract Algebras*. Universal Algebra, equational classes, equational logic.

1958 Jerzy Łoś and Roman Suszko, *Remarks on sentential logics*. Structural consequence operators.

1973 Ryszard Wójcicki, *Matrix approach in the methodology of sentential calculi*.

1974 Helena Rasiowa, *An algebraic approach to non-classical logics*.

1975 S.L. Bloom, *Some theorems on structural consequence*. *operations*.

1981 Janusz Czelakowski, *Equivalential logics I and II*.

1986 Willem J. Blok, Don L. Pigozzi, *Protoalgebraic logics*.

1989 Willem J. Blok, Don L. Pigozzi, *Algebraizable logics*.

1996 Josep Maria Font, Ramon Jansana, *A general algebraic semantics for sentential logics*.

2000 Janusz Czelakowski, *Protoalgebraic logics*.

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Basic syntactical notions – 1

Propositional language: a countable type $\mathcal{L}$, i.e. a function $ar\colon C_{\mathcal{L}} \to$ N, where $C_{\mathcal{L}}$ is a countable set of symbols called connectives, giving for each one its arity. Nullary connectives are also called truth-constants. We write $\langle c, n\rangle \in \mathcal{L}$ whenever $c \in C_{\mathcal{L}}$ and $ar(c) = n$.

Formulae: Let Var be a fixed infinite countable set of symbols called variables. The set $Fm_{\mathcal{L}}$ of formulae in $\mathcal{L}$ is the least set containing Var and closed under connectives of $\mathcal{L}$, i.e. for each $\langle c, n\rangle \in \mathcal{L}$ and every $\varphi_1, \ldots, \varphi_n \in Fm_{\mathcal{L}}$, $c(\varphi_1, \ldots, \varphi_n)$ is a formula.

Substitution: a mapping $\sigma\colon Fm_{\mathcal{L}} \to Fm_{\mathcal{L}}$, such that $\sigma(c(\varphi_1, \ldots, \varphi_n)) = c(\sigma(\varphi_1), \ldots, \sigma(\varphi_n))$ holds for each $\langle c, n\rangle \in \mathcal{L}$ and every $\varphi_1, \ldots, \varphi_n \in Fm_{\mathcal{L}}$.

Consecution: a pair $\Gamma \rhd \varphi$, where $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$.

Basic syntactical notions – 2

A set L of consecutions can be seen as relation between sets of formulae and formulae. We write '$\Gamma \vdash_{\mathrm{L}} \varphi$' instead of '$\Gamma \rhd \varphi \in \mathrm{L}$'.

Definition 1.8

A set L of consecutions in $\mathcal{L}$ is called a logic in $\mathcal{L}$ whenever

- If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathrm{L}} \varphi$. (Reflexivity)
- If $\Delta \vdash_{\mathrm{L}} \psi$ for each $\psi \in \Gamma$ and $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Cut)
- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each substitution σ. (Structurality)

Observe that reflexivity and cut entail:

- If $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Monotonicity)

The least logic Dumb is described as:

$$\Gamma \vdash_{\mathrm{Dumb}} \varphi \qquad \text{iff} \qquad \varphi \in \Gamma.$$

Basic syntactical notions – 3

Theorem: a consequence of the empty set (note that Dumb has no theorems).

Inconsistent logic Inc: the set all consecutions (equivalently: a logic where all formulae are theorems).

Almost Inconsistent logic AInc: the maximum logic without theorems (note that $\Gamma, \varphi \vdash_{\mathrm{AInc}} \psi$).

Theory: a set of formulae T such that if $T \vdash_{\mathrm{L}} \varphi$ then $\varphi \in T$. By $\mathrm{Th}(\mathrm{L})$ we denote the set of all theories of L.

Note that

- $\mathrm{Th}(\mathrm{L})$ can be seen as a closure system. By $\mathrm{Th}_{\mathrm{L}}(\Gamma)$ we denote the theory generated in $\mathrm{Th}(\mathrm{L})$ by Γ (i.e., the intersection of all theories containing Γ).
- $\mathrm{Th}_{\mathrm{L}}(\Gamma) = \{\varphi \in Fm_{\mathcal{L}} \mid \Gamma \vdash_{\mathrm{L}} \varphi\}$.
- The set of all theorems is the least theory and it is generated by the empty set.

