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Non-associative substructural logics and their semilinear extensions: axiomatization and completeness properties

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Abstract. Substructural logics extending the full Lambek calculus FL have largely benefited from a systematical algebraic approach based on the study of their algebraic counterparts: residuated lattices. Recently, a non-associative generalization of FL (which we call SL) has been studied by Galatos and Ono as the logic of lattice-ordered residuated unital groupoids.

This paper is based on an alternative Hilbert-style presentation for SL which is *almost* (MP)*-based*. This presentation is then used to obtain, in a uniform way applicable to most (both associative and non-associative) substructural logics, a form of local deduction theorem, description of filter generation, and proper forms of generalized disjunctions.

A special stress is put on semilinear substructural logics (i.e. logics complete w.r.t. linearly ordered algebras). Axiomatizations of the weakest semilinear logic over SL and other prominent substructural logics are provided and their completeness with respect to chains defined over the real unit interval is proved.

§1. Introduction Substructural logics form a wide family of non-classical logics that can be roughly defined as those logical systems such that, when presented by means of a Gentzen-style calculus, lack some of the *structural rules*, i.e. rules not involving any connective of the language (see e.g. Paoli (2002); Restall (2000); Schroeder-Heister & Dosen (1994)). As such they encompass a variety of systems independently developed since mid XXth century, including relevant logics (Anderson & Belnap, 1975) or many-valued logics like monoidal logic (Höhle, 1995) (not satisfying *contraction*), linear logic (Girard, 1987) (which, besides contraction, also fails to enjoy *weakening*) or Lambek calculus (Lambek, 1958) (which, besides the former two, does not satisfy *exchange* either). The study of such heterogenous landscape has greatly benefited from a uniform approach, developed in the last two decades in the tradition of Algebraic Logic, which deals with substructural logics as logics of residuated lattices, i.e. propositional logics algebraizable in the sense

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of Blok & Pigozzi (1989) whose equivalent algebraic semantics are classes of lattice-ordered residuated monoids (called *residuated lattices* for short). The weakest logic considered in this line of research is the full Lambek logic FL, whose equivalent algebraic semantics is the variety of all residuated lattices. Most results on the algebraic study of FL and its extensions are collected in the monograph Galatos et al. (2007).

These systematical efforts, nevertheless, have neglected the algebraic study of systems lacking another important structural rule: *associativity*. Indeed, FL does satisfy associativity and for this reason its algebraic semantics interprets (multiplicative) conjunction by a monoidal operation. Actually, there have been several studies on non-associative substructural logics, starting with the original Lambek non-associative calculus (Lambek, 1961) (without lattice connectives), and followed (in the full language) e.g. by Buszkowski & Farulewski (2009). Galatos & Ono (2010) introduced a Gentzen-style and a Hilbert-style calculus for the non-associative version of the Full Lambek calculus. They proved that it is an algebraizable logic with the variety of lattice-ordered residuated unital groupoids as its equivalent algebraic semantics; and thus they obtained a natural generalization of the approach used for FL and its extensions. In this paper we work with the bounded extension of this logic, denoted as SL.

Building on this, Cintula & Noguera (2011) presented a general algebraic framework to deal with substructural logics with SL as the base logic. The authors introduced, as a crucial tool, the notion of almost (MP)-based logic: a logic with a Hilbert-style presentation where *modus ponens* is the only binary rule, there are no rules with more than two premises, and all unary rules are of the form $\varphi \vdash \gamma(\varphi)$, for $\gamma \in \mathrm{DT}$, where the set of terms DT satisfies a natural technical condition. They proved that every almost (MP)-based substructural logic enjoys a local deduction theorem and a certain form of Proof by Cases Property (PCP), which can arguably be seen as the defining property of a reasonable generalized notion of disjunction (as studied by Abstract Algebraic Logic). From this, one can extract a number of interesting consequences for logical systems in general (Cintula & Noguera, 2013; Czelakowski, 2001) and for substructural logics in particular (Cintula & Noguera, 2011), such as (parameterized) local deduction theorem, description of intersection of filters, axiomatization of logics given by positive universal classes of algebras, or axiomatization of intersection of axiomatic extensions of a given logic. It was shown (Cintula & Noguera, 2011) that FL, and hence all its axiomatic extensions, are indeed almost (MP)-based and, therefore, the authors could apply their general theory to all these logics and obtain, in a uniform way, the mentioned consequences. However, the question whether SL and other non-associative logics are almost (MP)-based was left open.

The main purpose of the present paper is to answer this question. Indeed we present an alternative Hilbert-style axiomatization of SL which, besides *modus ponens*, has the following unary rules:[1]

[1] The rules (α) and (β) are taken from corresponding algebraic terms introduced by Botur (2011) where, in the context of the study of a non-associative version of Hájek's logic BL, they were used to describe filters in commutative integral lattice-ordered residuated unital groupoids. To cope with the lack of exchange and weakening (i.e. lack of commutativity and integrality in the algebras), we also need to consider unit-adjunction and a modified version of those rules: (α') and (β'). As stated later in

(Adj$_u$) $\varphi \vdash \varphi \wedge \overline{1}$
(α) $\varphi \vdash \delta \,\&\, \varepsilon \to \delta \,\&\, (\varepsilon \,\&\, \varphi)$
(α') $\varphi \vdash \delta \,\&\, \varepsilon \to (\delta \,\&\, \varphi) \,\&\, \varepsilon$
(β) $\varphi \vdash \delta \to (\varepsilon \to (\varepsilon \,\&\, \delta) \,\&\, \varphi)$
(β') $\varphi \vdash \delta \to (\varepsilon \rightsquigarrow (\delta \,\&\, \varepsilon) \,\&\, \varphi)$

We show that this axiomatization is indeed almost (MP)-based which allows us (as mentioned above) to naturally extend to the non-associative case many results so far only known for associative logics. This clearly demonstrates that the algebraic approach started by Galatos & Ono (2010) and Cintula & Noguera (2011) is the right generalization of that used for associative logics by Galatos et al. (2007).

