General Theories of Logical Systems
2nd lesson

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Completeness theorem for classical logic

- Suppose that $T \in \mathrm{Th}(\mathrm{CL})$ and $\varphi \notin T$ ($T \nvdash_{\mathrm{CL}} \varphi$). We want to show that $T \not\models \varphi$ in some meaningful semantics.
- $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle} \varphi$. 1st completeness theorem
- $\langle \alpha, \beta\rangle \in \Omega(T)$ iff $\alpha \leftrightarrow \beta \in T$ (congruence relation on $\boldsymbol{Fm}_{\mathcal{L}}$ compatible with T: if $\alpha \in T$ and $\langle \alpha, \beta\rangle \in \Omega(T)$, then $\beta \in T$).
- Lindenbaum-Tarski algebra: $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T)$ is a Boolean algebra and $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}/\Omega(T), T/\Omega(T)\rangle} \varphi$. 2nd completeness theorem
- Lindenbaum Lemma: If $\varphi \notin T$, then there is a maximal consistent $T' \in \mathrm{Th}(\mathrm{CL})$ such that $T \subseteq T'$ and $\varphi \notin T'$.
- $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T') \cong \mathbf{2}$ (subdirectly irreducible Boolean algebra) and $T \not\models_{\langle \mathbf{2}, \{1\}\rangle} \varphi$. 3rd completeness theorem

Leibniz congruence – 1

Definition 2.1

Let $\mathbf{A} = \langle A, F\rangle$ be an L-matrix. We define:

- the matrix preorder $\leq_{\mathbf{A}}$ of $\mathbf{A}$ as

$$a \leq_{\mathbf{A}} b \qquad \text{iff} \qquad a \to^{\boldsymbol{A}} b \in F$$

- the Leibniz congruence $\Omega_{\boldsymbol{A}}(F)$ of $\mathbf{A}$ as

$$\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F) \qquad \text{iff} \qquad a \leq_{\mathbf{A}} b \ \text{ and } \ b \leq_{\mathbf{A}} a.$$

A congruence θ of $\boldsymbol{A}$ is logical in a matrix $\langle \boldsymbol{A}, F\rangle$ if for each $a, b \in A$ if $a \in F$ and $\langle a, b\rangle \in \theta$, then $b \in F$.

Leibniz congruence – 2

Theorem 2.2

Let $\mathbf{A} = \langle A, F \rangle$ *be an* L*-matrix. Then:*

1. $\leq_{\mathbf{A}}$ *is a preorder.*
2. $\Omega_{\mathbf{A}}(F)$ *is the largest logical congruence of* $\mathbf{A}$*.*
3. $\langle a, b \rangle \in \Omega_{\mathbf{A}}(F)$ *iff for each* $\chi \in Fm_{\mathcal{L}}$ *and each* $\mathbf{A}$*-evaluation* e*:*

$$e[p{\to}a](\chi) \in F \qquad \textit{iff} \qquad e[p{\to}b](\chi) \in F.$$

Proof.

1. Take A-evaluation e such that $e(p) = a$, $e(q) = b$, and $e(r) = c$. Recall that in L we have: $\vdash_{\mathrm{L}} p \to p$ and $p \to q, q \to r \vdash_{\mathrm{L}} p \to r$. As $\mathbf{A} = \mathbf{MOD}(\mathrm{L})$ we have: $e(p \to p) \in F$, i.e., $a \leq_{\mathbf{A}} a$ and if $e(p \to q), e(q \to r) \in F$, then $e(p \to r) \in F$ i.e., if $a \leq_{\mathbf{A}} b$ and $b \leq_{\mathbf{A}} c$, then $a \leq_{\mathbf{A}} c$.

Leibniz congruence – 2

Theorem 2.2

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3. $\langle a, b \rangle \in \Omega_{\mathbf{A}}(F)$ *iff for each* $\chi \in Fm_{\mathcal{L}}$ *and each* $\mathbf{A}$*-evaluation* e*:*

$$e[p{\to}a](\chi) \in F \qquad \text{iff} \qquad e[p{\to}b](\chi) \in F.$$

Proof.

