General Theories of Logical Systems
5th lesson

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The scope restriction for this lecture

Unless said otherwise, any logic L is weakly implicative in a language $\mathcal{L}$ with an implication $\to$.

Order and Leibniz congruence

Recall

Let $\mathbf{A} = \langle A, F\rangle$ be an L-matrix. We define:

- the matrix preorder $\leq_{\mathbf{A}}$ of $\mathbf{A}$ as

$$a \leq_{\mathbf{A}} b \qquad \text{iff} \qquad a \to^{\boldsymbol{A}} b \in F$$

- the Leibniz congruence $\Omega_{\boldsymbol{A}}(F)$ of $\mathbf{A}$ as

$$\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F) \qquad \text{iff} \qquad a \leq_{\mathbf{A}} b \ \text{ and } \ b \leq_{\mathbf{A}} a.$$

Observation

The Leibniz congruence of $\mathbf{A}$ is the identity iff $\leq_{\mathbf{A}}$ is an order.

Thus all reduced matrices of L are ordered by $\leq_{\mathbf{A}}$.

Weakly implicative logics are the logics of ordered matrices.

Linear filters

Definition 5.1

Let $\mathbf{A} = \langle A, F\rangle \in \mathbf{MOD}(\mathrm{L})$. Then

- F is *linear* if $\leq_{\mathbf{A}}$ is a total preorder, i.e. for every $a, b \in A$, $a \to^{\mathbf{A}} b \in F$ or $b \to^{\mathbf{A}} a \in F$
- $\mathbf{A}$ is a *linearly ordered model* (or just a *linear model*) if $\leq_{\mathbf{A}}$ is a linear order (equivalently: F is linear and $\mathbf{A}$ is reduced).

We denote the class of all linear models as $\mathbf{MOD}^{\ell}(\mathrm{L})$.

A theory T is linear in L if $T \vdash_{\mathrm{L}} \varphi \to \psi$ or $T \vdash_{\mathrm{L}} \psi \to \varphi$, for all φ, ψ

Lemma 5.2

Let $\mathbf{A} \in \mathbf{MOD}(\mathrm{L})$. Then F is linear iff $\mathbf{A}^* \in \mathbf{MOD}^{\ell}(\mathrm{L})$. In particular: a theory T is linear iff $\mathbf{LindT}_T \in \mathbf{MOD}^{\ell}(\mathrm{L})$

For proof just recall that: $[a]_F \leq_{\mathbf{A}^*} [b]_F$ iff $a \to^{\mathbf{A}} b \in F$.

Definition 5.3

We say that $\to$ is *semilinear* if

$$\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^{\ell}(\mathrm{L})}.$$

We say that L is *semilinear* if it has a semilinear implication.

(Weakly implicative) *semilinear* logics are the logics of *linearly* ordered matrices.

Characterization of semilinearity via the Linear Extension Property LEP

Definition 5.4

We say that a L has the *Linear Extension Property* LEP if linear theories form a base of $\mathrm{Th}(\mathrm{L})$, i.e. for every theory $T \in Th(\mathrm{L})$ and every formula $\varphi \in Fm_{\mathcal{L}} \setminus T$, there is a linear theory $T' \supseteq T$ such that $\varphi \notin T'$.

Theorem 5.5

Let L *be a weakly implicative logic. TFAE:*

1. L *is semilinear.*
2. L *has the* LEP*.*

The proof

$1 \to 2$: If $T \nvdash_{\mathrm{L}} \chi$, then there is a $\mathbf{B} = \langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}^{\ell}(\mathrm{L})$ and a $\mathbf{B}$-evaluation e s.t. $e[T] \subseteq F$ and $e(\chi) \notin F$. We define $T' = e^{-1}[F]$: it is a theory (due to Lemma 1.12), $T \subseteq T'$, and $T' \nvdash_{\mathrm{L}} \chi$. Take φ, ψ and assume w.l.o.g. that $e(\varphi) \leq_{\mathbf{B}} e(\psi)$, thus $e(\varphi \to \psi) \in F$, i.e. $\varphi \to \psi \in T'$.

$2 \to 1$: assume that $\Gamma \nvdash_{\mathrm{L}} \varphi$ and set $T = \mathrm{Th}_{\mathrm{L}}(\Gamma)$. Then there is a linear theory $T' \supseteq T$ such that $T' \nvdash_{\mathrm{L}} \varphi$.