Among others we obtain a method to find an axiomatization of the minimum logic L^ℓ extending a given logic L which is complete with respect to linearly ordered L-algebras (usually simply called L-*chains*). The class of logics complete with respect to chains has been introduced in a very general framework by Cintula & Noguera (2010) under the name *semilinear logics*.[2] When restricted to the framework of substructural logics, semilinear logics form a distinctive subfamily that contains most systems referred to in the literature as *fuzzy logics*. The discipline that studies these systems, Mathematical Fuzzy Logic (Cintula et al., 2011), has shown an interest in finding the *basic fuzzy logic* contained in all others.

Several systems have been proposed as such and later replaced by weaker ones, for instance: Hájek's logic BL (Hájek, 1998; Cignoli et al., 2000), $\mathrm{FL}^\ell_{\mathrm{ew}}$ = MTL (Esteva & Godo, 2001; Jenei & Montagna, 2002), $\mathrm{FL}^\ell_{\mathrm{w}}$ = psMTL^r (Jenei & Montagna, 2003), and $\mathrm{FL}^\ell_{\mathrm{e}}$ = UL (Metcalfe & Montagna, 2007).[3] One can observe that the common feature of all the mentioned logics is that they enjoy a *standard completeness theorem*, i.e. completeness with respect to a semantics of algebras defined on the real unit interval $[0, 1]$, which is implicitly regarded by many authors (and sometimes even explicitly, e.g. by Metcalfe & Montagna (2007)) as an essential requirement for fuzzy logics. Interestingly enough, the logic FL^ℓ of FL-chains does not enjoy standard completeness (Wang & Zhao, 2009), therefore, for these authors it can hardly be taken as a good candidate for a really basic *fuzzy* logic (even though for some it is *fuzzy enough* (Běhounek & Cintula, 2006)). Moreover, one can also argue that FL^ℓ is still *not basic enough* because it satisfies a remaining structural rule: associativity. This brings us again to the main motivation of this paper, the algebraic study of non-associative logics, and now also the study of their semilinear extensions. Following the methods and results from previous works (Cintula & Noguera, 2011; Horčík, 2011) for semilinear associative substructural logics, the second aim of the present paper, thus, is to use the terms appearing in almost (MP)-based

Proposition 3.10., in the presence of associativity the rules (α) and (β) are trivialized and the terms used in (α') and (β') become (when $\delta = \overline{1}$) equivalent to the usual terms that appear in product normality rules; moreover, in the presence of exchange the terms in (α) and (β) become respectively equivalent to those in (α') and (β').

[2] The term *semilinear* refers to the fact that these logics can be characterized as those logics such that their (relatively) subdirectly irreducible algebras are linearly ordered.

[3] An alternative path in the search for weaker systems, instead of removing logical laws, has consisted in restricting the language by considering fragments of fuzzy logics (see e.g. Esteva et al. (2003); Cintula et al. (2007)).

presentations to obtain axiomatizations of semilinear non-associative substructural logics and study their standard completeness properties. In particular, we obtain a presentation of SL^ℓ and prove that it enjoys completeness with respect to residuated unital groupoids over $[0, 1]$. Therefore SL^ℓ can be seen as a new good candidate for a basic fuzzy logic, for it does not even satisfy associativity and is standard complete.

Structure of the paper In Section §2. we briefly recall the necessary syntactical and semantical preliminaries for the paper: 2.1. shows the Hilbert-style presentation of SL given by Galatos and Ono, lists some important syntactical properties that hold in the system and introduces prominent axiomatic extensions, while 2.2. introduces the semantics for these logics based on lattice-ordered residuated unital groupoids. Section §3. is devoted to the general algebraic study of (non-associative) substructural logics. Subsection 3.1. presents an alternative Hilbert-style axiomatic system for SL (the formal proof of the equivalence of this new system with the original one can be found in Appendix A) and uses it to show that it is an almost (MP)-based logic. Subsection 3.2. derives from this result a form of (parameterized) local deduction theorem for SL and its extensions and some results on filter generation. Subsection 3.3. extracts from the terms appearing in the almost (MP)-based presentation a description of a p-disjunction for SL, shows its simplifications in prominent extensions, and considers the aforementioned applications of these p-disjunctions. Section §4. is devoted to semilinear extensions of non-associative substructural logics, i.e. the logics given by their linearly ordered algebras. Subsection 4.1., as yet another application of almost (MP)-basedness and p-disjunctions, shows several equivalent ways to axiomatize these semilinear logics. Finally, Subsection 4.2. gives, by means of algebraic constructions, a proof of completeness of SL^ℓ and other non-associative logics with respect to their chains defined over the real and the rational unit intervals.

