Mathematical Fuzzy Logic – 1st lesson

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Logic is the science that studies correct reasoning.

It is studied as part of Philosophy, Mathematics, and Computer Science.

From XIXth century, it has become a formal science that studies symbolic abstractions capturing the formal aspects of inference: symbolic logic or mathematical logic.

What is a correct reasoning?

Example 1.1

“If God exists, He must be good and ommipotent. If God was good and omnipotent, He would not allow human suffering. But, there is human suffering. Therefore, God does not exist."

Is this a correct reasoning?

What is a correct reasoning?

Formalization

Atomic parts:

p: God exists
q: God is good
r: God is ommipotent
s: There is human suffering

The form of the reasoning:

$$\frac{\begin{array}{l} p \to q \wedge r \\ \neg(q \wedge r \wedge s) \\ s \end{array}}{\neg p}$$

Is this a correct reasoning?

Classical logic

Syntax:

Formulae $\boldsymbol{Fm}_{\mathcal{L}}$ built from atoms combined by connectives $\mathcal{L} = \{\neg, \wedge, \vee, \to\}$.

Classical logic

Semantics:

Bivalence Principle

Every proposition is either true or false.

Definition 1.2

The Boolean algebra of two elements, $\mathbf{2}$, is defined over the universe $\{0, 1\}$ with the following operations:

$\neg^{\mathbf{2}}$	
0	1
1	0

$\wedge^{\mathbf{2}}$	0	1
0	0	0
1	0	1

$\vee^{\mathbf{2}}$	0	1
0	0	1
1	1	1

$\to^{\mathbf{2}}$	0	1
0	1	1
1	0	1

$\mathbf{2} = \langle \{0, 1\}, \neg^{\mathbf{2}}, \wedge^{\mathbf{2}}, \vee^{\mathbf{2}}, \to^{\mathbf{2}} \rangle$

Correct reasoning in classical logic

Definition 1.3

Given $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$ we say that φ is a logical consequence of Γ, denoted $\Gamma \models_{\mathbf{2}} \varphi$, iff for every $\mathbf{2}$-evaluation e such that $e(\gamma) = 1$ for every $\gamma \in \Gamma$, we have $e(\varphi) = 1$.

Correct reasoning = logical consequence

Definition 1.4

Given $\psi_1, \ldots, \psi_n, \varphi \in \boldsymbol{Fm}_{\mathcal{L}}$ we say that $\langle \psi_1, \ldots, \psi_n, \varphi \rangle$ is a correct reasoning if $\{\psi_1, \ldots, \psi_n\} \models_{\mathbf{2}} \varphi$. In this case, $\psi_1, \ldots, \psi_n$ are the premises of the reasoning and φ is the conclusion.

Correct reasoning in classical logic

Remark

$$\frac{\begin{array}{c}\psi_1 \\ \psi_2 \\ \vdots \\ \psi_n\end{array}}{\varphi}$$

is a correct reasoning iff there is no interpretation making the premises true and the conclusion false.

Correct reasoning in classical logic

Example 1.5

Modus ponens:

$$\frac{\begin{array}{l} p \to q \\ p \end{array}}{q}$$

It is a correct reasoning (if $e(p \to q) = e(p) = 1$, then $e(q) = 1$).

Example 1.6

Abduction:

$$\frac{\begin{array}{l} p \to q \\ q \end{array}}{p}$$

It is NOT a correct reasoning (take: $e(p) = 0$ and $e(q) = 1$).

Correct reasoning in classical logic

Example 1.7

$p \to q \wedge r$

$\neg(q \wedge r \wedge s)$

s

$\neg p$

Assume $e(p \to q \wedge r) = e(\neg(q \wedge r \wedge s)) = e(s) = 1$. Then $e(q \wedge r \wedge s) = 0$, so $e(q \wedge r) = 0$. But, since $e(p \to q \wedge r) = 1$, we must have $e(p) = 0$, and therefore: $e(\neg p) = 1$.

It is a correct reasoning!

BUT, is this really a proof that God does not exist?

NO. We only know that if the premisses were true, then the conclusion would be true as well.

Is classical logic enough?

Because of the Bivalence Principle, in classical logic every predicate yields a perfect division between those objects it applies to, and those it does not. We call them *crisp*.

Examples: prime number, even number, monotonic function, continuous function, divisible group, ... (any mathematical predicate)

Therefore, classical logic is especially designed to capture the notion of correct reasoning in Mathematics.

Sorites paradox [Eubulides of Miletus, IV century BC]

A man who has no money is poor. If a poor man earns one euro, he remains poor. Therefore, a man who has one million euros is poor.