Take Lindenbaum–Tarski matrix $\mathbf{LindT}_{T'}$ and note that $\mathbf{LindT}_{T'} \in \mathbf{MOD}^{\ell}(\mathrm{L})$ (due to Lemma 5.2). Then take evaluation $e(v) = [v]_{T'}$ and observe that $e[\Gamma] \subseteq e[T'] = [T']_{T'}$ and as $\varphi \notin T'$ we get $e(\varphi) \notin [T']_{T'}$ (due to Lemma 1.12).

Semilinearity Property SLP and its transfer

Definition 5.6

We say that a L has the *Semilinearity Property* SLP if the following meta-rule is valid:

$$\frac{\Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \to \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}.$$

Theorem 5.7

Assume that L *satisfies the* SLP*. Then for each* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *and each set* $X \cup \{a, b\} \subseteq A$ *we have:*

$$\mathrm{Fi}(X, a \to b) \cap \mathrm{Fi}(X, b \to a) = \mathrm{Fi}(X).$$

Lemma 5.8

Let $\boldsymbol{A}$ an $\mathcal{L}$-algebra and F a linear filter. Then the set $[F, A] = \{G \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A}) \mid F \subseteq G\}$ is linearly ordered by inclusion.

Proof.

Take $G_1, G_2 \in [F, A]$ and elements $a_1 \in G_1 \setminus G_2$ and $a_2 \in G_2 \setminus G_1$. Assume w.l.o.g. that $a_1 \leq_{\langle \boldsymbol{A}, F\rangle} a_2$. Thus also $a_1 \to^{\boldsymbol{A}} a_2 \in F \subseteq G_1$ and so by (MP) also $a_2 \in G_1$—a contradiction. □

Lemma 5.9

Linear filters are finitely $\cap$-irred. i.e. $\mathbf{MOD}^{\ell}(\mathrm{L}) \subseteq \mathbf{MOD}^{*}(\mathrm{L})_{\mathrm{RFSI}}$.

Proof.

Let $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ be a linear filter and $F = G_1 \cap G_2$. Then $G_1, G_2 \in [F, A]$ which is linearly ordered by inclusion, therefore $F = G_1$ or $F = G_2$. The second claim follows from Theorem 2.6. □

Characterization of semilinear logics

Theorem 5.10

Let L *be a weakly implicative logic. TFAE:*

1. L *is semilinear.*
2. L *has the* LEP*.*

If L *is finitary the list can be expanded by:*

3. L *has the* SLP*.*
4. L *has the transferred* SLP*.*
5. *Linear filters coincide with finitely* $\cap$*-irreducible ones in each* $\mathcal{L}$*-algebra.*
6. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}} = \mathbf{MOD}^\ell(\mathrm{L})$.
7. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}} \subseteq \mathbf{MOD}^\ell(\mathrm{L})$.

(Every semilinear logic enjoys properties 3.–7.)

The proof

$1 \leftrightarrow 2$: Theorem 5.5

$2 \to 3$: assume that $T \nvdash_{\mathrm{L}} \chi$, let $T' \supseteq T$ be a linear theory s.t. $T' \nvdash_{\mathrm{L}} \chi$. Assume w.l.o.g. that $T' \vdash_{\mathrm{L}} \varphi \to \psi$, then obviously $T, \varphi \to \psi \nvdash_{\mathrm{L}} \chi$.

$3 \to 4$: Theorem 5.7.

$4 \to 5$: let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra. One direction is Lemma 5.9. Converse one: assume that F is not linear, i.e., there are $a, b \in A$ st. $a \to b \notin F$ and $b \to a \notin F$. Thus $F \subsetneq \mathrm{Fi}(F, a \to b)$ and $F \subsetneq \mathrm{Fi}(F, b \to a)$ and so $\mathrm{Fi}(F, a \to b) \cap \mathrm{Fi}(F, b \to a) = \mathrm{Fi}(F) = F$, i.e., F is finitely $\cap$-reducible.

$5 \to 6$: due to Theorem 2.6.

$6 \to 7$: trivial consequence.

$7 \to 1$: due to Theorem 2.8. Note only here we need finitarity

Classes of semilinear logics

Corollary 5.11

Every regularly implicative semilinear logic is also Rasiowa-implicative.

Proof.

