Mathematical Fuzzy Logic – 2nd lesson

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Outline

Syntax:

Formulae $\boldsymbol{Fm}_{\mathcal{L}}$ built from atoms combined by connectives $\mathcal{L} = \{\overline{0}, \wedge, \vee, \to\}$.

We also use defined connectives $\neg$, $\overline{1}$, and $\leftrightarrow$ defined as:

$$\neg\varphi = \varphi \to \overline{0} \qquad \overline{1} = \neg\overline{0} \qquad \varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$$

A proof system for Gödel–Dummett logic

Axioms:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \to (\psi \to \varphi)$
(A3)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(A4)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A5a)	$\varphi \wedge \psi \to \varphi$
(A5b)	$\varphi \wedge \psi \to \psi$
(A5c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
(A6a)	$\varphi \to \varphi \vee \psi$
(A6b)	$\psi \to \varphi \vee \psi$
(A6c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
(A7)	$\overline{0} \to \varphi$
(A8)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$

Inference rule: *modus ponens*.

We write $\Gamma \vdash_{\mathrm{G}} \varphi$ if there is a proof of φ from Γ.

Relation to intuitionistic logic

The intuitionistic logic has axioms:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \to (\psi \to \varphi)$
(A3)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(A5a)	$\varphi \wedge \psi \to \varphi$
(A5b)	$\varphi \wedge \psi \to \psi$
(A5c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
(A6a)	$\varphi \to \varphi \vee \psi$
(A6b)	$\psi \to \varphi \vee \psi$
(A6c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
(A7)	$\overline{0} \to \varphi$
(A8)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$

Inference rule: *modus ponens*.

Algebraic semantics

A *Heyting-algebra* is a structure $\boldsymbol{B} = (B, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}, \rightarrow^{\boldsymbol{B}}, \overline{0}^{\boldsymbol{B}}, \overline{1}^{\boldsymbol{B}})$ such that:

(1) $(B, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}, \overline{0}^{\boldsymbol{B}}, \overline{1}^{\boldsymbol{B}})$ is a bounded lattice,

(2) $z \leq x \rightarrow^{\boldsymbol{B}} y$ iff $x \wedge^{\boldsymbol{B}} z \leq y$, (residuation)

where $x \leq y$ is defined as $x \wedge y = x$

and can be shown equivalent to $x \rightarrow y = \overline{1}$.

We say $\boldsymbol{B}$ is

- Gödel algebra (or just G-algebra) whenever

$$(x \rightarrow y) \vee (y \rightarrow x) = \overline{1} \qquad \text{(prelinearity)}$$

- linearly ordered (or Heyting chain) if $\leq$ is a total order.

Note that each Heyting chain is G-algebra, so we also call it G-chain.

By $\mathbb{G}$ (or $\mathbb{G}_{lin}$ resp.) we denote the class of all G-algebras (G-chains resp.)

Standard semantics

Consider algebra $[0, 1]_\mathrm{G} = \langle [0, 1], \wedge^{[0,1]_\mathrm{G}}, \vee^{[0,1]_\mathrm{G}}, \to^{[0,1]_\mathrm{G}}, 0, 1\rangle$, where:

$$a \wedge^{[0,1]_\mathrm{G}} b = \min\{a, b\}$$

$$a \vee^{[0,1]_\mathrm{G}} b = \max\{a, b\}$$

$$a \to^{[0,1]_\mathrm{G}} b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

Exercise 3

(a) Prove that $[0, 1]_\mathrm{G}$ is the unique G-chain with the lattice reduct $\langle [0, 1], \min, \max, 0, 1\rangle$,

Semantical consequence

Definition 2.1

A $\boldsymbol{B}$-evaluation is a homomorphism e from $\boldsymbol{Fm}_{\mathcal{L}}$ to $\boldsymbol{B}$; i.e.:

- $e(\overline{0}) = \overline{0}^{\boldsymbol{B}}$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{\boldsymbol{B}} e(\psi)$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{\boldsymbol{B}} e(\psi)$
- $e(\varphi \to \psi) = e(\varphi) \to^{\boldsymbol{B}} e(\psi)$

Definition 2.2

Given $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$ and a class $\mathbb{K}$ of G-algebras, we say that φ is a logical consequence of Γ w.r.t. $\mathbb{K}$, denoted $\Gamma \models_{\mathbb{K}} \varphi$, iff for every $\boldsymbol{B} \in \mathbb{K}$ and every $\boldsymbol{B}$-evaluation e such that $e[\Gamma] \subseteq \{\overline{1}^{\boldsymbol{B}}\}$, we have $e(\varphi) = \overline{1}^{\boldsymbol{B}}$.

Completeness theorem for Gödel–Dummett logic

Theorem 2.3

The following are equivalent for every set of formulae $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{\mathrm{G}} \varphi$
2. $\Gamma \models_{\mathbb{G}} \varphi$
3. $\Gamma \models_{\mathbb{G}_{lin}} \varphi$
4. $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$

Completeness theorem for Gödel–Dummett logic

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2. $\Gamma \models_{\mathbb{G}} \varphi$
3. $\Gamma \models_{\mathbb{G}_{lin}} \varphi$
4. $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$

Exercise 4

(a) Prove the implications from top to bottom.

Some theorems of Gödel–Dummett logic

Proposition 2.4

(T1) $\varphi \to \varphi$

(T2) $(\varphi \to \psi) \vee (\psi \to \varphi)$

(T3) $\varphi \to (\psi \to \varphi \wedge \psi)$

(T4) $(\overline{1} \to \varphi) \leftrightarrow \varphi$

(T5) $(\overline{1} \leftrightarrow \varphi) \leftrightarrow \varphi$

(T6) $(\varphi \to \psi) \leftrightarrow (\varphi \wedge \psi \leftrightarrow \varphi)$

Proposition 2.5

$\varphi \wedge \psi \leftrightarrow \psi \wedge \varphi$

$\varphi \wedge (\psi \wedge \chi) \leftrightarrow (\varphi \wedge \psi) \wedge \chi$

$\varphi \wedge (\varphi \vee \psi) \leftrightarrow \varphi$

$\overline{1} \wedge \varphi \leftrightarrow \varphi$

$(\varphi \to (\psi \to \chi)) \leftrightarrow (\varphi \wedge \psi \to \chi)$

$\varphi \vee \psi \leftrightarrow \psi \vee \varphi$

$\varphi \vee (\psi \vee \chi) \leftrightarrow (\varphi \vee \psi) \vee \chi$

$\varphi \vee (\varphi \wedge \psi) \leftrightarrow \varphi$

$\overline{0} \vee \varphi \leftrightarrow \varphi$

The rule of substitution

Proposition 2.6

$$\vdash_G \varphi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi \vdash_G \psi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi, \psi \leftrightarrow \chi \vdash_G \varphi \leftrightarrow \chi$$

$$\varphi \leftrightarrow \psi \vdash_G (\varphi \wedge \chi) \leftrightarrow (\psi \wedge \chi) \qquad \varphi \leftrightarrow \psi \vdash_G (\varphi \vee \chi) \leftrightarrow (\psi \vee \chi)$$
$$\varphi \leftrightarrow \psi \vdash_G (\chi \wedge \varphi) \leftrightarrow (\chi \wedge \varphi) \qquad \varphi \leftrightarrow \psi \vdash_G (\chi \vee \varphi) \leftrightarrow (\chi \vee \psi)$$
$$\varphi \leftrightarrow \psi \vdash_G (\varphi \to \chi) \leftrightarrow (\psi \to \chi) \qquad \varphi \leftrightarrow \psi \vdash_G (\chi \to \varphi) \leftrightarrow (\chi \to \psi)$$