§2. Preliminaries

2.1. Syntactical properties The weakest logic we consider in this paper is the bounded version of the non-associative full Lambek calculus (Galatos & Ono, 2010). We call it SL and formulate it in the language $\mathcal{L}_{\mathrm{SL}} = \{\wedge, \vee, \&, \to, \rightsquigarrow, \overline{0}, \overline{1}, \bot, \top\}$ (we also use the defined connective $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$). When writing formulae in this language we will assume that the increasing binding order of connectives is: first &, then $\{\wedge, \vee\}$, and finally $\{\to, \rightsquigarrow\}$. This logic can be axiomatized by means of the following Hilbert-style calculus presented in Table 1 (it is obtained from that of (Galatos & Ono, 2010, Figure 5) by expanding its language with a new basic connective $\bot$ and derived connective $\top$ defined as $\bot \to \bot$ and by adding the axiom $\bot \to \varphi$).

Galatos & Ono (2010) provide a Gentzen-style calculus which can be easily extended to a calculus for SL. On the other hand, SL is implicitly presented by an alternative Hilbert-style system in (Cintula & Noguera, 2011, Definition 2.5.1). By using any of these presentations, one may obtain other well-known properties of substructural logics which already hold in SL:

(T)	$\chi \to \varphi, \varphi \to \psi \vdash \chi \to \psi$	$(\mathrm{Pf})_{\rightsquigarrow}$	$\psi \to \chi \vdash (\varphi \rightsquigarrow \psi) \to (\varphi \rightsquigarrow \chi)$
$(\mathrm{E}_{\rightsquigarrow 2})$	$\psi \to (\varphi \rightsquigarrow \chi) \vdash \varphi \to (\psi \to \chi)$	$(\mathrm{Adj}_{\mathrm{u}})$	$\varphi \vdash \varphi \wedge \overline{1}$
(Res_2)	$\varphi \,\&\, \psi \to \chi \vdash \psi \to (\varphi \to \chi)$	(Symm_2)	$\varphi \to \psi \vdash \varphi \rightsquigarrow \psi$

Table 1. *Axiomatic system of* SL

(R)	$\varphi \to \varphi$	(As)	$\varphi \vdash (\varphi \to \psi) \to \psi$
(MP)	$\varphi, \varphi \to \psi \vdash \psi$	($\mathrm{As}_{\ell\ell}$)	$\varphi \to ((\varphi \rightsquigarrow \psi) \to \psi)$
(Sf)	$\varphi \to \psi \vdash (\psi \to \chi) \to (\varphi \to \chi)$	(Symm_1)	$\varphi \rightsquigarrow \psi \vdash \varphi \to \psi$
(Pf)	$\psi \to \chi \vdash (\varphi \to \psi) \to (\varphi \to \chi)$	($\mathrm{E}_{\rightsquigarrow 1}$)	$\varphi \to (\psi \to \chi) \vdash \psi \to (\varphi \rightsquigarrow \chi)$
(Res_1)	$\psi \to (\varphi \to \chi) \vdash \varphi \,\&\, \psi \to \chi$	(R′)	$\overline{1} \to (\varphi \to \varphi)$
($\mathrm{Adj}_{\&}$)	$\varphi \to (\psi \to \psi \,\&\, \varphi)$	(Push)	$\varphi \to (\overline{1} \to \varphi)$
(Bot)	$\bot \to \varphi$	($\overline{1}$)	$\overline{1}$
($\wedge$1)	$\varphi \wedge \psi \to \varphi$	($\vee$1)	$\varphi \to \varphi \vee \psi$
($\wedge$2)	$\varphi \wedge \psi \to \psi$	($\vee$2)	$\psi \to \varphi \vee \psi$
($\wedge$3)	$(\chi \to \varphi) \wedge (\chi \to \psi) \to (\chi \to \varphi \wedge \psi)$	($\vee$3)	$(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$
(Adj)	$\varphi, \psi \vdash \varphi \wedge \psi$	($\vee 3_{\rightsquigarrow}$)	$(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \to (\varphi \vee \psi \rightsquigarrow \chi)$

We list some other properties that will be useful later; ($\mathrm{P_{SL}}2$)–($\mathrm{P_{SL}}24$) are taken from (Cintula & Noguera, 2011, Proposition 2.5.5) (where one can find their proofs), the remaining ones can be proved easily (e.g. in the Gentzen calculus for SL).