Basic syntactical notions – 4

Axiomatic system: a set $\mathcal{AS}$ of consecutions closed under substitutions. An element $\Gamma \rhd \varphi$ is an

- axiom if $\Gamma = \emptyset$,
- finitary deduction rule if Γ is a finite,
- infinitary deduction rule otherwise.

An axiomatic system is finitary if all its rules are finitary.

Proof: a proof of a formula φ from a set of formulae Γ in $\mathcal{AS}$ is a well-founded tree labeled by formulae such that

- its root is labeled by φ and leaves by axioms of $\mathcal{AS}$ or elements of Γ and
- if a node is labeled by ψ and $\Delta \neq \emptyset$ is the set of labels of its preceding nodes, then $\Delta \rhd \psi \in \mathcal{AS}$.

We write $\Gamma \vdash_{\mathcal{AS}} \varphi$ if there is a proof of φ from Γ in $\mathcal{AS}$.

Basic syntactical notions – 5

Lemma 1.9

Let $\mathcal{AS}$ be an axiomatic system. Then $\vdash_{\mathcal{AS}}$ is the least logic containing $\mathcal{AS}$.

Presentation: We say that $\mathcal{AS}$ is an axiomatic system for (or a presentation of) the logic L if $\mathrm{L} = \vdash_{\mathcal{AS}}$. A logic is said to be finitary if it has some finitary presentation.

Lemma 1.10

A logic L is finitary iff for each set of formulae $\Gamma \cup \{\varphi\}$ we have: if $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathrm{L}} \varphi$.

Note that $\mathrm{Inc}, \mathrm{AInc}, \mathrm{Dumb}$ are finitary because:

Inc	is axiomatized by	axioms $\{\varphi \mid \varphi \in Fm_{\mathcal{L}}\}$
AInc	is axiomatized by	unary rules $\{\varphi \rhd \psi \mid \varphi, \psi \in Fm_{\mathcal{L}}\}$
Dumb	is axiomatized by	by the empty set

Examples: classical logic CL and logic BCI

Finitary axiomatic system for CL in $\mathcal{L}_{\mathrm{CL}} = \{\to, \neg\}$

- A1 $\varphi \to (\psi \to \varphi)$
- A2 $(\chi \to (\varphi \to \psi)) \to ((\chi \to \varphi) \to (\chi \to \psi))$
- A3 $(\neg\psi \to \neg\varphi)) \to (\varphi \to \psi)$
- MP $\varphi, \varphi \to \psi \rhd \psi$

Finitary axiomatic system for BCI in $\mathcal{L}_{\mathrm{BCI}} = \{\to\}$

- B $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
- C $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
- I $\varphi \to \varphi$
- MP $\varphi, \varphi \to \psi \rhd \psi$

Basic syntactical notions – 6

Let $\mathcal{L}_1 \subseteq \mathcal{L}_2$ be propositional languages, L_i a logic in $\mathcal{L}_i$, and $\mathcal{S}$ a set of consecutions in $\mathcal{L}_2$.

- L_2 is the expansion of L_1 by $\mathcal{S}$ if it is the weakest logic in $\mathcal{L}_2$ containing L_1 and $\mathcal{S}$, i.e. the logic axiomatized by all $\mathcal{L}_2$-substitutional instances of consecutions from $\mathcal{S} \cup \mathcal{AS}$, for any presentation $\mathcal{AS}$ of L_1.
- L_2 is an expansion of L_1 if $\mathrm{L}_1 \subseteq \mathrm{L}_2$, i.e. it is the expansion of L_1 by $\mathcal{S}$, for some set of consecutions $\mathcal{S}$.
- L_2 is an axiomatic expansion of L_1 if it is an expansion obtained by adding a set of *axioms*.
- L_2 is a conservative expansion of L_1 if it is an expansion and for each consecution $\Gamma \rhd \varphi$ in $\mathcal{L}_1$ we have that $\Gamma \vdash_{\mathrm{L}_2} \varphi$ entails $\Gamma \vdash_{\mathrm{L}_1} \varphi$.