Corollary 2.7

$\varphi \leftrightarrow \psi \vdash_G \chi \leftrightarrow \chi'$, *where χ' results from χ by replacing some subformula φ by ψ.*

Deduction Theorem and Semilinearity Property

Theorem 2.8 (Deduction theorem)

For every set of formulae $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_{\mathrm{G}} \psi \textit{ iff } \Gamma \vdash_{\mathrm{G}} \varphi \to \psi$$

Lemma 2.9 (Semilinearity Property)

If $\Gamma, \varphi \to \psi \vdash_{\mathrm{G}} \chi$ *and* $\Gamma, \psi \to \varphi \vdash_{\mathrm{G}} \chi$, *then* $\Gamma \vdash_{\mathrm{G}} \chi$.

Proof.

By the deduction theorem: $\Gamma \vdash_{\mathrm{G}} (\varphi \to \psi) \to \chi$ and $\Gamma \vdash_{\mathrm{G}} (\psi \to \varphi) \to \chi$. Thus by (A6c) $\Gamma \vdash_{\mathrm{G}} (\varphi \to \psi) \vee (\psi \to \varphi) \to \chi$ and (T2) completes the proof. □

Linear Extension Property

Definition 2.10

A theory Γ is linear if $\Gamma \vdash_{\mathrm{G}} \varphi \to \psi$ or $\Gamma \vdash_{\mathrm{G}} \psi \to \varphi$ for each φ, ψ.

Lemma 2.11 (Linear Extension Property)

If $\Gamma \nvdash_{\mathrm{G}} \varphi$, then there is linear theory $\Gamma' \supseteq \Gamma$ s.t. $\Gamma' \nvdash_{\mathrm{G}} \varphi$.

Proof.

Enumerate all pairs of formulae: $\varphi_0, \psi_0, \psi_1, \varphi_1, \ldots$.
Construct theories $\Gamma_0, \Gamma_1, \ldots$ s.t. $\Gamma_0 = \Gamma$; $\Gamma_i \subseteq \Gamma_{i+1}$, and $\Gamma_i \nvdash_{\mathrm{G}} \varphi$:

- *if* $\Gamma_i, \varphi_i \to \psi_i \nvdash_{\mathrm{G}} \varphi$ *then* $\Gamma_{i+1} = \Gamma_i \cup \{\varphi_i \to \psi_i\}$
- *otherwise* $\Gamma_{i+1} = \Gamma_i \cup \{\psi_i \to \varphi_i\}$

Clearly $\Gamma_{i+1} \nvdash_{\mathrm{G}} \varphi$ (the 1st is obvious, in the 2nd would $\Gamma_{i+1} \vdash_{\mathrm{G}} \varphi$ entail $\Gamma_i \vdash_{\mathrm{G}} \varphi$ by the Semilinearity Property, a contradiction with the IH.
Define $\Gamma' = \bigcup \Gamma_i$. Clearly Γ' is linear, $\Gamma' \supseteq \Gamma$, and $\Gamma' \nvdash \varphi$. □

Lindenbaum–Tarski algebra

Definition 2.12

Let T be a theory. We define

$$[\varphi]_T = \{\psi \mid T \vdash_{\mathrm{G}} \varphi \leftrightarrow \psi\} \qquad L_T = \{[\varphi]_T \mid \varphi \in Fm_{\mathcal{L}}\}$$

The Lindenbaum–Tarski algebra of a theory T ($\mathbf{Lind}_T$) as an algebra with the domain L_T and operations:

$$\begin{aligned}
\overline{0}^{\mathbf{Lind}_T} &= [\overline{0}]_T\\
[\varphi]_T \rightarrow^{\mathbf{Lind}_T} [\psi]_T &= [\varphi \rightarrow \psi]_T\\
[\varphi]_T \vee^{\mathbf{Lind}_T} [\psi]_T &= [\varphi \vee \psi]_T\\
[\varphi]_T \wedge^{\mathbf{Lind}_T} [\psi]_T &= [\varphi \wedge \psi]_T
\end{aligned}$$

Exercise 5

Prove that the definition of $\mathbf{Lind}_T$ is sound.

Lindenbaum–Tarski algebra: basic properties

Proposition 2.13

1. $[\varphi]_T \leq [\psi]_T$ *iff* $T \vdash_{\mathrm{G}} \varphi \to \psi$.
2. $[\varphi]_T = \overline{1}^{\mathbf{Lind}_T}$ *iff* $T \vdash_{\mathrm{G}} \varphi$.
3. $\mathbf{Lind}_T$ *is a* G*-algebra.*
4. $\mathbf{Lind}_T$ *is a* G*-chain iff* T *is linear.*

Proof.

1. $[\varphi]_T \leq [\psi]_T$ iff $[\varphi]_T \wedge [\psi]_T = [\varphi]_T$ iff $[\varphi \wedge \psi]_T = [\varphi]_T$ iff $T \vdash_{\mathrm{G}} \varphi \wedge \psi \leftrightarrow \varphi$ iff (by (T6)) $T \vdash_{\mathrm{G}} \varphi \to \psi$.

2. Follows from 1 and (T4).

3. The fact that it is a Heyting algebra follows from Proposition **??** (residuation: $[\varphi]_T \leq [\psi]_T \to [\chi]_T$ iff $T \vdash \varphi \to (\psi \to \chi)$ iff $T \vdash_{\mathrm{G}} \varphi \wedge \psi \to \chi$ iff $[\varphi]_T \wedge [\psi]_T \leq [\chi]_T$) and it satisfies prelinearity due to (T2) and (T5).

4. Trivial. □

General/linear/standard completeness theorem

Theorem 2.14

The following are equivalent for every set of formulae $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{\mathrm{G}} \varphi$
2. $\Gamma \models_{\mathbb{G}} \varphi$
3. $\Gamma \models_{\mathbb{G}_{lin}} \varphi$
4. $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$

Proof.

(3) $\to$ (1) contrapositively: Assume that $T \nvdash_{\mathrm{G}} \varphi$. Let T' be a liner theory s.t. $T \subseteq T'$ and $T' \nvdash_{\mathrm{G}} \varphi$. Thus $\mathbf{Lind}_{T'}$ is a G-chain. To complete the proof take $\mathbf{Lind}_{T'}$-evaluation e, such that $e(\varphi) = [\varphi]_{T'}$ and observe that $e(\chi) = \overline{1}^{\mathbf{Lind}_{T'}}$ iff $T' \vdash \chi$. □

The proof of standard completeness theorem

Proof.