2. $\Omega_{\mathbf{A}}(F)$ is obviously an equivalence relation. It is a congruence due to (sCng) and logical due to (MP).
Take a logical congruence θ and $\langle a, b \rangle \in \theta$. Since $\langle a, a \rangle \in \theta$, we have $\langle a \to^{\mathbf{A}} a, a \to^{\mathbf{A}} b \rangle \in \theta$. As $a \to^{\mathbf{A}} a \in F$ and θ is logical we get $a \to^{\mathbf{A}} b \in F$, i.e., $a \leq_{\mathbf{A}} b$. The proof of $b \leq_{\mathbf{A}} a$ is analogous.

Leibniz congruence – 2

Theorem 2.2

Let $\mathbf{A} = \langle A, F\rangle$ *be an* L*-matrix. Then:*

1. $\leq_{\mathbf{A}}$ *is a preorder.*
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3. $\langle a, b\rangle \in \Omega_{\mathbf{A}}(F)$ *iff for each* $\chi \in Fm_{\mathcal{L}}$ *and each* $\mathbf{A}$*-evaluation* e*:*

$$e[p{\to}a](\chi) \in F \qquad \text{iff} \qquad e[p{\to}b](\chi) \in F.$$

Proof.

3. One direction is a corollary of Theorem 1.16 and (MP).
The converse one: set $\chi = p \to q$ and $e(q) = b$: then $a \to^{\mathbf{A}} b \in F$ iff $b \to^{\mathbf{A}} b \in F$, thus $a \leq_{\mathbf{A}} b$. The proof of $b \leq_{\mathbf{A}} a$ is analogous (using $e(q) = a$). □

Algebraic counterpart

Definition 2.3

An L-matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ is reduced, $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ in symbols, if $\Omega_{\boldsymbol{A}}(F)$ is the identity relation $\mathrm{Id}_{\boldsymbol{A}}$.

An algebra $\boldsymbol{A}$ is L-algebra, $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})$ in symbols, if there a set $F \subseteq A$ such that $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$.

Note that $\Omega_{\boldsymbol{A}}(A) = A^2$. Thus from $\mathcal{F}i_{\mathrm{Inc}}(\boldsymbol{A}) = \{A\}$ we obtain:

$$\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{Inc}) \qquad \text{iff} \qquad A \text{ is a singleton}$$

Exercise 3

Classical logic: prove that for any Boolean algebra $\boldsymbol{A}$:

$$\Omega_{\boldsymbol{A}}(\{1\}) = \mathrm{Id}_{\boldsymbol{A}} \qquad \text{i.e., } \boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{CL}).$$

On the other hand, show that:

$$\Omega_{\mathbf{4}}(\{a,1\}) = \mathrm{Id}_{\boldsymbol{A}} \cup \{\langle 1,a\rangle, \langle 0,\neg a\rangle\} \text{ i.e. } \langle \mathbf{4}, \{a,1\}\rangle \notin \mathbf{MOD}^*(\mathrm{CL}).$$

BCI: recall the algebra $\boldsymbol{M}$ defined via:

$\to^{\boldsymbol{M}}$	$\top$	t	f	$\bot$
$\top$	$\top$	$\bot$	$\bot$	$\bot$
t	$\top$	t	f	$\bot$
f	$\top$	$\bot$	t	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$

Show that:

$$\Omega_{\boldsymbol{M}}(\{t,\top\}) = \Omega_{\boldsymbol{M}}(\{t,f,\top\}) = \mathrm{Id}_{\boldsymbol{M}} \qquad \text{i.e.} \quad \boldsymbol{M} \in \mathbf{ALG}^*(\mathrm{BCI}).$$

Factorizing matrices – 1

Let us take $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$. We write:

- $\boldsymbol{A}^*$ for $\boldsymbol{A}/\Omega_{\boldsymbol{A}}(F)$
- $[\cdot]_F$ for the canonical epimorphism of $\boldsymbol{A}$ onto $\boldsymbol{A}^*$ defined as:

$$[a]_F = \{b \in A \mid \langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)\}$$

- $\mathbf{A}^*$ for $\langle \boldsymbol{A}^*, [F]_F\rangle$.