($\mathrm{P_{SL}}2$) $\vdash \varphi \,\&\, (\varphi \to \psi) \to \psi$
($\mathrm{P_{SL}}8$) $\varphi \to \psi \vdash \chi \,\&\, \varphi \to \chi \,\&\, \psi$
($\mathrm{P_{SL}}9$) $\varphi \to \psi \vdash \varphi \,\&\, \chi \to \psi \,\&\, \chi$
($\mathrm{P_{SL}}10$) $\varphi_1 \to \psi_1, \varphi_2 \to \psi_2 \vdash \varphi_1 \,\&\, \varphi_2 \to \psi_1 \,\&\, \psi_2$
($\mathrm{P_{SL}}20$) $\vdash \chi \,\&\, (\varphi \vee \psi) \leftrightarrow (\chi \,\&\, \varphi) \vee (\chi \,\&\, \psi)$
($\mathrm{P_{SL}}21$) $\vdash (\varphi \vee \psi) \,\&\, \chi \leftrightarrow (\varphi \,\&\, \chi) \vee (\psi \,\&\, \chi)$
($\mathrm{P_{SL}}22$) $\vdash (\varphi \wedge \overline{1}) \,\&\, (\psi \wedge \overline{1}) \to \varphi \wedge \overline{1}$
($\mathrm{P_{SL}}23$) $\vdash (\varphi \wedge \overline{1}) \,\&\, (\psi \wedge \overline{1}) \to \psi \wedge \overline{1}$
($\mathrm{P_{SL}}24$) $\vdash (\varphi \to \psi) \wedge \overline{1} \to (\varphi \wedge \overline{1} \to \psi \wedge \overline{1})$
($\mathrm{P_{SL}}25$) $\vdash (\varphi \to \psi) \wedge \overline{1} \to (\varphi \vee \chi \to \psi \vee \chi)$
($\mathrm{P_{SL}}26$) $\vdash (\varphi \to \psi) \wedge \overline{1} \to (\varphi \vee \psi \to \psi)$
($\mathrm{P_{SL}}27$) $\vdash (\psi \to \varphi) \wedge \overline{1} \to (\varphi \vee \psi \to \varphi)$
($\mathrm{P_{SL}}28$) $\vdash \varphi \wedge \overline{1} \to (\varphi \wedge \overline{1}) \wedge \overline{1}$

Some important extensions of SL are obtained by adding the axioms $\mathrm{a}_1, \mathrm{a}_2, \mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}$ corresponding to structural rules (see Table 2). Given any $S \subseteq \{\mathrm{a}_1, \mathrm{a}_2, \mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$, by SL_S we denote the axiomatic extension of SL by S. If $\{\mathrm{a}_1, \mathrm{a}_2\} \subseteq S$, then instead of them we write the symbol 'a'. Analogously if $\{\mathrm{i}, \mathrm{o}\} \subseteq S$, instead of them we write the symbol 'w'. Equivalent ways to formulate these axioms may be found e.g. in (Cintula & Noguera, 2011, Theorem 2.5.7).

$\mathrm{SL_a}$ is, in fact, the bounded version of full Lambek logic FL, i.e., our framework encompasses the associative systems as well. For the sake of simplicity we keep the language fixed and we only consider finitary logics (i.e. logics enjoying a Hilbert-style calculus where all rules have finitely-many premises). Therefore we set the following convention to delimit, in this paper (!), the class of substructural logics. The reader can check (Cintula & Noguera, 2011; Galatos et al., 2007) for other possible conventional definitions of this family of logics.

CONVENTION 2.1. *A logic is* substructural *if it is a finitary extension of* SL*.*

Table 2. *Structural rules*

a_1	$\varphi \mathbin{\&} (\psi \mathbin{\&} \chi) \to (\varphi \mathbin{\&} \psi) \mathbin{\&} \chi$	*re-associate to the left*
a_2	$(\varphi \mathbin{\&} \psi) \mathbin{\&} \chi \to \varphi \mathbin{\&} (\psi \mathbin{\&} \chi)$	*re-associate to the right*
e	$\varphi \mathbin{\&} \psi \to \psi \mathbin{\&} \varphi$	*exchange*
c	$\varphi \to \varphi \mathbin{\&} \varphi$	*contraction*
i	$\psi \to (\varphi \to \psi)$	*left weakening*
o	$\overline{0} \to \varphi$	*right weakening*

2.2. Algebraic semantics In this subsection we present the algebraic semantics for SL and other substructural logics. For this we need to recall several algebraic notions and fix some notation and terminology.[4]

A *poset* $\boldsymbol{P} = \langle P, \leq \rangle$ is a partially ordered set. If $\leq$ is a total order, then $\boldsymbol{P}$ is called a *chain.* A map $\gamma\colon P \to P$ is said to be a *closure operator* on $\boldsymbol{P}$ if it is expanding ($x \leq \gamma(x)$), monotone ($x \leq y$ implies $\gamma(x) \leq \gamma(y)$) and idempotent ($\gamma(\gamma(x)) = \gamma(x)$). Dually, a map $\sigma\colon P \to P$ is called an *interior operator* on $\boldsymbol{P}$ provided that it is contracting ($\sigma(x) \leq x$), monotone and idempotent. The elements in the image $\gamma[P]$ (resp. $\sigma[P]$) are called γ*-closed* (resp. σ*-open*).

Let $\boldsymbol{P}$ be a poset. A map $f\colon P \to P$ is *residuated* if there is a map $f^\dagger\colon P \to P$ such that for all $x, y \in P$ we have $f(x) \leq y$ iff $x \leq f^\dagger(y)$. Equivalently, f is residuated iff f is monotone and the inverse image of any principal downset is a principal downset as well (recall that a principal downset is a subset of P of the form $\{y \in P \mid y \leq x\}$ for some $x \in P$). A binary operation $\circ\colon P^2 \to P$ is residuated if it is residuated component-wise, i.e. for every $a \in P$ the maps given by $x \mapsto a \circ x$ and $x \mapsto x \circ a$ are residuated. Equivalently, there are maps $\backslash\colon P^2 \to P$ and $/\colon P^2 \to P$ such that for all $a, b, c \in P$ we have

$$a \circ b \leq c \quad \text{iff} \quad b \leq a\backslash c \quad \text{iff} \quad a \leq c/b\,.$$

The maps $\backslash, /$ are called respectively *left* and *right residual* of $\circ$.