Contrapositively: assume that $T \not\vdash_{\mathrm{G}} \varphi$. We know that there is a countable G-chain $\boldsymbol{B}$ and $\boldsymbol{B}$-evaluation e such that $e[T] \subseteq \{\overline{1}^{\boldsymbol{B}}\}$ and $e(\varphi) \neq \overline{1}^{\boldsymbol{B}}$.
There has to be (because every countable order can be monotonously embedded into a dense one) a mapping $f \colon B \rightarrow [0,1]$ such that $f(\overline{0}) = 0$, $f(\overline{1}) = 1$, and for each $a, b \in B$ we have:

$$a \leq b \quad \text{iff} \quad f(a) \leq f(a)$$

We define $[0,1]_{\mathrm{G}}$-evaluation $\overline{e}(v) = f(e(v))$ and prove (by induction):

$$\overline{e}(\psi) = f(e(\psi))$$

Then $\overline{e}(\psi) = 1$ iff $e(\psi) = \overline{1}^{\boldsymbol{B}}$ and so $\overline{e}[T] \subseteq \{1\}$ and $\overline{e}(\varphi) \neq 1$. □

Outline

Syntax:

Formulae $\boldsymbol{Fm}_{\mathcal{L}}$ built from atoms combined by connectives $\mathcal{L} = \{\overline{0}, \wedge, \vee, \to\}$.

We also use defined connectives $\neg$, $\overline{1}$, and $\leftrightarrow$ defined as:

$$\neg\varphi = \varphi \to \overline{0} \qquad \overline{1} = \neg\overline{0} \qquad \varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$$

But also connectives $\oplus$ and $\&$ defined as:

$$\varphi \oplus \psi = \neg\varphi \to \psi \qquad \varphi \,\&\, \psi = \neg(\varphi \to \neg\psi)$$

A proof system for Łukasiewicz logic

Axioms:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \to (\psi \to \varphi)$
(A3)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(A4)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A5a)	$\varphi \wedge \psi \to \varphi$
(A5b)	$\varphi \wedge \psi \to \psi$
(A5c)	$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to \varphi \wedge \psi))$
(A6a)	$\varphi \to \varphi \vee \psi$
(A6b)	$\psi \to \varphi \vee \psi$
(A6c)	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
(A7)	$\overline{0} \to \varphi$
(A8)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

Inference rule: *modus ponens*.

We write $\Gamma \vdash_{Ł} \varphi$ if there is a proof of φ from Γ.

Algebraic semantics

An MV-*algebra* is a structure $\boldsymbol{B} = \langle B, \oplus, \neg, \overline{0} \rangle$ such that:

(1) $\langle B, \oplus, \overline{0} \rangle$ is a commutative monoid,

(2) $\neg\neg x = x$,

(3) $x \oplus \neg\overline{0} = \neg\overline{0}$,

(4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

In each MV-algebra we define additional operations:

$x \to y$	is	$\neg x \oplus y$	implication
$x \mathbin{\&} y$	is	$\neg(\neg x \oplus \neg y)$	strong conjunction
$x \wedge y$	is	$x \mathbin{\&} (x \to y)$	min-conjunction
$x \vee y$	is	$\neg(\neg x \wedge \neg y)$	max-disjunction
$\overline{1}$	is	$\neg\overline{0}$	top

Exercise 6

Prove that $\langle B, \wedge, \vee, \overline{0}, \overline{1} \rangle$ is a bounded lattice.

We say that an MV-algebra $\boldsymbol{B}$ is linearly ordered (or MV-chain) if its lattice reduct is.

By $\mathbb{MV}$ (or $\mathbb{MV}_{lin}$ resp.) we denote the class of all MV-algebras (MV-chains resp.)

Consider the algebra $[0,1]_{Ł} = \langle [0,1], \oplus, \neg, 0\rangle$, with operations defined as:

$$\neg a = 1 - a \qquad\qquad a \oplus b = \min\{1, a+b\}.$$

Exercise 3

(b) Prove that $[0,1]_{Ł}$ is the unique (up to isomorphism) MV-chain with the lattice reduct $\langle [0,1], \min, \max, 0, 1\rangle$.

Semantical consequence

Definition 2.15

A $\boldsymbol{B}$-evaluation is a homomorphism e from $\boldsymbol{Fm}_{\mathcal{L}}$ to $\boldsymbol{B}$; i.e.:

- $e(\overline{0}) = \overline{0}^{\boldsymbol{B}}$
- $e(\varphi \to \psi) = e(\varphi) \to^{\boldsymbol{B}} e(\psi) = \neg^{\boldsymbol{B}} e(\varphi) \oplus^{\boldsymbol{B}} e(\psi)$
- $e(\varphi \wedge \psi) = e(\varphi) \wedge^{\boldsymbol{B}} e(\psi) = \cdots$
- $e(\varphi \vee \psi) = e(\varphi) \vee^{\boldsymbol{B}} e(\psi) = \cdots$

Definition 2.16

Given $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$ and a class $\mathbb{K}$ of MV-algebras, we say that φ is a logical consequence of Γ w.r.t. $\mathbb{K}$, denoted $\Gamma \models_{\mathbb{K}} \varphi$, iff for every $\boldsymbol{B} \in \mathbb{K}$ and every $\boldsymbol{B}$-evaluation e such that $e(\Gamma) = \overline{1}$ for every $\gamma \in \Gamma$, we have $e(\varphi) = \overline{1}$.

General/linear/standard completeness theorem

Theorem 2.17

The following are equivalent for every set of formulae $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{\L} \varphi$
2. $\Gamma \models_{\mathbb{MV}} \varphi$
3. $\Gamma \models_{\mathbb{MV}_{lin}} \varphi$

If Γ *is finite we can add:*

4. $\Gamma \models_{[0,1]_{\L}} \varphi$

General/linear/standard completeness theorem

Theorem 2.17

The following are equivalent for every set of formulae $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{Ł} \varphi$
2. $\Gamma \models_{\mathbb{MV}} \varphi$
3. $\Gamma \models_{\mathbb{MV}_{lin}} \varphi$

If Γ *is finite we can add:*

4. $\Gamma \models_{[0,1]_{Ł}} \varphi$

Exercise 4

(b) Prove the implications from top to bottom.