Lemma 2.4

Let $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$ *and* $a, b \in A$. *Then:*

1. $a \in F$ *iff* $[a]_F \in [F]_F$.
2. $\mathbf{A}^* \in \mathbf{MOD}(\mathrm{L})$.
3. $[a]_F \leq_{\mathbf{A}^*} [b]_F$ *iff* $a \to^{\boldsymbol{A}} b \in F$.
4. $\mathbf{A}^* \in \mathbf{MOD}^*(\mathrm{L})$.

Factorizing matrices – 2

Proof.

1. One direction is trivial. Conversely: $[a]_F \in [F]_F$ implies that $[a]_F = [b]_F$ for some $b \in F$; thus $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$ and, since $\Omega_{\boldsymbol{A}}(F)$ is a logical congruence, we obtain $a \in F$.
2. Recall that the second claim of Lemma 1.12 says that for a surjective $g\colon \boldsymbol{A} \to \boldsymbol{B}$ and $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ we get $g[F] \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{B})$, whenever $g(x) \in g[F]$ implies $x \in F$.
3. $[a]_F \leq_{\mathbf{A}^*} [b]_F$ iff $[a]_F \to^{\boldsymbol{A}^*} [b]_F \in [F]_F$ iff $[a \to^{\boldsymbol{A}} b]_F \in [F]_F$ iff $a \to^{\boldsymbol{A}} b \in F$.
4. Assume that $\langle [a]_F, [b]_F\rangle \in \Omega_{\boldsymbol{A}^*}([F]_F)$, i.e., $[a]_F \leq_{\mathbf{A}^*} [b]_F$ and $[b]_F \leq_{\mathbf{A}^*} [a]_F$. Therefore $a \to^{\boldsymbol{A}} b \in F$ and $b \to^{\boldsymbol{A}} a \in F$, i.e., $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$. Thus $[a]_F = [b]_F$. □

Lindenbaum–Tarski matrix

Let L be a weakly implicative logic in $\mathcal{L}$ and $T \in Th(\mathrm{L})$. For every formula φ, we define the set

$$[\varphi]_T = \{\psi \in Fm_{\mathcal{L}} \mid \varphi \leftrightarrow \psi \subseteq T\}.$$

The Lindenbaum–Tarski matrix with respect to L and T, $\mathbf{LindT}_T$, has the filter $\{[\varphi]_T \mid \varphi \in T\}$ and algebraic reduct with the domain $\{[\varphi]_T \mid \varphi \in Fm_{\mathcal{L}}\}$ and operations:

$$c^{\mathbf{LindT}_T}([\varphi_1]_T, \ldots, [\varphi_n]_T) = [c(\varphi_1, \ldots, \varphi_n)]_T$$

Clearly, for every $T \in \mathrm{Th}(\mathrm{L})$ we have:

$$\mathbf{LindT}_T = \langle \boldsymbol{Fm}_{\mathcal{L}}, T \rangle^*.$$

The second completeness theorem

Theorem 2.5

Let L *be a weakly implicative logic. Then for any set* Γ *of formulae and any formula* φ *the following holds:*

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi.$$

Proof.

Using just the soundness part of the first completeness theorem it remains to prove:

$$\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi \quad \text{implies} \quad \Gamma \vdash_{\mathrm{L}} \varphi.$$