A *lattice* is a poset where every pair of elements x, y has a greatest lower bound $x \wedge y$ and a least upper bound $x \vee y$. A lattice $\boldsymbol{A}$ is called *bounded* provided that it has a minimum $\bot$ and a maximum $\top$. We call an algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \overline{0}, \overline{1}, \bot, \top \rangle$ a *doubly pointed bounded lattice* (shortly *dpb*-lattice) if $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice endowed with additional constants $\overline{0}, \overline{1}$. Let $\mathrm{T} \subseteq \{\mathrm{i}, \mathrm{o}\}$. Then a *dpb*-lattice is said to be a dpb_{T}-lattice provided that $\overline{1} = \top$ if $\mathrm{i} \in \mathrm{T}$ and $\overline{0} = \bot$ if $\mathrm{o} \in \mathrm{T}$. We apply the same convention also for chains, i.e., a dpb_{T}-chain is a chain which is also a dpb_{T}-lattice.

An algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \overline{0}, \overline{1}, \bot, \top \rangle$ is called a *(semiunital) residuated lattice ordered groupoid* (shortly $r\ell$-groupoid) if $\langle A, \wedge, \vee, \overline{0}, \overline{1}, \bot, \top \rangle$ is a *dpb*-lattice satisfying $x \leq (\overline{1} \cdot x) \wedge (x \cdot \overline{1})$, the groupoid operation $\cdot$ is residuated, and its residuals are the operations $\backslash$ and $/$. Note that the element $\overline{1}$ is assumed to be only a semiunit, i.e., we have $x \leq \overline{1} \cdot x$ and $x \leq x \cdot \overline{1}$. Although we are interested mainly in the cases where $\overline{1}$ is actually a unit (see Definition 2.2.), we need this more

[4] For any unexplained notions, notations, and terminology of Universal Algebra used in the paper see e.g. Burris & Sankappanavar (1981).

general definition due to technical reasons which become apparent in Section 4.2. Let $S \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$. An $r\ell$-groupoid $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \overline{0}, \overline{1}, \bot, \top\rangle$ is said to be an $r\ell_{\mathrm{S}}$*-groupoid* provided that

- if $\mathrm{e} \in S$, then $x \cdot y = y \cdot x$ for all $x, y \in A$,[5]
- if $\mathrm{c} \in S$, then $x \leq x \cdot x$ for all $x \in A$,
- $\langle A, \wedge, \vee, \overline{0}, \overline{1}, \bot, \top\rangle$ is a dpb_{T}-lattice for $\mathrm{T} = \mathrm{S} \setminus \{\mathrm{e}, \mathrm{c}\}$.

DEFINITION 2.2. *Let* $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \overline{0}, \overline{1}, \bot, \top\rangle$ *be an* $r\ell_{\mathrm{S}}$*-groupoid for some* $\mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$. *We say that* $\boldsymbol{A}$ *is*

- totally ordered *(or just* rt_{S}-groupoid*, for short) if* $\langle A, \wedge, \vee\rangle$ *forms a chain.*
- unital *if* $\overline{1}$ *is a neutral element for the groupoid operation, i.e.,* $\overline{1} \cdot x = x = x \cdot \overline{1}$.

Unital (totally ordered) $r\ell_{\mathrm{S}}$*-groupoids are also called* SL_{S}-algebras *(resp.* SL_{S}-chains*).*

For $S = \emptyset$ we speak about SL-*algebras* and SL-*chains*. Observe that the residuation condition together with the fact that $\overline{1}$ is a neutral element implies that for every SL-algebra $\boldsymbol{A}$ and each $a, b \in A$ we have

$$a \leq b \quad \text{iff} \quad \overline{1} \leq a\backslash b \quad \text{iff} \quad \overline{1} \leq b/a\,.$$

Given an SL-algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \overline{0}, \overline{1}, \bot, \top\rangle$ an $\boldsymbol{A}$*-evaluation* is an homomorphism from the algebra of formulae to $\boldsymbol{A}$ such that the connectives $\wedge, \vee, \&, \to, \rightsquigarrow, \overline{0}, \overline{1}, \bot, \top$ are respectively interpreted by the operations $\wedge, \vee, \cdot, \backslash, /, \overline{0}, \overline{1}, \bot, \top$. By means of this notion, we can give, more generally, the following definition for the algebraic counterpart of any substructural logic, which can easily be seen to encompass the previous cases.

DEFINITION 2.3. *Let* L *be the substructural logic obtained by adding a set of axioms* AX *and a set of rules* R *to* SL. $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \overline{0}, \overline{1}, \bot, \top\rangle$ *is an* L*-algebra if it is an* SL*-algebra such that:*

- *for every* $\varphi \in AX$ *and every* $\boldsymbol{A}$*-evaluation* e, $e(\varphi) \geq \overline{1}$,
- *for every* $\langle \Gamma, \varphi\rangle \in R$ *and every* $\boldsymbol{A}$*-evaluation* e, *if* $e(\psi) \geq \overline{1}$ *for every* $\psi \in \Gamma$, *then* $e(\varphi) \geq \overline{1}$.