Some theorems of Łukasiewicz logic

Proposition 2.18

(T1) $\varphi \to \varphi$

(T2) $(\varphi \to \psi) \vee (\psi \to \varphi)$

(T3) $\varphi \to (\psi \to \varphi \wedge \psi)$

(T4) $(\overline{1} \to \varphi) \leftrightarrow \varphi$

(T5) $(\overline{1} \leftrightarrow \varphi) \leftrightarrow \varphi$

(T6) $(\varphi \to \psi) \leftrightarrow (\varphi \wedge \psi \leftrightarrow \varphi)$

(T7) $\varphi \vee \chi \to ((\varphi \to \psi) \vee \chi \to \psi \vee \chi)$

(T8) $\varphi \vee \varphi \to \varphi$

(T9) $\varphi \vee \psi \to \psi \vee \varphi$

Proposition 2.19

$\varphi \oplus \psi \leftrightarrow \psi \oplus \varphi$

$\varphi \oplus (\psi \oplus \chi) \leftrightarrow (\varphi \oplus \psi) \oplus \chi$

$\overline{0} \oplus \varphi \leftrightarrow \varphi$

$\neg\neg\varphi \leftrightarrow \varphi$

$\varphi \oplus \neg\overline{0} \leftrightarrow \neg\overline{0}$

$\neg(\neg\varphi \oplus \psi) \oplus \psi \leftrightarrow \neg(\neg\psi \oplus \varphi) \oplus \varphi$

The rule of substitution

Proposition 2.20

$$\vdash_{Ł} \varphi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi \vdash_{Ł} \psi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi, \psi \leftrightarrow \chi \vdash_{Ł} \varphi \leftrightarrow \chi$$

$$\varphi \leftrightarrow \psi \vdash_{Ł} (\varphi \wedge \chi) \leftrightarrow (\psi \wedge \chi) \qquad \varphi \leftrightarrow \psi \vdash_{Ł} (\varphi \vee \chi) \leftrightarrow (\psi \vee \chi)$$
$$\varphi \leftrightarrow \psi \vdash_{Ł} (\chi \wedge \varphi) \leftrightarrow (\chi \wedge \varphi) \qquad \varphi \leftrightarrow \psi \vdash_{Ł} (\chi \vee \varphi) \leftrightarrow (\chi \vee \psi)$$
$$\varphi \leftrightarrow \psi \vdash_{Ł} (\varphi \to \chi) \leftrightarrow (\psi \to \chi) \qquad \varphi \leftrightarrow \psi \vdash_{Ł} (\chi \to \varphi) \leftrightarrow (\chi \to \psi)$$

Corollary 2.21

$\varphi \leftrightarrow \psi \vdash_{Ł} \chi \leftrightarrow \chi'$, *where χ' results from χ be replacing some subformula φ by ψ.*

Proof by Cases Property

Theorem 2.22 (Proof by Cases Property)

If $\Gamma, \varphi \vdash_{Ł} \chi$ *and* $\Gamma, \psi \vdash_{Ł} \chi$*, then* $\Gamma, \varphi \vee \psi \vdash_{Ł} \chi$*.*

Proof.

Claim If $\Gamma \vdash_{Ł} \varphi$, then $\Gamma \vee \chi \vdash_{Ł} \delta \vee \chi$ for each formula χ and each δ appearing in the proof of φ from Γ.

Proof of the claim: trivial for $\delta \in \Gamma$ or δ an axiom; if we used MP, then by IH there has to be η st.

$\Gamma \vee \chi \vdash_{Ł} \eta \vee \chi$ $\quad$ $\Gamma \vee \chi \vdash_{Ł} (\eta \to \delta) \vee \chi$ thus (T7) completes the proof.

Now using the claim: $\Gamma \vee \psi, \varphi \vee \psi \vdash_{Ł} \chi \vee \psi$ and $\Gamma \vee \chi, \psi \vee \chi \vdash_{Ł} \chi \vee \chi$. Using (A6a), (T8), and (T9) we get $\Gamma, \varphi \vee \psi \vdash_{Ł} \psi \vee \chi$ and $\Gamma, \psi \vee \chi \vdash_{Ł} \chi$ and the rest is trivial. □

Semilinearity and Linear Extensions Properties

Lemma 2.23 (Semilinearity Property)

If $\Gamma, \varphi \to \psi \vdash_{Ł} \chi$ *and* $\Gamma, \psi \to \varphi \vdash_{Ł} \chi$, *then* $\Gamma \vdash_{Ł} \chi$.

Proof.

By the Proof by Cases Property and (T2). □

Definition 2.24

A theory Γ is linear if $\Gamma \vdash_{Ł} \varphi \to \psi$ or $\Gamma \vdash_{Ł} \psi \to \varphi$ for each φ, ψ.

Lemma 2.25 (Linear Extension Property)

If $\Gamma \nvdash_{Ł} \varphi$, *then there is linear theory* $\Gamma' \supseteq \Gamma$ *s.t.* $\Gamma' \nvdash_{Ł} \varphi$.

Proof.

The same as in the case of Gödel-Dummett logic. □

Lindenbaum–Tarski algebra

Definition 2.26

Let T be a theory. We define

$$[\varphi]_T = \{\psi \mid T \vdash_{Ł} \varphi \leftrightarrow \psi\} \qquad L_T = \{[\varphi]_T \mid \varphi \in Fm_{\mathcal{L}}\}$$

The Lindenbaum–Tarski algebra of a theory T ($\mathbf{Lind}_T$) as an algebra with the domain L_T and operations:

$$\overline{0}^{\mathbf{Lind}_T} = [\overline{0}]_T$$

$$\neg^{\mathbf{Lind}_T}[\varphi]_T = [\varphi \to \overline{0}]_T$$

$$[\varphi]_T \oplus^{\mathbf{Lind}_T} [\psi]_T = [\neg\varphi \to \psi]_T$$

Lindenbaum–Tarski algebra: basic properties

Proposition 2.27

1. $[\varphi]_T \leq [\psi]_T$ *iff* $T \vdash_{Ł} \varphi \to \psi$.
2. $[\varphi]_T = \overline{1}^{\mathbf{Lind}_T}$ *iff* $T \vdash_{Ł} \varphi$.
3. $\mathbf{Lind}_T$ *is an* MV*-algebra.*
4. $\mathbf{Lind}_T$ *is an* MV*-chain iff* T *is linear.*

Proof.

1. The same as in the case of Gödel–Dummett logic (i.e. $[\varphi]_T \leq [\psi]_T$ iff $[\varphi]_T \wedge [\psi]_T = [\varphi]_T$ iff $[\varphi \wedge \psi]_T = [\varphi]_T$ iff $T \vdash_{Ł} \varphi \wedge \psi \leftrightarrow \varphi$ iff (by (T6)) $T \vdash \varphi \to \psi$).

2. Follows from 1 and (T4).

3. Obvious from Proposition **??**.

4. Trivial. □

General/linear/standard completeness theorem

Theorem 2.28

The following are equivalent for every set of formulae $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$*:*

1. $\Gamma \vdash_{\text{Ł}} \varphi$
2. $\Gamma \models_{\mathbb{MV}} \varphi$
3. $\Gamma \models_{\mathbb{MV}_{lin}} \varphi$

If Γ *is finite we can add:*

4. $\Gamma \models_{[0,1]_{\text{Ł}}} \varphi$

The proof of the equivalence of the first three claims is the same as in the case of Gödel–Dummett logic.

We give a proof of 3. implies 4. but first . . .

MV-algebras and LOAGs

A lattice ordered Abelian group (*LOAG* for short) is a structure $\langle G, +, 0, -, \leq \rangle$ s.t. $\langle G, +, 0, - \rangle$ is an Abelian group and:

(i) $\langle G, \leq \rangle$ *is a lattice,*
(ii) *if* $x \leq y$, *then* $x + z \leq y + z$ *for all* $z \in G$.