Take Lindenbaum–Tarski matrix $\mathbf{LindT}_{\mathrm{Th_L}(\Gamma)} = \langle \boldsymbol{Fm}_{\mathcal{L}}, \mathrm{Th_L}(\Gamma)\rangle^*$ and evaluation $e(\psi) = [\psi]_{\mathrm{Th_L}(\Gamma)}$. As clearly $e[\Gamma] \subseteq e[\mathrm{Th_L}(\Gamma)] = [\mathrm{Th_L}(\Gamma)]_{\mathrm{Th_L}(\Gamma)}$, then, as $\mathbf{LindT}_{\mathrm{Th_L}(\Gamma)}$ is an L-model, we have: $e(\varphi) = [\varphi]_{\mathrm{Th_L}(\Gamma)} \in [\mathrm{Th_L}(\Gamma)]_{\mathrm{Th_L}(\Gamma)}$, and so $\varphi \in \mathrm{Th_L}(\Gamma)$ i.e., $\Gamma \vdash_{\mathrm{L}} \varphi$. □

Closure systems and closure operators – 1

Closure system over a set A: a collection of subsets $\mathcal{C} \subseteq \mathcal{P}(A)$ closed under arbitrary intersections and such that $A \in \mathcal{C}$. The elements of $\mathcal{C}$ are called closed sets.

Closure operator over a set A: a mapping $C\colon \mathcal{P}(A) \to \mathcal{P}(A)$ such that for every $X, Y \subseteq A$:

1. $X \subseteq C(X)$,
2. $C(X) = C(C(X))$, and
3. if $X \subseteq Y$, then $C(X) \subseteq C(Y)$.

Exercise 4

If C is a closure operator, $\{X \subseteq A \mid C(X) = X\}$ is a closure system.

If $\mathcal{C}$ is closure system, $C(X) = \bigcap\{Y \in \mathcal{C} \mid X \subseteq Y\}$ is a closure operator.

Closure systems and closure operators – 2

A closure operator C is finitary if for every $X \subseteq A$, $C(X) = \bigcup\{C(Y) \mid Y \subseteq X \text{ and } Y \text{ is finite}\}$.

A closure system $\mathcal{C}$ is called inductive if it is closed under unions of upwards directed families (i.e. families $\mathcal{D} \neq \emptyset$ such that for every $A, B \in \mathcal{D}$, there is $C \in \mathcal{D}$ such that $A \cup B \subseteq C$).

Theorem 2.6 (Schmidt Theorem)

A closure operator C is finitary if, and only if, its associated closure system $\mathcal{C}$ is inductive.

Closure systems and closure operators – 3

Each logic L determines a closure system $\mathrm{Th}(\mathrm{L})$ and a closure operator Th_{L}.

Conversely, given a structural closure operator C over $Fm_{\mathcal{L}}$ (for every σ, if $\varphi \in C(\Gamma)$, then $\sigma(\varphi) \in C(\sigma[\Gamma])$), there is a logic L such that $C = \mathrm{Th}_{\mathrm{L}}$.

L is a finitary logic iff Th_{L} is a finitary closure operator.

The set of all L-filters over a given algebra $\boldsymbol{A}$, $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is a closure system over A. Its associated closure operator is $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}$.

Transfer theorem for finitarity

Corollary 2.7

Given a logic L *in a language* $\mathcal{L}$*, the following conditions are equivalent:*

1. L *is finitary.*
2. $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}$ *is a finitary closure operator for any* $\mathcal{L}$*-algebra* $\boldsymbol{A}$*.*
3. $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ *is an inductive closure system for any* $\mathcal{L}$*-algebra* $\boldsymbol{A}$*.*

Closure systems and closure operators – 4

A base of a closure system $\mathcal{C}$ over A is any $\mathcal{B} \subseteq \mathcal{C}$ satisfying one of the following equivalent conditions:

1. $\mathcal{C}$ is the coarsest closure system containing $\mathcal{B}$.
2. For every $T \in \mathcal{C} \setminus \{A\}$, there is a $\mathcal{D} \subseteq \mathcal{B}$ such that $T = \bigcap \mathcal{D}$.
3. For every $T \in \mathcal{C} \setminus \{A\}$, $T = \bigcap\{B \in \mathcal{B} \mid T \subseteq B\}$.
4. For every $Y \in \mathcal{C}$ and $a \in A \setminus Y$ there is $Z \in \mathcal{B}$ such that $Y \subseteq Z$ and $a \notin Z$.