Formalization:

p_n: *A man who has exactly* n *euros is poor*

$$\begin{array}{l} p_0 \\ p_0 \to p_1 \\ p_1 \to p_2 \\ p_2 \to p_3 \\ \vdots \\ p_{999999} \to p_{1000000} \\ \hline p_{1000000} \end{array}$$

Sorites paradox [Eubulides of Miletus, IV century BC]

- There is no doubt that the premise p_0 is true.
- There is no doubt that the conclusion $p_{1000000}$ is false.
- For each i, the premise $p_i \to p_{i+1}$ seems to be true.
- The reasoning is logically correct (application of *modus ponens* one million times).
- We have a paradox!

Vagueness

The predicates that generate this kind of paradoxes are called *vague*.

Remark

A predicate is vague iff it has borderline cases, i.e. there are objects for which we cannot tell whether they fall under the scope of the predicate.

Example: Consider the predicate *tall*. Is a man measuring 1.78 meters tall?

Vagueness

- It is not a problem of ambiguity. Once we fix an unambiguous context, the problem remains.
- It is not a problem of uncertainty. Uncertainty typically appears when some relevant information is not known. Even if we assume that all relevant information is known, the problem remains.
- It cannot be solved by establishing a crisp definition of the predicate. The problem is: with the meaning that the predicate *tall* has in the natural language, whatever it might be, is a man measuring 1.78 meters tall?

Solutions in Analytical Philosophy

(1) Nihilist solution: *Vague predicates have no meaning*. If they would have, sorites paradox would lead to a contradiction.

(2) Epistemicist solution: *Vagueness is a problem of ignorance*. All predicates are crisp, but our epistemological constitution makes us unable to know the exact extension of a vague predicate. Some premise $p_i \to p_{i+1}$ is false.

(3) Supervaluationist solution: The meaning of vague predicate is the set of its precisifications (possible ways to make it crisp). *Truth is supertruth*, i.e. true under all precisifications. Some premise $p_i \to p_{i+1}$ is false.

Solutions in Analytical Philosophy

(4) Pragmatist solution: *Vague predicates do not have a univocal meaning*. A vague language is a set of crisp languages. For every utterance of a sentence involving a vague predicate, pragmatical conventions endow it with some particular crisp meaning. Some premise $p_i \to p_{i+1}$ is false.

(5) Degree-based solution: *Truth comes in degrees*. p_0 is completely true and $p_{1000000}$ is completely false. The premises $p_i \to p_{i+1}$ are very true, but not completely.

Many-valued logics to deal with vagueness

1949 Sören Halldén in *The Logic of Nonsense* proposes a three-valued logic to model vague predicates.

1955 Stephan Körner proposes an alternative three-valued treatment.

1965 Lotfi Zadeh proposes Fuzzy Set Theory (FST) as a mathematical treatment of vagueness and imprecision. FST becomes an extremely popular paradigm for engineering applications, known also as *Fuzzy Logic*.

1969 Goguen shows how to combine Zadeh's fuzzy sets and Łukasiewicz logic to solve sorites paradox.

Fuzzy sets as a model for vague predicates

Formally a fuzzy set is a pair $\langle X, \mu\rangle$ where X is a classical set and $\mu\colon X \to [0,1]$ is a function (called *membership function*) that maps every object $x \in X$ to its membership degree $\mu(x) \in [0,1]$.

Example: For the predicate *tall* take $X := [0.3, 2.4]$ (containing all possible heights) and

$$\mu(x) = \begin{cases} 0 & \text{if } x \leq 1.2, \\ \frac{5}{3}x - 2 & \text{if } 1.2 \leq x \leq 1.8, \\ 1 & \text{if } x \geq 1.8. \end{cases}$$

Changing the semantics

Take the infinite set of truth values $[0, 1]$, but let us try to keep the classical interpretation of connectives as much as possible.

If $a, b \in [0, 1]$, we define:

$$a \wedge b = \min\{a, b\}$$

$$a \vee b = \max\{a, b\}$$

$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

$$\neg a = a \to 0 = \begin{cases} 1 & \text{if } a = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We denote the resulting algebra as $[0, 1]_{\mathrm{G}}$.