A strong unit u is an element s.t.

$$(\forall x \in G)(\exists n \in N)(x \leq nu)$$

For LOAG $\boldsymbol{G} = \langle G, +, 0, -, \leq \rangle$ and strong unit u we define algebra $\Gamma(\boldsymbol{G}, u) = \langle [0, u], \oplus, \neg, \overline{0} \rangle$, where $x \oplus y = \min\{u, x + y\}$, $\neg x = u - x$, $\overline{0} = 0$.

By $\boldsymbol{R}$ we denote the additive LOAG of reals.

Proposition 2.29

$\Gamma(\boldsymbol{G}, u)$ *is an* MV*-algebra and for each* $u > 0$ *is* $\Gamma(\boldsymbol{R}, u)$ *isomorphic to the standard* MV*-algebra* $[0, 1]_{Ł}$.

Proof of std. completeness of Łukasiewicz logic

If $T \not\vdash_{Ł} \varphi$ we know that there is countable MV-chain $\boldsymbol{B}$ s.t. $T \not\models_{\boldsymbol{B}} \varphi$. Let $x_1, \ldots, x_n$ be variables occurring in $T \cup \{\varphi\}$. Then:

$$\not\models_{\boldsymbol{B}} (\forall x_1, \ldots, x_n) \bigwedge_{\psi \in T} (\psi \approx \overline{1}) \Rightarrow (\varphi \approx \overline{1})$$

Let us define algebra $\boldsymbol{B}' = \langle Z \times B, +, -, 0 \rangle$ as:

$$\langle i, x \rangle + \langle j, y \rangle = \begin{cases} \langle i + j, x \oplus y \rangle & \text{if } x \mathbin{\&} y = 0 \\ \langle i + j + 1, x \mathbin{\&} y \rangle & \text{otherwise} \end{cases}$$

$$-\langle i, x \rangle = \langle -i - 1, \neg x \rangle \text{ and } 0 = \langle 0, \overline{0} \rangle$$

Proposition 2.30

$\boldsymbol{B}'$ is a LOAG and $\boldsymbol{B} = \Gamma(\boldsymbol{B}', \langle 1, \overline{0} \rangle)$.

Proof of std. completeness of Łukasiewicz logic

Let us fix an extra variable u, we define a translation of MV-terms into LOAG-terms:

$$x' = x \qquad \overline{0}' = 0 \qquad (\neg t)' = u - t' \qquad (t_1 \oplus t_2)' = (t_1' + t_2') \wedge u.$$

Recall that we have:

$$\not\models_{\boldsymbol{B}} (\forall x_1, \dots, x_n) \bigwedge_{\psi \in T} (\psi \approx \overline{1}) \Rightarrow (\varphi \approx \overline{1}),$$

Thus also:

$$\not\models_{\boldsymbol{B}'} (\forall u)(\forall x_1, \dots, x_n)[(0 < u) \wedge \bigwedge_{i \leq n} (x_i \leq u) \wedge (0 \leq x_i) \wedge \bigwedge_{\psi \in T} (\psi' \approx \overline{1}) \Rightarrow (\varphi' \approx \overline{1})$$

Proof of std. completeness of Łukasiewicz logic

Gurevich–Kokorin theorem: each $\forall_1$-sentence of LOAGs is true in additive LOAG of reals iff it is true in all linearly ordered LOAGs. Thus

$$\not\models_{\boldsymbol{R}} (\forall u)(\forall x_1, \dots, x_n)[(0 < u) \wedge \bigwedge_{i \leq n} (x_i \leq u) \wedge (0 \leq x_i) \wedge \bigwedge_{\psi \in T} (\psi' \approx \overline{1}) \Rightarrow (\varphi' \approx \overline{1})]$$

And so

$$\not\models_{\Gamma(\boldsymbol{R},u)} (\forall x_1, \dots, x_n) \bigwedge_{\psi \in T} (\psi \approx \overline{1}) \Rightarrow (\varphi \approx \overline{1})$$

And so

$$\not\models_{[0,1]_{Ł}} (\forall x_1, \dots, x_n) \bigwedge_{\psi \in T} (\psi \approx \overline{1}) \Rightarrow (\varphi \approx \overline{1})$$

i.e., $T \not\models_{[0,1]_{Ł}} \varphi$

Outline

The classical case

Theorem 2.31 (Functional completeness)

Every Boolean function is (i.e. any function $f\colon \{0,1\}^n \to \{0,1\}$ for some $n \geq 1$) is representable by some formula of classical logic.

The fuzzy case

Let L be either Ł of G.

Definition 2.32

A function $f\colon [0,1]^n \to [0,1]$ is *represented* by a formula $\varphi(v_1,\dots,v_n)$ in L if $e(\varphi) = f(e(v_1), e(v_2), \dots, e(v_n))$ for each $[0,1]_{\mathrm{L}}$-evaluation e.

Definition 2.33

The *functional representation* of L is the set $\mathcal{F}_{\mathrm{L}}$ of all functions from any power of $[0,1]$ into $[0,1]$ that are represented in L by some formula.

We denote by $\mathcal{F}^n_{\mathrm{L}}$ the subset of functions of $\mathcal{F}_{\mathrm{L}}$ in n variables.

Relation with Lindenbaum–Tarski algebra

Let us fix $\mathrm{L} = \text{Ł}$.
Let f_i be functions of n_i variables, $i \in \{1, 2\}$. We say that $f_1 = f_2$ iff $f_1(x_1, x_2, \ldots, x_{n_1}) = f_2(x_1, x_2, \ldots, x_{n_2})$ for every $x_j \in [0, 1]$. Let us for each $f \in \mathcal{F}_{\text{Ł}}$ define a class

$$[f] = \{g \in \mathcal{F}_{\text{Ł}} \mid f = g\} \qquad F = \{[f] \mid f \in \mathcal{F}_{\text{Ł}}\}$$

We define an MV-algebra $\boldsymbol{F}$ with domain F and operations:

$$\overline{0}^{\boldsymbol{F}} = [0] \quad \neg^{\boldsymbol{F}}[f] = [1 - f]_T \quad [f] \oplus^{\boldsymbol{F}} [g] = [\min\{1, f + g\}]$$

Theorem 2.34

The algebras $\boldsymbol{F}$ and $\mathbf{Lind}_{\emptyset}$ are isomorphic.

In the case of G, the definitions and the result are analogous.

A proof

Let the atoms be enumerated as $v_1, v_2, \ldots$. Any formula with variables with maximal index n is viewed as formula in variables $v_1, \ldots, v_n$.
We define the homomorphism:

$$g \colon L_\emptyset \to F \text{ as } g([\varphi]) = [f_\varphi] \text{ where } f_\varphi \text{ is the function represented by } \varphi.$$

Then:

- the definition is sound and g is one-one: $[\varphi] = [\psi]$ iff $\vdash_{Ł} \varphi \leftrightarrow \psi$ iff (due to the standard completeness theorem) $e(\varphi) = e(\psi)$ for each e $[0,1]_{Ł}$-evaluation iff $[f_\varphi] = [f_\psi]$.
- g is a homomorphism:
  $g([\varphi] \oplus [\psi]) = g([\varphi \oplus \psi]) = [f_{\varphi\oplus\psi}] = [f_\varphi \oplus f_\psi] = [f_\varphi] \oplus [f_\psi]$.
- g is onto (obvious).