Exercise 5

Show that the four definitions are equivalent.

An element X of a closure system $\mathcal{C}$ over A is called (finitely) $\cap$-irreducible if for each (finite non-empty) set $\mathcal{Y} \subseteq \mathcal{C}$ such that $X = \bigcap_{Y \in \mathcal{Y}} Y$, there is $Y \in \mathcal{Y}$ such that $X = Y$.

Abstract Lindenbaum Lemma

An element X of a closure system $\mathcal{C}$ over A is called maximal w.r.t. an element a if it is a maximal element of the set $\{Y \in \mathcal{C} \mid a \notin Y\}$ w.r.t. the order given by inclusion.

Proposition 2.8

Let $\mathcal{C}$ be a closure system over a set A and $T \in \mathcal{C}$. Then, T is maximal w.r.t. an element if, and only if, T is $\cap$-irreducible.

Lemma 2.9

Let C be a finitary closure operator and $\mathcal{C}$ its corresponding closure system. If $T \in \mathcal{C}$ and $a \notin T$, then there is $T' \in \mathcal{C}$ such that $T \subseteq T'$ and T' is maximal with respect to a. $\cap$-irreducible closed sets form a base.

Operations on matrices – 1

$\langle \boldsymbol{A}, F\rangle$: first-order structure in the equality-free predicate language with function symbols from $\mathcal{L}$ and a unique unary predicate symbol interpreted by F.

Submatrix: $\langle \boldsymbol{A}, F\rangle \subseteq \langle \boldsymbol{B}, G\rangle$ if $\boldsymbol{A} \subseteq \boldsymbol{B}$ and $F = A \cap G$. Operator: $\mathbf{S}(\langle \boldsymbol{A}, F\rangle)$ is the class of all subalgebras of $\langle \boldsymbol{A}, F\rangle$.
Homomorphic image: $\langle \boldsymbol{B}, G\rangle$ is a homomorphic image of $\langle \boldsymbol{A}, F\rangle$ if it exists $h\colon \boldsymbol{A} \to \boldsymbol{B}$ homomorphism of algebras such that $h[F] \subseteq G$. Operator $\mathbf{H}$.
Strict homomorphic image: $\langle \boldsymbol{B}, G\rangle$ is a strict homomorphic image of $\langle \boldsymbol{A}, F\rangle$ if it exists $h\colon \boldsymbol{A} \to \boldsymbol{B}$ homomorphism of algebras such that $h[F] \subseteq G$ and $h[A \setminus F] \subseteq B \setminus G$. Operator $\mathbf{H}_{\mathrm{S}}$.
Isomorphic image: Image by a bijective strict homomorphism. Operator $\mathbf{I}$.

Operations on matrices – 2

Direct product: Given matrices $\{\langle \boldsymbol{A}_i, F_i\rangle \mid i \in I\}$, their direct product is $\langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A} = \prod_{i\in I} \boldsymbol{A}_i$, $f^{\boldsymbol{A}}(a_1,\dots,a_n)(i) = f^{\boldsymbol{A}_i}(a_1(i),\dots,a_n(i))$. $F = \prod_{i\in I} F_i$. $\pi_j : \boldsymbol{A} \twoheadrightarrow \boldsymbol{A}_j$. Operator $\mathbf{P}$.

Exercise 6

Let L be a weakly implicative logic. Then:

1. $\mathbf{SP}(\mathbf{MOD}(\mathrm{L})) \subseteq \mathbf{MOD}(\mathrm{L})$.
2. $\mathbf{SP}(\mathbf{MOD}^*(\mathrm{L})) \subseteq \mathbf{MOD}^*(\mathrm{L})$.

Subdirect products and subdirect irreducibility

A matrix $\mathbf{A}$ is said to be representable as a subdirect product of the family of matrices $\{\mathbf{A}_i \mid i \in I\}$ if there is an embedding homomorphism α from $\mathbf{A}$ into the direct product $\prod_{i\in I} \mathbf{A}_i$ such that for every $i \in I$, the composition of α with the i-th projection, $\pi_i \circ \alpha$, is a surjective homomorphism. In this case, α is called a subdirect representation, and it is called finite if I is finite.