Trivially: $\varphi, \psi \to \varphi \vdash \psi \to \varphi$ and from regularity also: $\varphi, \varphi \to \psi \vdash \psi \to \varphi$. Thus, by the SLP, we derive $\varphi \vdash \psi \to \varphi$. □

Example 5.12

$Ł_3^{\leq}$ (the degree-preserving version of $Ł_3$) is is weakly implicative semilinear logic but it is not algebraically implicative.

Example 5.13

Logic of linear residuated lattices is algebraically implicative semilinear logic but it is not regularly implicative.

Intuitionistic logic is not semilinear

Example 5.14

Intuitionistic logic is not semilinear w.r.t. any implication.

Proposition 5.15

All axiomatic extensions of a semilinear logic are semilinear too.

If L can be axiomatically extended to IL, then it is not semilinear.

The least semilinear extension

Corollary 5.16

The intersection of a family of semilinear logics in the same language is a semilinear logic.

As Inc is trivially semilinear we can soundly define:

Definition 5.17 (Logic L^{ℓ})

Given a weakly implicative logic L, we denote by L^{ℓ} the least semilinear logic extending L.

Proposition 5.18

If L is a finitary weakly implicative logic, then so is L^{ℓ}.

The least semilinear extension—semantics

Proposition 5.19

Let L *be a weakly implicative logic. Then* $\mathrm{L}^{\ell} = \models_{\mathbf{MOD}^{\ell}(\mathrm{L})}$ *and* $\mathbf{MOD}^{\ell}(\mathrm{L}^{\ell}) = \mathbf{MOD}^{\ell}(\mathrm{L})$.

Proof.

Let L' be any extension of L, then $\mathbf{MOD}^{\ell}(\mathrm{L}') \subseteq \mathbf{MOD}^{\ell}(\mathrm{L})$. Thus in particular:

$$\mathbf{MOD}^{\ell}(\mathrm{L}^{\ell}) \subseteq \mathbf{MOD}^{\ell}(\mathrm{L}) \text{ and so } \models_{\mathbf{MOD}^{\ell}(\mathrm{L})} \subseteq \models_{\mathbf{MOD}^{\ell}(\mathrm{L}^{\ell})} = \mathrm{L}^{\ell}$$

As $\models_{\mathbf{MOD}^{\ell}(\mathrm{L})}$ is clearly semilinear we have the first claim.
The second inclusion of the second claim is trivial

(as $\mathbb{K} \subseteq \mathbf{MOD}^{*}(\models_{\mathbb{K}})$)

□

The least semilinear extension—axiomatization

Theorem 5.20 (Axiomatization of L^ℓ)

Let L *be a finitary p-disjunctional weakly implicative logic. Then* L^ℓ *is the extension of* L *with the axiom(s):*

$$(\mathrm{P}_\nabla) \quad \vdash_{\mathrm{L}} (\varphi \to \psi) \nabla (\psi \to \varphi).$$

Proof.

Using the previous proposition we know that $\mathrm{L}^\ell = \models_{\mathbf{MOD}^\ell(\mathrm{L})}$. The proof is completed by Theorem 4.37; we only need to observe that a matrix $\mathbf{A} \in \mathbf{MOD}^\ell(\mathrm{L})$ iff $\mathbf{A} \models P$, where P is the positive clause $F(\varphi \to \psi) \vee F(\psi \to \varphi)$. □

The axiom(s) (P_∇) is (are) called the *prelinearity axiom(s)*.

Semilinearity and (generalized) disjunction

How to proceed if we do not know any p-disjunction of L?

Idea: choose a *suitable* p-protodisjunction ∇, extend L to L^{∇}, and proceed as above.

Problem: what if $\mathrm{L}^{\nabla} \not\subseteq \mathrm{L}^{\ell}$? To overcome it, we define:

$$(\mathrm{MP}_{\nabla}) \quad \varphi \to \psi, \varphi \nabla \psi \vdash_{\mathrm{L}} \psi \qquad \text{and} \qquad \varphi \to \psi, \psi \nabla \varphi \vdash_{\mathrm{L}} \psi.$$

Proposition 5.21

Let ∇ be a p-protodisjunction in L.

1. *If L is p-disjunctional, then (MP_{∇}) is satisfied.*
2. *If L is semilinear, then (P_{∇}) is satisfied.*

Proof.