The class of all SL-algebras, denoted as $\mathbb{SL}$, is well-known to be a variety and it gives a semantics for the logic SL. In general, for every substructural logic L the class $\mathbb{L}$ of L-algebras (clearly, a subquasivariety of $\mathbb{SL}$) gives a semantics for L. To formulate the corresponding completeness theorems, we need to define a notion of semantical consequence. Given a class $\mathbb{K} \subseteq \mathbb{SL}$, a set of formulae Γ and a formula φ, $\Gamma \models_{\mathbb{K}} \varphi$ if for every $\boldsymbol{A} \in \mathbb{K}$ and every $\boldsymbol{A}$-evaluation e, if $e(\psi) \geq \overline{1}$ for every $\psi \in \Gamma$, then $e(\varphi) \geq \overline{1}$.

THEOREM 2.4. *Let* L *be a substructural logic. Then for every set of formulae* Γ *and every formula* φ *we have:* $\Gamma \vdash_{\mathrm{L}} \varphi$ *if, and only if,* $\Gamma \models_{\mathbb{L}} \varphi$.

Technically speaking, SL is an algebraizable logic in the sense of Blok & Pigozzi (1989), $\mathbb{SL}$ is its equivalent algebraic semantics, and the translations are $E(p, q) = \{p \to q, q \to p\}$ and $\mathcal{E}(p) = \{p \wedge \overline{1} \approx \overline{1}\}$. The same holds for every substructural logic L and its corresponding quasivariety $\mathbb{L}$.

[5] Note that in this case the residuals coincide and we so we can denote them both by $\to$.

Given a substructural logic L and an $\mathcal{L}_{\mathrm{SL}}$-algebra $\boldsymbol{A}$, a set $F \subseteq A$ is an L-*filter* if for every set of formulae Γ and every formula φ such that $\Gamma \vdash_{\mathrm{L}} \varphi$ and every $\boldsymbol{A}$-evaluation e it holds: if $e[\Gamma] \subseteq F$, then $e(\varphi) \in F$. By $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ we denote the set of all L-filters over $\boldsymbol{A}$. Since $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is a closure system (it clearly contains A and is closed under arbitrary intersections), one can define a notion of generated filter. Given $X \subseteq A$, the L-filter generated by X, denoted as $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X)$ is the least L-filter containing X (we omit the indexes when clear from the context).

We will need the following generic characterization for membership in the filter generated by a set (later we will show more usual algebraic descriptions of filters).

Proposition 2.5. *Let* L *be the substructural logic obtained by adding a set of axioms AX and a set of rules R to* SL. *Furthermore let $\boldsymbol{A}$ be an* L*-algebra and $X \cup \{a\} \subseteq A$. Let us define sets $V_{AX} \subseteq A$ and $V_R \subseteq \mathcal{P}(A) \times A$ as*[6]

$$V_{AX} = \{e(\psi) \mid e \textit{ is an } \boldsymbol{A}\textit{-evaluation and } \psi \in AX\}$$
$$V_R = \{\langle e[\Gamma], e(\psi)\rangle \mid e \textit{ is an } \boldsymbol{A}\textit{-evaluation and } \langle \Gamma, \psi\rangle \in R\}$$

Then $a \in \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X)$ iff there is a finite sequence $\langle a_0, \ldots, a_n\rangle$ (called proof of a from X*) of elements of A such that*

- *$a_n = a$,*
- *for each $i \leq n$, either $a_i \in X \cup V_{AX}$ or there is a non-empty $Z \subseteq \{a_0, \ldots, a_{i-1}\}$ such that $\langle Z, a_i\rangle \in V_R$.*

Algebraizability gives a correspondence between filters and (relative) congruences in L-algebras. Let $\boldsymbol{Con}_{\mathbb{L}}(\boldsymbol{A})$ denote the lattice of congruences of $\boldsymbol{A}$ relative to $\mathbb{L}$, i.e. giving a quotient in $\mathbb{L}$. If $\mathbb{L}$ is a variety, then $\boldsymbol{Con}_{\mathbb{L}}(\boldsymbol{A})$ contains all congruences of $\boldsymbol{A}$. The *Leibniz operator* $\Omega_{\boldsymbol{A}}$ is a mapping defined, for any $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, as $\Omega_{\boldsymbol{A}}(F) = \{\langle a, b\rangle \in A^2 \mid a\backslash b \in F \text{ and } b\backslash a \in F\}$.

Proposition 2.6. *Let* L *be a substructural logic and $\boldsymbol{A}$ an* L*-algebra. The Leibniz operator $\Omega_{\boldsymbol{A}}$ is a lattice isomorphism from $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ to $\boldsymbol{Con}_{\mathbb{L}}(\boldsymbol{A})$. Its inverse is the function that maps any $\theta \in \boldsymbol{Con}_{\mathbb{L}}(\boldsymbol{A})$ to the filter $\{a \in A \mid \langle a \wedge \overline{1}, \overline{1}\rangle \in \theta\}$.*

Observe that the minimum filter is the one generated by the emptyset, $\mathrm{Fi}(\emptyset)$, and it must correspond to the identity congruence $Id_{\boldsymbol{A}}$. Therefore, using the previous proposition, we obtain that, on any L-algebra $\boldsymbol{A}$, $\mathrm{Fi}(\emptyset) = \{a \in A \mid a \geq \overline{1}\}$. This set is, of course, contained in any other filter.