Operator $\mathbf{P}_{\mathrm{SD}}(\mathbb{K})$.

A matrix $\mathbf{A} \in \mathbb{K}$ is (finitely) subdirectly irreducible relative to $\mathbb{K}$ if for every (finite non-empty) subdirect representation α of $\mathbf{A}$ with a family $\{\mathbf{A}_i \mid i \in I\} \subseteq \mathbb{K}$ there is $i \in I$ such that $\pi_i \circ \alpha$ is an isomorphism. The class of all (finitely) subdirectly irreducible matrices relative to $\mathbb{K}$ is denoted as $\mathbb{K}_{\mathrm{R(F)SI}}$.

$\mathbb{K}_{\mathrm{RSI}} \subseteq \mathbb{K}_{\mathrm{RFSI}}$.

Characterization of RSI and RFSI reduced models

Theorem 2.10

Given a weakly implicative logic L *and* $\mathbf{A} = \langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}^*(\mathrm{L})$, *we have:*

1. $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ *iff* F *is* $\cap$*-irreducible in* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$.
2. $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ *iff* F *is finitely* $\cap$*-irreducible in* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$.

Subdirect representation

Theorem 2.11

If L *is a finitary weakly implicative logic, then*

$$\mathbf{MOD}^*(\mathrm{L}) = \mathbf{P}_{\mathrm{SD}}(\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}),$$

in particular every matrix in $\mathbf{MOD}^*(\mathrm{L})$ *is representable as a subdirect product of matrices in* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$.

The third completeness theorem

Theorem 2.12

Let L *be a finitary weakly implicative logic. Then*

$$\vdash_{\mathrm{L}} \; = \; \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}}.$$

Leibniz operator

Leibniz operator: the function giving for each $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ the Leibniz congruence $\Omega_{\boldsymbol{A}}(F)$.

Proposition 2.13

Let L *be a weakly implicative logic* L *and* $\boldsymbol{A}$ *an* $\mathcal{L}$*-algebra. Then*

1. $\Omega_{\boldsymbol{A}}$ *is monotone: if* $F \subseteq G$ *then* $\Omega_{\boldsymbol{A}}(F) \subseteq \Omega_{\boldsymbol{A}}(G)$.
2. $\Omega_{\boldsymbol{A}}$ *commutes with inverse images by homomorphisms: for every* $\mathcal{L}$*-algebra* $\boldsymbol{B}$*, homomorphism* $h\colon \boldsymbol{A} \to \boldsymbol{B}$*, and* $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{B})$*:*

$$\Omega_{\boldsymbol{A}}(h^{-1}[F]) = h^{-1}[\Omega_{\boldsymbol{B}}(F)] = \{\langle a, b\rangle \mid \langle h(a), h(b)\rangle \in \Omega_{\boldsymbol{B}}(F)\}.$$

3. $\Omega_{\boldsymbol{A}}[\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})] = \boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$.

$\boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$ is the set ordered by inclusion of congruences of $\boldsymbol{A}$ giving a quotient in $\mathbf{ALG}^*(\mathrm{L})$.

Example

Recall that for the algebra $\boldsymbol{M} \in \mathbf{ALG}^*(\mathrm{BCI})$ defined via:

$\to^{\boldsymbol{M}}$	$\top$	t	f	$\bot$
$\top$	$\top$	$\bot$	$\bot$	$\bot$
t	$\top$	t	f	$\bot$
f	$\top$	$\bot$	t	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$

we have

$$\Omega_{\boldsymbol{M}}(\{t, \top\}) = \Omega_{\boldsymbol{M}}(\{t, f, \top\}) = \mathrm{Id}_{\boldsymbol{M}} \qquad \text{i.e., } \Omega_{\boldsymbol{M}} \text{ is not injective}$$