1. Using PCP for $\varphi, \varphi \to \psi \vdash \psi$ and $\psi, \varphi \to \psi \vdash \psi$.
2. Using SLP for $\varphi \to \psi \vdash_{\mathrm{L}} (\varphi \to \psi) \nabla (\psi \to \varphi)$ and
$\psi \to \varphi \vdash_{\mathrm{L}} (\varphi \to \psi) \nabla (\psi \to \varphi))$. □

Lemma 5.22

Let ∇ be a p-protodisjunction and $\boldsymbol{A}$ an $\mathcal{L}$-algebra.

1. *If L fulfils (MP_∇), then each linear filter in $\boldsymbol{A}$ is ∇-prime.*
2. *If L fulfils (P_∇), then each ∇-prime filter in $\boldsymbol{A}$ is linear.*

Proof.

1. Assume that F is linear ($a \to^{\boldsymbol{A}} b \in F$ or $b \to^{\boldsymbol{A}} a \in F$) and $a \nabla^{\boldsymbol{A}} b \subseteq F$. Thus from (MP_∇) we obtain: $b \in F$ or $a \in F$.

2. Assume that F is not linear, i.e. there are elements a, b st. $x = a \to^{\boldsymbol{A}} b \notin F$ and $y = b \to^{\boldsymbol{A}} a \notin F$. From (P_∇) we obtain $x \nabla^{\boldsymbol{A}} y = (a \to^{\boldsymbol{A}} b) \nabla^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} a) \subseteq F$, i.e., F is not ∇-prime. □

(P_∇) and (MP_∇): natural binding conditions – 2

Theorem 5.23 (Interplay of p-disjunctions and semilinearity)

Let L *be a finitary and* ∇ *a p-protodisjunction. TFAE:*

1. L *is p-disjunctional and satisfies* (P_∇).
2. L *is semilinear and satisfies* (MP_∇).

Thus in particular:

- *If* L *satisfies* (P_∇) *and* (MP_∇)*:* L *is semilinear iff it is p-disjunctional.*
- *If* L *is p-disjunctional:* L *is semilinear iff* L *satisfies* (P_∇).
- *If* L *is semilinear:* L *is p-disjunctional iff* L *satisfies* (MP_∇).

Proof.

(MP_∇) follows from Proposition 5.21. From (P_∇) we know that ∇-prime theories are linear and as we have PEP, we get LEP. The converse direction is analogous. □

Corollary 5.24

Let L *be a finitary logic and* ∇ *a p-protodisjunction* satisfying (MP_∇). *Then* L^ℓ *is the extension of* L^∇ *by* (P_∇).

Proof.

Since $\mathrm{L}^\nabla + (\mathrm{P}_\nabla)$ is an axiomatic extension of L^∇, ∇ remains a p-disjunction there. Thus, by Theorem 5.23, it is a semilinear logic.

Let L' be a finitary semilinear extension of L. Clearly L' satisfies (MP_∇) as well and thus by Theorem 5.23 it is a p-disjunctional logic and satisfies (P_∇). Thus $\mathrm{L}^\nabla \subseteq \mathrm{L}'$ and so

$$\mathrm{L}^\nabla + (\mathrm{P}_\nabla) \subseteq \mathrm{L}' + (\mathrm{P}_\nabla) = \mathrm{L}'. \quad \square$$

Axiomatization of expansions

Corollary 5.25

Let L_1 *be a semilinear logic with a p-protodisjunction which satisfies* (MP_∇) *and* L_2 *its* finitary *weakly implicative expansion by a set of consecutions* $\mathcal{C}$*. TFAE:*

- L_2 *is semilinear.*
- $\Gamma \nabla \chi \vdash_{\mathrm{L}_2} \varphi \nabla \chi$ *for each consecution* $\Gamma \rhd \varphi \in \mathcal{C}$*.*

Corollary 5.26

Let L *be a semilinear logic with a p-protodisjunction which satisfies* (MP_∇)*. Then all its weakly implicative axiomatic expansions are semilinear as well.*

Summary: Abstract Algebraic Logic

In this course we have tried to demonstrate that AAL provides powerful tools to:

- understand the several ways by which a logic can be given an algebraic semantics
- build a general and abstract theory of non-classical logics based on their relation to algebras
- understand the rôle of connectives in (non-)classical logics
- classify non-classical logics
- find general results connecting logical and algebraic properties (bridge theorems)
- generalize properties from syntax to semantics (transfer theorems)
- advance the study of particular (families of) non-classical logics by using the abstract notions and results