Changing the semantics

Definition 1.8

A $[0,1]_{\mathrm{G}}$-evaluation is a homomorphism e from $\boldsymbol{Fm}_{\mathcal{L}}$ to $[0,1]_{\mathrm{G}}$; i.e.:

- $e(\neg\varphi) = \neg^{[0,1]_{\mathrm{G}}} e(\varphi)$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{[0,1]_{\mathrm{G}}} e(\psi)$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{[0,1]_{\mathrm{G}}} e(\psi)$
- $e(\varphi \to \psi) = e(\varphi) \to^{[0,1]_{\mathrm{G}}} e(\psi)$

Definition 1.9

Given $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$ we say that φ is a logical consequence of Γ, denoted $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$, iff for every $[0,1]_{\mathrm{G}}$-evaluation e such that $e(\gamma) = 1$ for every $\gamma \in \Gamma$, we have $e(\varphi) = 1$.

Changing the semantics

Some classical properties fail in $[0,1]_{\mathrm{G}}$:

- $\neg\neg\varphi \to \varphi \quad \neg\neg\frac{1}{2} \to \frac{1}{2} = 1 \to \frac{1}{2} = \frac{1}{2}$
- $\varphi \vee \neg\varphi \quad \frac{1}{2} \vee \neg\frac{1}{2} = \frac{1}{2}$
- $\neg(\neg\varphi \wedge \neg\psi) \to \varphi \vee \psi \quad \neg(\neg\frac{1}{2} \wedge \neg\frac{1}{2}) \to \frac{1}{2} \vee \frac{1}{2} = 1 \to \frac{1}{2} = \frac{1}{2}$
- $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$
  $((\frac{1}{2} \to 0) \to 0) \to ((0 \to \frac{1}{2}) \to \frac{1}{2}) = 1 \to \frac{1}{2} = \frac{1}{2}$

A proof system for classical logic

Axioms:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \to (\psi \to \varphi)$
(A3)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(A4)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A5a)	$\varphi \wedge \psi \to \varphi$
(A5b)	$\varphi \wedge \psi \to \psi$
(A5c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
(A6a)	$\varphi \to \varphi \vee \psi$
(A6b)	$\psi \to \varphi \vee \psi$
(A6c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
(A7a)	$\neg\varphi \to (\varphi \to \psi)$
(A7b)	$(\varphi \to \neg\psi) \to (\psi \to \neg\varphi))$
(A8)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$
(A9)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

Inference rule: *modus ponens* from $\varphi \to \psi$ and φ infer ψ.

A proof system for classical logic

Proof: a proof of a formula φ from a set of formulae Γ is a finite sequence of formulae $\langle \psi_1, \ldots, \psi_n \rangle$ such that:

- $\psi_n = \varphi$
- for every $i < n$, either $\psi_i \in \Gamma$, or ψ_i is an instance of an axiom, or there are $j, k < i$ such that $\psi_k = \psi_j \to \psi_i$.

We write $\Gamma \vdash_{\mathrm{CL}} \varphi$ if there is a proof of φ from Γ.

The proof system is finitary: if $\Gamma \vdash_{\mathrm{CL}} \varphi$, then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathrm{CL}} \varphi$.

Completeness theorem for classical logic

Theorem 1.10

For every set of formulae $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$ *we have:*

$$\Gamma \vdash_{\mathrm{CL}} \varphi \text{ if, and only if, } \Gamma \models_{\mathbf{2}} \varphi.$$

Exercise 1

Prove it.

A proof system for Gödel–Dummett logic

Axioms:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \to (\psi \to \varphi)$
(A3)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(A4)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A5a)	$\varphi \wedge \psi \to \varphi$
(A5b)	$\varphi \wedge \psi \to \psi$
(A5c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
(A6a)	$\varphi \to \varphi \vee \psi$
(A6b)	$\psi \to \varphi \vee \psi$
(A6c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
(A7a)	$\neg\varphi \to (\varphi \to \psi)$
(A7b)	$(\varphi \to \neg\psi) \to (\psi \to \neg\varphi))$
(A8)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$

Inference rule: *modus ponens*.

We write $\Gamma \vdash_{\mathrm{G}} \varphi$ if there is a proof of φ from Γ.

Completeness theorem for Gödel–Dummett logic

Theorem 1.11

For every set of formulae $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$ *we have:*

$$\Gamma \vdash_{\mathrm{G}} \varphi \text{ if, and only if, } \Gamma \models_{[0,1]_{\mathrm{G}}} \varphi.$$

Exercise 2

(a) Prove the implication from left to right (soundness of the proof system for Gödel–Dummett logic).

(Global) Deduction Theorem for Gödel–Dummett logic

Theorem 1.12

For every set of formulae $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_{\mathrm{G}} \psi \text{ iff } \Gamma \vdash_{\mathrm{G}} \varphi \to \psi$$

A solution to *sorites* paradox?