Interesting equivalence

Theorem 2.14

Given any weakly implicative logic L*, TFAE:*

1. *For every* $\mathcal{L}$*-algebra* $\boldsymbol{A}$*, the Leibniz operator* $\Omega_{\boldsymbol{A}}$ *is a lattice isomorphism from* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ *to* $\boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{A})$*.*
2. *For every* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$*,* F *is the least* L*-filter on* $\boldsymbol{A}$*.*
3. *The Leibniz operator* $\Omega_{\boldsymbol{Fm}_{\mathcal{L}}}$ *is a lattice isomorphism from* $\mathrm{Th}(\mathrm{L})$ *to* $\boldsymbol{Con}_{\mathbf{ALG}^*(\mathrm{L})}(\boldsymbol{Fm}_{\mathcal{L}})$*.*
4. *There is a set of equations* $\mathcal{T}$ *in one variable such that*
   $$(\mathrm{Alg}) \quad p \dashv\vdash_{\mathrm{L}} \{\mu(p) \leftrightarrow \nu(p) \mid \mu \approx \nu \in \mathcal{T}\}.$$
5. *There is a set of equations* $\mathcal{T}$ *in one variable such that for each* $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$ *and each* $a \in A$ *holds:* $a \in F$ *if, and only if,* $\mu^{\boldsymbol{A}}(a) = \nu^{\boldsymbol{A}}(a)$ *for every* $\mu \approx \nu \in \mathcal{T}$*.*

In the last two items the sets $\mathcal{T}$ *can be taken the same.*

Algebraically implicative logics

Definition 2.15

We say that a logic L is algebraically implicative if it is weakly implicative and satisfies one of the equivalent conditions from the previous theorem.
In this case, $\mathbf{ALG}^*(\mathrm{L})$ is called an equivalent algebraic semantics for L and the set $\mathcal{T}$ is called a truth definition.

Example 2.16

In many cases, one equation is enough for the truth definition. For instance, in classical logic, intuitionism, t-norm based fuzzy logics, etc. the truth definition is $\{p \approx \overline{1}\}$. Linear logic is algebraically implicative with $\mathcal{T} = \{p \wedge \overline{1} \approx \overline{1}\}$.

Exercise 7

$\mathcal{L} = \{\neg, \to\}$. Algebra $\boldsymbol{A}$ with domain $\{0, \frac{1}{2}, 1\}$ and operations:

	$\neg$
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

$\to$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

$Ł_3 = \models_{\langle \boldsymbol{A}, \{1\}\rangle}$ [three-valued Łukasiewicz logic]

$J_3 = \models_{\langle \boldsymbol{A}, \{\frac{1}{2}, 1\}\rangle}$ [Da Costa, D'Ottaviano]

Defined connectives: $\overline{1} = p \to p$, $\diamond p = \neg p \to p$

$Ł_3$ and J_3 are both algebraically implicative with

L	$\mathbf{ALG}^*(\mathrm{L})$	$\mathcal{T}(p)$
$Ł_3$	$\mathbf{Q}(\boldsymbol{A})$	$\{p \approx \overline{1}\}$
J_3	$\mathbf{Q}(\boldsymbol{A})$	$\{\diamond p \approx \overline{1}\}$

Equational consequence

An equation in the language $\mathcal{L}$ is a formal expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in Fm_{\mathcal{L}}$.

We say that an equation $\varphi \approx \psi$ is a consequence of a set of equations Π w.r.t. a class $\mathbb{K}$ of $\mathcal{L}$-algebras if for each $\boldsymbol{A} \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e we have $e(\varphi) = e(\psi)$ whenever $e(\alpha) = e(\beta)$ for each $\alpha \approx \beta \in \Pi$; we denote it by $\Pi \models_{\mathbb{K}} \varphi \approx \psi$.