If $\mathcal{L}_1 = \mathcal{L}_2$, we use ‘extension’ instead ‘expansion’.

Note that CL is the axiomatic expansion of BCI by A1–A3.

Basic semantical notions – 1

$\mathcal{L}$-algebra: $\boldsymbol{A} = \langle A, \langle c^{\boldsymbol{A}} \mid c \in C_{\mathcal{L}} \rangle \rangle$, where $A \neq \emptyset$ (universe) and $c^{\boldsymbol{A}} \colon A^n \to A$ for each $\langle c, n \rangle \in \mathcal{L}$.

Algebra of formulae: the algebra $\boldsymbol{Fm}_{\mathcal{L}}$ with domain $Fm_{\mathcal{L}}$ and operations $c^{\boldsymbol{Fm}_{\mathcal{L}}}$ for each $\langle c, n \rangle \in \mathcal{L}$ defined as:

$$c^{\boldsymbol{Fm}_{\mathcal{L}}}(\varphi_1, \ldots, \varphi_n) = c(\varphi_1, \ldots, \varphi_n).$$

$\boldsymbol{Fm}_{\mathcal{L}}$ if the absolutely free algebra in language $\mathcal{L}$ with generators Var.

Homomorphism of algebras: a mapping $f \colon A \to B$ such that for every $\langle c, n \rangle \in \mathcal{L}$ and every $a_1, \ldots, a_n \in A$,

$$f(c^{\boldsymbol{A}}(a_1, \ldots, a_n)) = c^{\boldsymbol{B}}(f(a_1), \ldots, f(a_n)).$$

Note that substitutions are exactly endomorphisms of $\boldsymbol{Fm}_{\mathcal{L}}$.

Basic semantical notions – 2

$\mathcal{L}$-matrix: a pair $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra called the algebraic reduct of $\mathbf{A}$, and F is a subset of A called the filter of $\mathbf{A}$. The elements of F are called designated elements of $\mathbf{A}$.

A matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ is

- trivial if $F = A$.
- finite if A is finite.
- Lindenbaum if $\boldsymbol{A} = \boldsymbol{Fm}_{\mathcal{L}}$.

$\boldsymbol{A}$-evaluation: a homomorphism from $\boldsymbol{Fm}_{\mathcal{L}}$ to $\boldsymbol{A}$, i.e. a mapping $e\colon Fm_{\mathcal{L}} \to A$, such that for each $\langle c, n\rangle \in \mathcal{L}$ and each n-tuple of formulae $\varphi_1, \ldots, \varphi_n$ we have:

$$e(c(\varphi_1, \ldots, \varphi_n)) = c^{\boldsymbol{A}}(e(\varphi_1), \ldots, e(\varphi_n)).$$

Basic semantical notions – 3

Semantical consequence: A formula φ is a semantical consequence of a set Γ of formulae w.r.t. a class $\mathbb{K}$ of $\mathcal{L}$-matrices if for each $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e, we have $e(\varphi) \in F$ whenever $e[\Gamma] \subseteq F$; we denote it by $\Gamma \models_{\mathbb{K}} \varphi$.

Exercise 1

Let $\mathbb{K}$ a class of $\mathcal{L}$-matrices. Then $\models_{\mathbb{K}}$ is a logic in $\mathcal{L}$.

Lemma 1.11

Furthermore if $\mathbb{K}$ is a finite class of finite matrices, then the logic $\models_{\mathbb{K}}$ is finitary.

L-matrix: Let L be a logic in $\mathcal{L}$ and $\mathbf{A}$ an $\mathcal{L}$-matrix. We say that $\mathbf{A}$ is an L-matrix if $\mathrm{L} \subseteq \models_{\mathbf{A}}$. We denote the class of L-matrices by $\mathbf{MOD}(\mathrm{L})$.