How do the functions from $\mathcal{F}_{Ł}$ look like?

Observations

- they are all continuous
- they are piece-wise linear
- all pieces have integer coefficients
- if $x_1, \dots, x_n \in \{0, 1\}^n$, then $f(x_1, \dots, x_n) \in \{0, 1\}$
- if $x_1, \dots, x_n \in ([0, 1] \cap \mathrm{Q})^n$, then $f(x_1, \dots, x_n) \in [0, 1] \cap \mathrm{Q}$

Definition 2.35

A McNaughton function $f\colon [0, 1]^n \to [0, 1]$ is a continuous piece-wise linear function, where each of the pieces has integer coefficients.

Theorem 2.36 (McNaughton theorem)

$\mathcal{F}_{Ł}$ is the set of all McNaughton functions.

A lemma

Lemma 2.37

Let $f\colon [0,1]^n \to$ R *be an* integer linear polynomial, *i.e. of the form*

$$f(x_1,\ldots,x_n) = \sum_{i=1}^{n} a_i x_i + b \qquad \text{for some } a_1,\ldots,a_n, b \in \text{Z}$$

Then there is a formula φ_f *representing the function*

$$f^{\#} = \max\{0, \min\{1, f\}\}.$$

Proof.

By induction on $m = \sum_{i=1}^{n} |a_i|$. If $m = 0$ then $f^{\#}$ is either constantly 0 or 1, then we can take as φ either the term $\overline{0}$ or $\overline{1}$, respectively. Assume now $m > 0$ and let a_j be s.t. $|a_j| = \max_{i=1}^{n} |a_i|$. WLOG we can assume $a_j > 0$: indeed otherwise we consider $f' = 1 - f$, here $a_j > 0$ and so we have φ_{1-f}. Note that clearly $\varphi_f = \neg\varphi_{1-f}$. ...

A lemma: continuation of the proof

Let us consider the function $g = f - x_j$: by IH we have formulae φ_g and φ_{g+1}. If we show that

$$(g + x_j)^{\#} = (g^{\#} \oplus x_j) \odot (g + 1)^{\#} \tag{1}$$

the proof is done as:

$$\varphi_f = \varphi_{g+x_j} = (\varphi_g \oplus x_j) \mathbin{\&} \varphi_{g+1}.$$

So we need to prove (**??**). Let L and R be its left/right side :

- if $|g(\vec{x})| > 1$ then $L = R = 1$ or $L = R = 0$
- $0 \leq g(\vec{x}) \leq 1$ then $L = \min\{1, g(\vec{x}) + x_j\}$, $g(\vec{x}) = g^{\#}(\vec{x})$ and $(g+1)^{\#}(\vec{x}) = 1$. Hence $R = g(\vec{x}) \oplus x_j = \min\{1, g(\vec{x}) + x_j\} = L$.
- $-1 \leq g(\vec{x}) \leq 0$ then $L = \max\{0, g(\vec{x}) + x_j\}$, $g^{\#}(\vec{x}) = 0$ and $(g+1)^{\#}(\vec{x}) = g(\vec{x}) + 1$. Hence $g^{\#}(\vec{x}) \oplus x_j = x_j$ and so $R = \max\{0, x_j + g(\vec{x}) + 1 - 1\} = \max\{0, x_j + g(\vec{x})\} = L$.

The proof for one variable functions

Definition 2.38

Let $a, b \in [0, 1] \cap \mathrm{Q}$. Then any McNaughton function f s.t. $f(x) = 1$ iff $x \in [a, b]$ is called *pseudo characteristic function* of interval $[a, b]$.

Exercise 7

Prove that each interval has a pseudo characteristic function and find a formula representing it.

Lemma 2.39

Let $a, b \in [0, 1] \cap \mathrm{Q}$. Then for each $\epsilon > 0$ there is a pseudo characteristic function of the interval $[a, b]$, s.t. $f(x) = 0$ for $x \in [0, a - \epsilon] \cup [b + \epsilon, 1]$.

Proof.

Just note that if f is a pseudo characteristic function of some interval then so is f^n for each n. □

The proof for one variable functions

Let p be a McNaughton function of one variable given by n integer linear polynomials $p_1, \ldots, p_n$. For each $i \in \{1, 2, \ldots n\}$ let $P_i = [a_i, b_i]$ be the interval in which p uses p_i. Note that:

- $[0, 1] = \bigcup_i P_i$
- $a_i, b_i \in [0, 1] \cap \mathbb{Q}$
- there is a pseudo characteristic function f_i of $[a_i, b_i]$ such that $p(x) \geq (f_i \,\&\, p_i^{\#})(x)$ for each $x \notin P_i$.

Then

$$p(x) = \bigvee_i (f_i \,\&\, p_i^{\#})(x) \text{ and thus } \varphi_p = \bigvee_i \varphi_{f_i} \,\&\, \varphi_{p_i}.$$

Outline

The classical case, FMP and decidability

CL is complete with respect to a finite algebra, $\mathbf{2}$.

Definition 2.40

A logic has the finite model property (FMP) if it is complete with respect to a set of finite algebras.

From the FMP, we obtain decidability:

- Thanks to the finitary Hilbert-style proof system, the set of theorems is recursively enumerable.
- For each formula, if it is not a theorem we can check it in each algebra of n elements for each $n \in \mathrm{N}$ until we find a countermodel (thanks to FMP). Thus the set of non-theorems is also recursively enumerable.
- Therefore, theoremhood is a decidable problem.
- Provability from finitely-many premises is also decidable (using deduction theorem).

Lemma 2.41

Let $\boldsymbol{A}_1$ *be an* MV*- or* G*-algebra and* $\boldsymbol{A}_2$ *a subalgebra of* $\boldsymbol{A}_1$*. Then* $\models_{\boldsymbol{A}_1} \subseteq \models_{\boldsymbol{A}_2}$.

Exercise 8

(a) Prove that each n-valued G-chain is isomorphic to the *canonical n-valued G-chain:* $\boldsymbol{G}_n = \langle\{\frac{i}{n-1} \mid i \leq n-1\}, \min, \max, \rightarrow, 0, 1\rangle$.

(b) Prove that each n-valued MV-chain is isomorphic to the *canonical n-valued MV-chain:* $\text{Ł}_n = \langle\{\frac{i}{n-1} \mid i \leq n-1\}, \oplus, \neg, 0\rangle$.

Let us denote by $\mathbb{FIN}_{\mathrm{L}}$ the class of finite L-chains.

Lemma 2.42

$$\boldsymbol{G}_n \subseteq \boldsymbol{G}_m \quad \textit{iff} \quad n \leq m.$$

$$\text{Ł}_n \subseteq \text{Ł}_m \quad \textit{iff} \quad n-1 \textit{ divides } m-1.$$

The case of Gödel–Dummett logic

Theorem 2.43

Let φ be a formula with $n-2$ variables. Then: $\vdash_{\mathrm{G}} \varphi$ iff $\models_{\boldsymbol{G}_n} \varphi$.