- $X = \{0, 1, 2, \dots, 10^6\}$, $\mu : X \to [0, 1]$.
- The truth value of p_n will be $\mu(n)$.
- $\mu(0) = 1$ and $\mu(10^6) = 0$ (the first premise is completely true, the conclusion is completely false).
- Take $\varepsilon = 10^{-6}$ and $\mu(n) = 1 - n\varepsilon$.
- Then the value of $p_n \to p_{n+1}$ is $\mu(n) \to \mu(n+1) = \mu(n+1) = 1 - n\varepsilon$. It tends to 0 as well!

This semantics does not give a good interpretation of the *sorites* paradox, because it does not explain why the premises are seemingly true.

Changing the semantics again

Take the infinite set of truth values $[0, 1]$, but let us try to keep the classical interpretation of connectives as much as possible. If $a, b \in [0, 1]$, we define:

$$a \wedge b = \max\{a, b\}$$

$$a \vee b = \max\{a, b\}$$

$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ 1 - a + b & \text{otherwise.} \end{cases}$$

$$\neg a = a \to 0 = 1 - a$$

We denote the resulting algebra as $[0, 1]_{Ł}$.

Changing the semantics again

Some classical properties fail in $[0,1]_{Ł}$:

- $\varphi \vee \neg\varphi \quad \frac{1}{2} \vee \neg\frac{1}{2} = \frac{1}{2}$
- $(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$
  $(\frac{1}{2} \to (\frac{1}{2} \to 0)) \to (\frac{1}{2} \to 0) = 1 \to \frac{1}{2} = \frac{1}{2}$

BUT other classical properties hold, e.g.:

- $\neg\neg\varphi \to \varphi$
- $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$
- all De Morgan laws involving $\neg, \vee, \wedge$

Changing the semantics again

Definition 1.13

A $[0,1]_{Ł}$-evaluation is a homomorphism e from $\boldsymbol{Fm}_{\mathcal{L}}$ to $[0,1]_{Ł}$; i.e.:

- $e(\neg\varphi) = \neg^{[0,1]_{Ł}} e(\varphi)$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{[0,1]_{Ł}} e(\psi)$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{[0,1]_{Ł}} e(\psi)$
- $e(\varphi \to \psi) = e(\varphi) \to^{[0,1]_{Ł}} e(\psi)$

Definition 1.14

Given $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$ we say that φ is a logical consequence of Γ, denoted $\Gamma \models_{[0,1]_{Ł}} \varphi$, iff for every $[0,1]_{Ł}$-evaluation e such that $e(\gamma) = 1$ for every $\gamma \in \Gamma$, we have $e(\varphi) = 1$.

Fuzzy Logic solution to sorites paradox

- $X = \{0, 1, 2, \ldots, 10^6\}$, $\mu : X \to [0, 1]$.
- The truth value of p_n will be $\mu(n)$.
- $\mu(0) = 1$ and $\mu(10^6) = 0$ (the first premise is completely true, the conclusion is completely false).
- Take $\varepsilon = 10^{-6}$ and $\mu(n) = 1 - n\varepsilon$.
- Compute the value of $p_n \to p_{n+1}$ by means of Łukasiewicz implication.
- $\mu(n) \to \mu(n+1) = 1 - \mu(n) + \mu(n+1) = 1 - (1 - n\varepsilon) + (1 - (n+1)\varepsilon) = 1 - \varepsilon < 1$ (all these premises have the same truth value: almost completely true).

A proof system for Łukasiewicz logic

Axioms:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \to (\psi \to \varphi)$
(A3)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(A4)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A5a)	$\varphi \wedge \psi \to \varphi$
(A5b)	$\varphi \wedge \psi \to \psi$
(A5c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
(A6a)	$\varphi \to \varphi \vee \psi$
(A6b)	$\psi \to \varphi \vee \psi$
(A6c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
(A7a)	$\neg\varphi \to (\varphi \to \psi)$
(A7b)	$(\varphi \to \neg\psi) \to (\psi \to \neg\varphi))$
(A9)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

Inference rule: *modus ponens*.

We write $\Gamma \vdash_{Ł} \varphi$ if there is a proof of φ from Γ.

Completeness theorem for Łukasiewicz logic

Theorem 1.15

For every finite set of formulae $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$ *we have:*

$$\Gamma \vdash_{\text{Ł}} \varphi \text{ if, and only if, } \Gamma \models_{[0,1]_{\text{Ł}}} \varphi.$$

Exercise 2

(b) Prove the implication from left to right (soundness of the proof system for Łukasiewicz logic).