Lemma 1.12

Let L *be a logic in* $\mathcal{L}$ *and a mapping* $g\colon \boldsymbol{A} \to \boldsymbol{B}$ *be a homomorphism of* $\mathcal{L}$*-algebras* $\boldsymbol{A}, \boldsymbol{B}$*. Then:*

- $\langle \boldsymbol{A}, g^{-1}[G]\rangle \in \mathbf{MOD}(\mathrm{L})$*, whenever* $\langle \boldsymbol{B}, G\rangle \in \mathbf{MOD}(\mathrm{L})$*.*
- $\langle \boldsymbol{B}, g[F]\rangle \in \mathbf{MOD}(\mathrm{L})$*, whenever* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$ *and* g *is surjective and* $g(x) \in g[F]$ *implies* $x \in F$*.*

Logical filter: Given a logic L in $\mathcal{L}$ and an $\mathcal{L}$-algebra $\boldsymbol{A}$, a subset $F \subseteq A$ is an L-*filter* if $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$. By $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ we denote the set of all L-filters over $\boldsymbol{A}$.

$\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is a closure system and can be given a lattice structure by defining for any $F, G \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, $F \wedge G = F \cap G$ and $F \vee G = \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(F \cup G)$.

Generated filter: Given a set $X \subseteq A$, the logical filter generated by X is $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X) = \bigcap\{F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A}) \mid X \subseteq F\}$.

$$\mathcal{F}i_{\mathrm{Dumb}}(\boldsymbol{A}) = \mathcal{P}(A) \qquad \mathcal{F}i_{\mathrm{AInc}}(\boldsymbol{A}) = \{\emptyset, A\} \qquad \mathcal{F}i_{\mathrm{Inc}}(\boldsymbol{A}) = \{A\}$$

Examples: classical logic CL and logic BCI

Exercise 2

1. Classical logic: Let $\boldsymbol{A}$ be a Boolean algebra. Then $\mathcal{F}i_{\mathrm{CL}}(\boldsymbol{A})$ is the class of lattice filters on $\boldsymbol{A}$, in particular for the two-valued Boolean algebra $\mathbf{2}$:

$$\mathcal{F}i_{\mathrm{CL}}(\mathbf{2}) = \{\{1\}, \{0, 1\}\}.$$

2. The logic BCI: By $\boldsymbol{M}$ we denote the $\mathcal{L}_{\mathrm{BCI}}$-algebra with domain $\{\bot, \top, t, f\}$ and:

$\to^{\boldsymbol{M}}$	$\top$	t	f	$\bot$
$\top$	$\top$	$\bot$	$\bot$	$\bot$
t	$\top$	t	f	$\bot$
f	$\top$	$\bot$	t	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$

Check that

$$\mathcal{F}i_{\mathrm{BCI}}(\boldsymbol{M}) = \{\{t, \top\}, \{t, f, \top\}, M\}.$$

The first completeness theorem

Proposition 1.13

For any logic L *in a language* $\mathcal{L}$*,* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{Fm}_{\mathcal{L}}) = \mathrm{Th}(\mathrm{L})$*.*

Theorem 1.14

Let L *be a logic. Then for each set* Γ *of formulae and each formula* φ *the following holds:* $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\Gamma \models_{\mathbf{MOD}(\mathrm{L})} \varphi$*.*

Completeness theorem for classical logic

- Suppose that $T \in \mathrm{Th}(\mathrm{CL})$ and $\varphi \notin T$ ($T \nvdash_{\mathrm{CL}} \varphi$). We want to show that $T \not\models \varphi$ in some meaningful semantics.
- $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}, T \rangle} \varphi$. 1st completeness theorem
- $\langle \alpha, \beta \rangle \in \Omega(T)$ iff $\alpha \leftrightarrow \beta \in T$ (congruence relation on $\boldsymbol{Fm}_{\mathcal{L}}$ compatible with T: if $\alpha \in T$ and $\langle \alpha, \beta \rangle \in \Omega(T)$, then $\beta \in T$).
- Lindenbaum–Tarski algebra: $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T)$ is a Boolean algebra and $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}/\Omega(T), T/\Omega(T) \rangle} \varphi$.
  2nd completeness theorem
- Lindenbaum Lemma: If $\varphi \notin T$, then there is a maximal consistent $T' \in \mathrm{Th}(\mathrm{CL})$ such that $T \subseteq T'$ and $\varphi \notin T'$.
- $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T') \cong \mathbf{2}$ (subdirectly irreducible Boolean algebra) and $T \not\models_{\langle \mathbf{2}, \{1\} \rangle} \varphi$. 3rd completeness theorem