Proposition 2.17

Let L *be a weakly implicative logic and* $\Pi \cup \{\varphi \approx \psi\}$ *a set of equations. Then*

$$\Pi \models_{\mathbf{ALG}^*(\mathrm{L})} \varphi \approx \psi \quad \textit{iff} \quad \{\alpha \leftrightarrow \beta \mid \alpha \approx \beta \in \Pi\} \vdash_{\mathrm{L}} \varphi \leftrightarrow \psi.$$

Alternatively, using translation $\rho[\Pi] = \bigcup_{\alpha\approx\beta\in\Pi}(\alpha \leftrightarrow \beta)$*:*

$$\Pi \models_{\mathbf{ALG}^*(\mathrm{L})} \varphi \approx \psi \quad \textit{iff} \quad \rho[\Pi] \vdash_{\mathrm{L}} \rho(\varphi \approx \psi).$$

Characterizations of algebraically implicative logics

We have defined a translation ρ from (sets of) equations to sets of formulae using $\leftrightarrow$.
Analogously we define a translation τ from (sets of) formulae to sets of equations using the truth definition $\mathcal{T}$:

$$\tau[\Gamma] = \{\alpha(\varphi) \approx \beta(\varphi) \mid \varphi \in \Gamma \text{ and } \alpha \approx \beta \in \mathcal{T}\}$$

Theorem 2.18

Given any weakly implicative logic L*, TFAE:*

1. L *is algebraically implicative with the truth definition* $\mathcal{T}$*.*
2. *There is a set of equations* $\mathcal{T}$ *in one variable such that:*
   1. $\Pi \models_{\mathbf{ALG}^*(\mathrm{L})} \varphi \approx \psi$ *iff* $\rho[\Pi] \vdash_{\mathrm{L}} \rho(\varphi \approx \psi)$
   2. $p \dashv\vdash_{\mathrm{L}} \rho[\tau(p)]$
3. *There is a set of equations* $\mathcal{T}$ *in one variable such that:*
   1. $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\tau[\Gamma] \models_{\mathbf{ALG}^*(\mathrm{L})} \tau(\varphi)$
   2. $p \approx q \dashv\models_{\mathbf{ALG}^*(\mathrm{L})} \tau[\rho(p \approx q)]$

Finitary algebraically implicative logics and quasivarieties

A quasivariety is a class of algebras described by quasiequations, formal expressions of the form $\bigwedge_{i=1}^{n} \alpha_i \approx \beta_i \Rightarrow \varphi \approx \psi$, where $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \varphi, \psi \in Fm_{\mathcal{L}}$.

Proposition 2.19

If L *is a finitary algebraically implicative logic, then it has a finite truth definition and* $\mathbf{ALG}^*(\mathrm{L})$ *is a quasivariety.*

Definition 2.20

We say that a weakly implicative logic L is

- regularly implicative if:

  (Reg) $\varphi, \psi \vdash_{\mathrm{L}} \psi \to \varphi$.

- Rasiowa-implicative if:

  (W) $\varphi \vdash_{\mathrm{L}} \psi \to \varphi$.

Proposition 2.21

A weakly implicative logic L *is regularly implicative iff all the filters of the matrices in* $\mathbf{MOD}^*(\mathrm{L})$ *are singletons.*

Proposition 2.22

A regularly implicative logic L *is Rasiowa-implicative iff for each* $\mathbf{A} = \langle \boldsymbol{A}, \{t\} \rangle \in \mathbf{MOD}^*(\mathrm{L})$ *the element* t *is the maximum of* $\leq_{\mathbf{A}}$*.*

Proposition 2.23

Each Rasiowa-implicative logic is regularly implicative and each regularly implicative logic is algebraically implicative.

Examples

The following logics are Rasiowa-implicative:

- classical logic
- global modal logics
- intuitionistic and superintuitionistic logics
- many fuzzy logics (Łukasiewicz, Gödel-Dummett, product logics, HL, MTL, ...)
- substructural logics with weakening
- inconsistent logic
- ...

Example 2.24

- The equivalence fragment of classical logic is a regularly implicative but not Rasiowa-implicative logic.
- Linear logic is algebraically, but not regularly, implicative.
- The logic BCI is weakly, but not algebraically, implicative.