Splitting of conjunction properties

In classical logic one can define conjunction in different ways:

$$\varphi \wedge \psi \quad \equiv_{\mathrm{CL}} \quad \neg(\varphi \to \neg\psi) \quad \equiv_{\mathrm{CL}} \quad \neg((\psi \to \varphi) \to \neg\psi)$$

In Ł:

$$\neg(\tfrac{1}{2} \to \neg\tfrac{1}{2}) = 0 \qquad \neg((\tfrac{1}{2} \to \tfrac{1}{2}) \to \neg\tfrac{1}{2}) = \tfrac{1}{2}$$

Thus we define *two* different conjunctions:

$$\varphi \mathbin{\&} \psi = \neg(\varphi \to \neg\psi) \qquad e(\varphi \mathbin{\&} \psi) = \max\{0, e(\varphi) + e(\psi) - 1\}$$

$$\varphi \wedge \psi = \neg((\psi \to \varphi) \to \neg\psi) \qquad e(\varphi \wedge \psi) = \min\{e(\varphi), e(\psi)\}$$

Exercise 2

(c) Check the interpretations of both conjunctions.

Splitting of conjunction properties

The two conjunctions play two different algebraic roles:

1. $a \mathbin{\&} b \leq c$ iff $b \leq a \to c$ (residuation)
2. $a \to b = 1$ iff $a \wedge b = a$ iff $a \leq b$ ($\wedge = \min$)

Failure of strong completeness

- $\varphi \oplus \psi := \neg(\neg\varphi \,\&\, \neg\psi)$
- $a \oplus b = \min\{a + b, 1\}$
- $\Sigma = \{p \oplus \overset{n}{\ldots} \oplus p \to q \mid n \geq 1\} \cup \{\neg p \to q\}$
- $\Sigma \models_{[0,1]_{Ł}} q$
- For every finite $\Sigma_0 \subseteq \Sigma$, $\Sigma_0 \not\models_{[0,1]_{Ł}} q$.
- $\Sigma \nvdash_{Ł} q$

Local Deduction Theorem for Łukasiewicz logic

Theorem 1.16

For every set of formulae $\Gamma \cup \{\varphi, \psi\}$*, there is* $n \geq 1$ *such that:*

$$\Gamma, \varphi \vdash_{Ł} \psi \textit{ iff } \Gamma \vdash_{Ł} \varphi \,\&\, \ldots^{n} \,\&\, \varphi \to \psi$$

1913 L.E.J. Brouwer proposes intuitionism as a new (genuine) form of mathematics.

1920 Jan Łukasiewicz publishes the first work ever on many-valued logic (a three-valued logic to deal with future contingents).

1922 He generalizes it to an n-valued logic for each $n \geq 3$.

1928 Heyting considers the logic behind intuitionism and endowes it with a Hilbert-style calculus.

1930 Together with Alfred Tarski, Łukasiewicz generalizes his logics to a $[0, 1]$-valued logic. They also provide a Hilbert-style calculus with 5 axioms and *modus ponens* and conjecture that it is complete w.r.t. the infinitely-valued logic.

1932 Kurt Gödel studies an infinite family of finite linearly ordered matrices for intuitionistic logic. They are not a complete semantics.

1934 Gentzen introduces natural deduction and sequent calculus for intuitionistic logic.

1935 Mordchaj Wajsberg claims to have proved Łukasiewicz's conjecture, but he never shows the proof.

1937 Tarski and Stone develop topological interpretations of intuitionistic logic.

1958 Rose and Rosser publish a proof of completeness of Łukasiewicz logic based on syntactical methods.

1959 Meredith shows that the fifth axiom of Łukasiewicz logic is redundant.

1959 Chang publishes a proof of completeness of Łukasiewicz logic based on algebraic methods.

1959 Michael Dummett resumes Gödel's work from 1932 and proposes a denumerable linearly ordered matrix for intuitionism. He gives a correct and complete Hilbert-style calculus for this matrix which turns out to be an axiomatic extension of intuitionism: Gödel-Dummett logic.

1963 Hay shows the finite strong completeness of Łukasiewicz logic.

1965 Saul Kripke introduces his relational semantics for intuitionistic logic.

1965 Lotfi Zadeh proposes Fuzzy Set Theory (FST) as a mathematical treatment of vagueness and imprecision. FST becomes an extremely popular paradigm for engineering applications, known also as *Fuzzy Logic*.

1969 Goguen shows how to combine Zadeh's fuzzy sets and Łukasiewicz logic to solve some vagueness logical paradoxes.