Weakly implicative logics

Definition 1.15

A logic L in a language $\mathcal{L}$ is weakly implicative if there is a binary connective $\to$ (primitive or definable) such that:

(R)	$\vdash_{\mathrm{L}} \varphi \to \varphi$
(MP)	$\varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi$
(T)	$\varphi \to \psi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \to \chi$
(sCng)	$\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \chi_i, \varphi, \ldots, \chi_n) \to c(\chi_1, \ldots, \chi_i, \psi, \ldots, \chi_n)$
	for each $\langle c, n\rangle \in \mathcal{L}$ and each $0 \leq i < n$.

Examples

The following logics are weakly implicative:

- CL, BCI, and Inc
- global modal logics
- intuitionistic and superintuitionistic logic
- linear logic and its variants
- (the most of) fuzzy logics
- substructural logics

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The following logics are not weakly implicative:

- local modal logics — we will see why later today
- Dumb, AInc, and the conjunction-disjunction fragment of classical logic — as they have no theorems
- logics of ortholattices — lesson 3

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Congruence Property

Conventions

Unless said otherwise, L is a weakly implicative in a language $\mathcal{L}$ with an implication $\to$. We write:

- $\varphi \leftrightarrow \psi$ instead of $\{\varphi \to \psi, \psi \to \varphi\}$
- $\Gamma \vdash \Delta$ whenever $\Gamma \vdash \chi$ for each $\chi \in \Delta$
- $\Gamma \dashv\vdash \Delta$ whenever $\Gamma \vdash \Delta$ and $\Delta \vdash \Gamma$.

Theorem 1.16

Let φ, ψ, χ be formulae. Then:

- $\vdash_{\mathrm{L}} \varphi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \psi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \delta, \delta \leftrightarrow \psi \vdash_{\mathrm{L}} \varphi \leftrightarrow \psi$
- $\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \chi \leftrightarrow \hat{\chi}$, *where $\hat{\chi}$ is obtained from χ by replacing some occurrences of φ in χ by ψ.*

Corollaries

Corollary 1.17

Let $\to'$ *be a connective satisfying* (R), (MP), (T), (sCng). *Then*

$$\varphi \leftrightarrow \psi \dashv\vdash_{\mathrm{L}} \varphi \leftrightarrow' \psi.$$

Proof.

Consider formulas $\chi = \varphi \to' \varphi$ and $\hat{\chi} = \varphi \to' \psi$ and the proof

$$\varphi \leftrightarrow \psi, \dots, \ (\varphi \to' \varphi) \to (\varphi \to' \psi), \ \varphi \to' \varphi, \ \varphi \to' \psi.$$

Analogously for $\hat{\chi} = \psi \to' \varphi$ we can write

$$\varphi \leftrightarrow \psi, \dots, \ (\varphi \to' \varphi) \to (\psi \to' \varphi), \ \varphi \to' \varphi, \ \psi \to' \varphi. \qquad \square$$

So we have shown $\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \varphi \leftrightarrow' \psi$. The reverse direction is fully analogous.

Corollary 1.17

Let $\to'$ be a connective satisfying (R), (MP), (T), (sCng). *Then*

$$\varphi \leftrightarrow \psi \dashv\vdash_{\mathrm{L}} \varphi \leftrightarrow' \psi.$$

Corollary 1.18

Local modal logic T^l is not weakly implicative.

Proof.

Let $\to'$ be a 'good' implication in T^l. Then $\to'$ (along with classical implication $\to$) is an implication in global T^g. Thus $\top \to \varphi, \varphi \to \top \vdash_{\mathrm{T}^g} \top \leftrightarrow' \varphi$ and so $\Box^n(\top \to \varphi) \vdash_{\mathrm{T}^l} \top \leftrightarrow' \varphi$ for some n. Consider the proof in T^l: $\Box^n\varphi, \ \ldots, \ \Box^n(\top \to \varphi), \ \ldots, \ \top \leftrightarrow' \varphi, \ \ldots, \ \Box^{n+1}\top \leftrightarrow' \Box^{n+1}\varphi, \ \Box^{n+1}\top, \ \Box^{n+1}\varphi$, a contradiction. □