Proof.

Contrapositively: assume that $\nvdash_{\mathrm{G}} \varphi$ and let e be a $[0,1]_{\mathrm{G}}$-evaluation s.t. $e(\varphi) \neq 1$. Let $X = \{0,1\} \cup \{e(v_i) \mid 1 \leq i \leq n-2\}$ and note that it is a subuniverse of $[0,1]_{\mathrm{G}}$, thus e can be seen as an $\boldsymbol{X}$-evaluation and so $\not\models_{\boldsymbol{X}} \varphi$. The previous exercise and lemma complete the proof. □

Theorem 2.44

For every finite set of formulae $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$, TFAE:

1. $\Gamma \vdash_{\mathrm{G}} \varphi$
2. $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$
3. $\Gamma \models_{\mathbb{FIN}_{\mathrm{G}}} \varphi$

Theorem 2.45

For every finite set of formulae $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}_{\mathcal{L}}$*, TFAE:*

1. $\Gamma \vdash_{\text{Ł}} \varphi$
2. $\Gamma \models_{[0,1]_{\text{Ł}}} \varphi$
3. $\Gamma \models_{\mathbb{FIN}_{\text{MV}}} \varphi$

Proof: we show it for one variable v.

Let us define the set E of $[0,1]_{\text{Ł}}$-evaluations s.t. $e[\Gamma] \subseteq \{1\}$. Note that E can be seen as a union of real intervals. Assume that there is $e \in E$ s.t. $e(\varphi) \neq 1$. If we show that there is an evaluation f, s.t. $f(v) = \frac{p}{n-1}$ and $f(\varphi) \neq 1$ we are done as f can be seen as **Ł**$_n$-evaluation.

- Either e lies on the border of some interval, then $f = e$ OR
- there has to be a neighborhood $X \subseteq E$ s.t. $f(\varphi) \neq \bar{1}$ for each $f \in X$, then there has to be such f. □

Outline

The classical case

- $\varphi \in \mathrm{SAT}(\mathrm{CL})$ if there is a $\mathbf{2}$-evaluation e such that $e(\varphi) = 1$.
- $\varphi \in \mathrm{TAUT}(\mathrm{CL})$ if for each $\mathbf{2}$-evaluation e holds $e(\varphi) = 1$.

Observe:

$$\varphi \in \mathrm{TAUT}(\mathrm{CL}) \quad \text{iff} \quad \neg\varphi \notin \mathrm{SAT}(\mathrm{CL})$$
$$\varphi \in \mathrm{SAT}(\mathrm{CL}) \quad \text{iff} \quad \neg\varphi \notin \mathrm{TAUT}(\mathrm{CL}).$$

Both problems, $\mathrm{SAT}(\mathrm{CL})$ and $\mathrm{TAUT}(\mathrm{CL})$, are decidable.

But how difficult are their computations?

Complexity classes

$f, g \colon \mathrm{N} \to \mathrm{N}$. $f \in O(g)$ iff there are $c, n_0 \in \mathrm{N}$ such that for each $n \geq n_0$ we have $f(n) \leq c\, g(n)$.

- **TIME**(f): the class of problems P such that there is a deterministic Turing machine M that accepts P and operates in time $O(f)$.
- **NTIME**(f): analogous class for nondeterministic Turing machines.
- **SPACE**(f): the class of problems P such that there is a deterministic Turing machine M that accepts P and operates in space $O(f)$.
- **NSPACE**(f): the analogous class for nondeterministic Turing machines.

Complexity classes

$$\mathbf{P} = \bigcup_{k \in \mathsf{N}} \mathbf{TIME}(n^k)$$

$$\mathbf{NP} = \bigcup_{k \in \mathsf{N}} \mathbf{NTIME}(n^k)$$

$$\mathbf{PSPACE} = \bigcup_{k \in \mathsf{N}} \mathbf{SPACE}(n^k)$$

If $\mathbf{C}$ is a complexity class, we denote $\mathbf{coC} = \{P \mid \overline{P} \in \mathbf{C}\}$, the class of complements of problems in $\mathbf{C}$.

Complexity classes

- Each deterministic complexity class $\mathbf{C}$ is closed under complementation: if $P \in \mathbf{C}$, then also $\overline{P} \in \mathbf{C}$.
- Is $\mathbf{NP}$ closed under complementation?
- $\mathbf{P} \subseteq \mathbf{NP}$, $\mathbf{P} \subseteq \mathbf{coNP}$, $\mathbf{NP} \subseteq \mathbf{PSPACE}$.
- Are the inclusions $\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE}$ proper?
- Each of the classes $\mathbf{P}$, $\mathbf{NP}$, $\mathbf{coNP}$, and $\mathbf{PSPACE}$ is closed under finite unions and intersections.

Complexity classes

A problem P is said to be $\mathbf{C}$-hard iff any decision problem P' in $\mathbf{C}$ is reducible to P.

A problem P is $\mathbf{C}$-complete iff P is $\mathbf{C}$-hard and $P \in \mathbf{C}$.

The classical case

- $\mathrm{SAT}(\mathrm{CL}) \in \mathbf{NP}$: guess an evaluation and check whether it satisfies the formula (a polynomial matter).
- $\mathrm{TAUT}(\mathrm{CL}) \in \mathbf{coNP}$: $\varphi \in \mathrm{TAUT}(\mathrm{CL})$ iff $\neg\varphi \in \overline{\mathrm{SAT}(\mathrm{CL})}$.
- Cook Theorem: Let $\mathrm{SAT}^{\mathrm{CNF}}(\mathrm{CL})$ be the SAT problem for formulae in conjunctive normal form. Then: $\mathrm{SAT}^{\mathrm{CNF}}(\mathrm{CL})$ is $\mathbf{NP}$-complete.
- $\mathrm{SAT}^{\mathrm{CNF}}(\mathrm{CL})$ is a fragment of $\mathrm{SAT}(\mathrm{CL})$, therefore $\mathrm{SAT}(\mathrm{CL})$ is $\mathbf{NP}$-complete and $\mathrm{TAUT}(\mathrm{CL})$ is $\mathbf{coNP}$-complete.

The fuzzy case: basic definitions

Let L be either Łukasiewicz Ł or Gödel logic G. We define:

- $\varphi \in \mathrm{SAT}(\mathrm{L})$ if there is an evaluation e such that $e(\varphi) = 1$.
- $\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathrm{L})$ if there is an evaluation e such that $e(\varphi) > 0$.
- $\varphi \in \mathrm{TAUT}(\mathrm{L})$ if for each evaluation e holds $e(\varphi) = 1$.
- $\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{L})$ if for each evaluation e holds $e(\varphi) > 0$.

Lemma 2.46

$\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{L})$ *iff* $\neg\varphi \notin \mathrm{SAT}(\mathrm{L})$
$\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathrm{L})$ *iff* $\neg\varphi \notin \mathrm{TAUT}(\mathrm{L})$.