We can also use the previous proposition to show that for any axiomatic extension L of SL (i.e., substructural logic L such that $\mathbb{L}$ is a variety), $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ forms a distributive lattice. Indeed the lattice of congruences on $\boldsymbol{A}$ is distributive (as it has a lattice reduct) and the $\mathbb{L}$-relative congruences and congruences on $\boldsymbol{A}$ are the same. This reasoning cannot be used when $\mathbb{L}$ is not a variety because in this case the relative congruences do not form a sublattice of the congruence lattice (Raftery, 2001).

Given a class of algebras $\mathbb{K}$ a non-trivial algebra $\boldsymbol{A}$ is *(finitely) subdirectly irreducible relative to* $\mathbb{K}$ if for every (finite non-empty) subdirect representation α of $\boldsymbol{A}$ with a family $\{\boldsymbol{A}_i \mid i \in I\} \subseteq \mathbb{K}$ there is $i \in I$ such that $\pi_i \circ \alpha$ is an isomorphism. The class of all (finitely) subdirectly irreducible algebras relative to $\mathbb{K}$ is denoted as $\mathbb{K}_{\mathrm{R(F)SI}}$. Of course $\mathbb{K}_{\mathrm{RSI}} \subseteq \mathbb{K}_{\mathrm{RFSI}}$.

[6] Note that if $\boldsymbol{A} = \boldsymbol{Fm}_{\mathcal{L}}$, then $V_{AX} = AX$ and $V_R = R$.

§3. Almost (MP)**-based non-associative substructural logics** In this section we present our new general results on non-associative substructural logics. They are based on the notion of almost (MP)-based logic, which has been firstly introduced and studied by Cintula & Noguera (2011). Before we recall this notion, we need to introduce same technical notions. Let *Var* be the fixed set of propositional variables in which we are writing the formulae of the language $\mathcal{L}_{\mathrm{SL}}$ and $\star$ be a new symbol, which acts as placeholder for a special kind of substitutions. A $\star$-formula is built using variables $Var \cup \{\star\}$ and a $\star$-substitution is a substitution in the extended language. Let φ be a $\star$-formula, δ be a $\star$-formula, and σ a $\star$-substitution defined as $\sigma(\star) = \varphi$ and $\sigma p = p$ for $p \in Var$. By $\delta(\varphi)$ we denote the $\star$-formula $\sigma\delta$; note that if φ is a formula in the original set of variables, so is $\delta(\varphi)$.

DEFINITION 3.7. *Given a set of* $\star$*-formulae* Γ*, we define the set* Γ^* *of* $\star$*-formulae as the smallest set such that*

- $\star \in \Gamma^*$ *and*
- $\delta(\gamma) \in \Gamma^*$ *for each* $\delta \in \Gamma$ *and each* $\gamma \in \Gamma^*$*.*

We are ready now to give the formal definition of almost (MP)-based logic.

DEFINITION 3.8. *Let* bDT *be a set of* $\star$*-formulae closed under all* $\star$*-substitutions* σ *such that* $\sigma(\star) = \star$*. A substructural logic* L *is* almost (MP)-based w.r.t. the set of basic deduction terms bDT *if:*

- L *has a presentation where the only deduction rules are* modus ponens *and those from* $\{\langle\varphi, \gamma(\varphi)\rangle \mid \varphi \in Fm_{\mathcal{L}_{\mathrm{SL}}}, \gamma \in \mathrm{bDT}\}$*, and*
- *for each* $\beta \in \mathrm{bDT}$ *and each formulae* φ, ψ*, there exist* $\beta_1, \beta_2 \in \mathrm{bDT}^*$ *such that:*[7]

$$\vdash_{\mathrm{L}} \beta_1(\varphi \to \psi) \to (\beta_2(\varphi) \to \beta(\psi)).$$

L *is called* (MP)-based *if it admits the empty set as a set of basic deduction terms.*

The goal of the first subsection is to obtain an equivalent Hilbert-style presentation of SL showing that this logic and thus its axiomatic extensions are almost (MP)-based. In the second subsection we use this result to obtain local deduction theorems and a descriptions of generated filters in non-associative substructural logics. Finally, in the third subsection the terms appearing in the unary rules of almost (MP)-based presentations are used to build a generalized disjunction connective satisfying the Proof by Cases Property. Using this and following general results by Cintula & Noguera (2013), we obtain other logical and algebraic properties.

3.1. Almost (MP)***-based presentations of substructural logics*** We start by providing (in Table 3) an alternative system for SL. Appendix A contains all (rather tedious) formal proofs necessary to prove the next theorem.

THEOREM 3.9. *The axiomatic system from Table 3 is a presentation of* SL*.*

[7] Here we are deviating from the original definition from Cintula & Noguera (2011) where β_1, β_2 were required to be in bDT. This alteration has no effect on the notion of almost (MP)-based logic as shown by claim 2 in Lemma 3.16. which can be read as: if bDT is a set of basic deduction terms in the sense just defined, then bDT^* is a set of basic deduction terms in the original sense, so the logic remains almost (MP)-based. This new definition of bDT will however allow us to obtain stronger results in Subsection 4.1.