The fuzzy case: basic definitions

Lemma 2.47

$$\varphi \in \mathrm{SAT}(Ł) \quad \textit{iff} \quad \neg\varphi \notin \mathrm{TAUT}_{\mathrm{pos}}(Ł)$$
$$\varphi \in \mathrm{TAUT}(Ł) \quad \textit{iff} \quad \neg\varphi \notin \mathrm{SAT}_{\mathrm{pos}}(Ł).$$

Exercise 9

Show that the first of the equivalencies hold also in G while the second one fails. (Hint: for the first part use properties of these sets proved in the next few slides).

The case of Łukasiewicz logic

Theorem 2.48

The sets $\mathrm{SAT}(\text{Ł})$ *and* $\mathrm{SAT}_{\mathrm{pos}}(\text{Ł})$ *are* $\mathbf{NP}$*-complete. Therefore the sets* $\mathrm{TAUT}(\text{Ł})$ *and* $\mathrm{TAUT}_{\mathrm{pos}}(\text{Ł})$ *are* $\mathbf{coNP}$*-complete.*

We prove it in a series of lemmata. First we show that $\mathrm{SAT}(\text{Ł})$ is $\mathbf{NP}$-hard:

Lemma 2.49

Let φ *be a formula with variables* $p_1, \ldots p_n$.

$$\varphi \in \mathrm{SAT}(\mathrm{CL}) \quad \textit{IFF} \quad \varphi \wedge \bigwedge_{i=1}^{n} (p_i \vee \neg p_i) \in \mathrm{SAT}(\text{Ł}).$$

$\mathrm{SAT_{pos}}(\text{Ł})$ is **NP**-hard

Lemma 2.50

Let φ *be a formula with variables* $p_1, \ldots p_n$ *built using:* $\wedge, \vee, \neg$.

$$\varphi \in \mathrm{SAT(CL)} \quad \textit{IFF} \quad \varphi^2 \wedge \bigwedge_{i=1}^{n} (p_i \vee \neg p_i)^2 \in \mathrm{SAT_{pos}}(\text{Ł}).$$

Proof.

Let e positively satisfy the right-hand formula. Then $e((p_i \vee \neg p_i)^2) > 0$ ergo $e(p_i) \neq 0.5$. We define the evaluation

$$e'(p_i) = \begin{cases} 1 & \text{if } e(p_i) > 0.5 \\ 0 & \text{if } e(p_i) < 0.5 \end{cases}$$

Clearly this can be extended to φ. And, since $e(\varphi^2) > 0$, we have $e(\varphi) > 0.5$ and so $e'(\varphi) = 1$. □

Lemma 2.51

$$e(\varphi \to \psi) \geq r \quad \textit{IFF} \quad \exists i,j \in [0,1] \quad \begin{array}{rcl} e(\varphi) & \leq & i \\ e(\psi) & \geq & j \\ r+i-j & \leq & 1 \end{array}$$

$$e(\varphi \to \psi) \leq r \quad \textit{IFF} \quad \exists i,j \in [0,1], y \in \{0,1\} \quad \begin{array}{rcl} e(\varphi) & \geq & i \\ e(\psi) & \leq & j \\ y-r & \leq & 0 \\ y+i & \leq & 1 \\ y-j & \leq & 0 \\ y+r+i-j & \geq & 1 \end{array}$$

Using this lemma we can reduce the question of (positive) satisfiability to the question of Mixed Integer Programming (MIP) which is known to be in **NP**:

For SAT(Ł) start with $e(\varphi) \geq 1$ for SAT_{pos}(Ł) start with $\begin{array}{rcl} e(\varphi) & \geq & i_0 \\ i_0 & > & 0 \end{array}$

The case of Gödel–Dummett logic

Lemma 2.52

The mapping $f\colon [0,1] \to \{0,1\}$ *defined as* $f(0) = 0$ *and* $f(x) = 1$ *if* $x \neq 0$ *is a homomorphism from* $[0,1]_{\mathrm{G}}$ *to* $\mathbf{2}$*.*

Corollary 2.53

$$\mathrm{SAT}_{\mathrm{pos}}(\mathrm{G}) \subseteq \mathrm{SAT}(\mathrm{CL}) \qquad \mathrm{TAUT}(\mathrm{CL}) \subseteq \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{G}).$$

The case of Gödel–Dummett logic

Corollary 2.54

$$\varphi \in \mathrm{SAT_{pos}(G)} \quad \textit{iff} \quad \varphi \in \mathrm{SAT(G)} \quad \textit{iff} \quad \varphi \in \mathrm{SAT(CL)}$$
$$\varphi \in \mathrm{TAUT_{pos}(G)} \quad \textit{iff} \quad \neg\neg\varphi \in \mathrm{TAUT(G)} \quad \textit{iff} \quad \varphi \in \mathrm{TAUT(CL)}$$

Proof.

Just observe that:

$$\mathrm{SAT(G)} \subseteq \mathrm{SAT_{pos}(G)} \subseteq \mathrm{SAT(CL)} \subseteq \mathrm{SAT(G)}.$$

And that

$$\varphi \in \mathrm{TAUT_{pos}(G)} \Rightarrow \neg\varphi \notin \mathrm{SAT(G)} \Rightarrow \neg\varphi \notin \mathrm{SAT_{pos}(G)}$$

$$\Rightarrow \neg\neg\varphi \in \mathrm{TAUT(G)} \Rightarrow \varphi \in \mathrm{TAUT(CL)} \Rightarrow \varphi \in \mathrm{TAUT_{pos}(G)}.$$

□

The case of Gödel–Dummett logic

Corollary 2.54

$\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathrm{G})$ *iff* $\varphi \in \mathrm{SAT}(\mathrm{G})$ *iff* $\varphi \in \mathrm{SAT}(\mathrm{CL})$
$\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathrm{G})$ *iff* $\neg\neg\varphi \in \mathrm{TAUT}(\mathrm{G})$ *iff* $\varphi \in \mathrm{TAUT}(\mathrm{CL})$

Theorem 2.55

The sets $\mathrm{SAT}(\mathrm{G})$ *and* $\mathrm{SAT}_{\mathrm{pos}}(\mathrm{G})$ *are* **NP***-complete and the sets* $\mathrm{TAUT}(\mathrm{G})$ *and* $\mathrm{TAUT}_{\mathrm{pos}}(\mathrm{G})$ *are* **coNP***-complete.*

Proof.

The only non clear case is $\mathrm{TAUT}(\mathrm{G})$: it is **NP**-hard due to the last reduction of the previous corollary. We present a non-deterministic polynomial 'algorithm' (sound due to Theorem **??**) for $Fm_{\mathcal{L}} \setminus \mathrm{TAUT}(\mathrm{G})$:

`Step 1`: guess a $\boldsymbol{G}_n$-evaluation e (assuming that φ has $n-2$ variables)
`Step 2`: compute the value of $e(\varphi)$ (clearly in polynomial time)
`Output`: if $e(\varphi) \neq 1$ output $\varphi \notin \mathrm{TAUT}(\mathrm{G})$. □