Table 3. *New axiomatic system for* SL

($\mathrm{Adj}_{\&}$)	$\varphi \to (\psi \to \psi \,\&\, \varphi)$	(Bot)	$\bot \to \varphi$
($\mathrm{Adj}_{\&\rightsquigarrow}$)	$\varphi \to (\psi \rightsquigarrow \varphi \,\&\, \psi)$	($\&\wedge$)	$(\varphi \wedge \overline{1}) \,\&\, (\psi \wedge \overline{1}) \to \varphi \wedge \psi$
($\wedge 1$)	$\varphi \wedge \psi \to \varphi$	($\vee 1$)	$\varphi \to \varphi \vee \psi$
($\wedge 2$)	$\varphi \wedge \psi \to \psi$	($\vee 2$)	$\psi \to \varphi \vee \psi$
($\wedge 3$)	$(\chi \to \varphi) \wedge (\chi \to \psi) \to (\chi \to \varphi \wedge \psi)$	($\vee 3$)	$(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$
(Res$'$)	$\psi \,\&\, (\varphi \,\&\, (\varphi \to (\psi \to \chi))) \to \chi$	(Push)	$\varphi \to (\overline{1} \to \varphi)$
($\mathrm{Res}'_{\rightsquigarrow}$)	$(\varphi \,\&\, (\varphi \to (\psi \rightsquigarrow \chi))) \,\&\, \psi \to \chi$	(Pop)	$(\overline{1} \to \varphi) \to \varphi$
(T$'$)	$(\varphi \to (\varphi \,\&\, (\varphi \to \psi)) \,\&\, (\psi \to \chi)) \to (\varphi \to \chi)$		
($\mathrm{T}'_{\rightsquigarrow}$)	$(\varphi \rightsquigarrow ((\varphi \rightsquigarrow \psi) \,\&\, \varphi) \,\&\, (\psi \to \chi)) \to (\varphi \rightsquigarrow \chi)$		
(MP)	$\varphi, \varphi \to \psi \vdash \psi$	($\mathrm{Adj_u}$)	$\varphi \vdash \varphi \wedge \overline{1}$
(α)	$\varphi \vdash \delta \,\&\, \varepsilon \to \delta \,\&\, (\varepsilon \,\&\, \varphi)$	(β)	$\varphi \vdash \delta \to (\varepsilon \to (\varepsilon \,\&\, \delta) \,\&\, \varphi)$
(α')	$\varphi \vdash \delta \,\&\, \varepsilon \to (\delta \,\&\, \varphi) \,\&\, \varepsilon$	(β')	$\varphi \vdash \delta \to (\varepsilon \rightsquigarrow (\delta \,\&\, \varepsilon) \,\&\, \varphi)$

Let us introduce a convenient notation for the terms appearing on the right-hand side of the rules (α), (α'), (β), and (β'). Given arbitrary formulae δ, ε, we define the following $\star$-formulae:

$$\alpha_{\delta,\varepsilon} = \delta \,\&\, \varepsilon \to \delta \,\&\, (\varepsilon \,\&\, \star) \qquad \beta_{\delta,\varepsilon} = \delta \to (\varepsilon \to (\varepsilon \,\&\, \delta) \,\&\, \star)$$
$$\alpha'_{\delta,\varepsilon} = \delta \,\&\, \varepsilon \to (\delta \,\&\, \star) \,\&\, \varepsilon \qquad \beta'_{\delta,\varepsilon} = \delta \to (\varepsilon \rightsquigarrow (\delta \,\&\, \varepsilon) \,\&\, \star)$$

Note that these terms (those in the second line) generalize the terms appearing in the well-known left and right product normality rules used in associative logics (Galatos et al., 2007, page 124):[8]

$$\lambda_\varepsilon = \varepsilon \to \star \,\&\, \varepsilon \qquad \rho_\varepsilon = \varepsilon \rightsquigarrow \varepsilon \,\&\, \star$$

The next, not difficult to prove, proposition shows how these terms, and hence the axiomatic systems in which they appear, can be simplified in stronger substructural logics (e.g. in presence of exchange we can omit the prime version of the rules and associativity allows us to replace α, α', β, β' by the rules $\varphi \vdash \rho_\varepsilon(\varphi)$ and $\varphi \vdash \lambda_\varepsilon(\varphi)$).

PROPOSITION 3.10. *We have*

1. $\vdash_{\mathrm{SL}} \gamma_{\overline{1},\overline{1}}(\varphi) \leftrightarrow \varphi$ *for each* $\gamma \in \{\alpha, \alpha', \beta, \beta'\}$
2. $\vdash_{\mathrm{SL_e}} \alpha_{\delta,\varepsilon}(\varphi) \leftrightarrow \alpha'_{\varepsilon,\delta}(\varphi)$ *and* $\vdash_{\mathrm{SL_e}} \beta_{\delta,\varepsilon}(\varphi) \leftrightarrow \beta'_{\delta,\varepsilon}(\varphi)$
3. $\vdash_{\mathrm{SL_a}} \varphi \to \gamma_{\delta,\varepsilon}(\varphi)$ *for each* $\gamma \in \{\alpha, \beta\}$
4. $\vdash_{\mathrm{SL_a}} \lambda_\varepsilon(\varphi) \to \alpha'_{\delta,\varepsilon}(\varphi)$ *and* $\vdash_{\mathrm{SL_a}} \rho_\varepsilon(\varphi) \to \beta'_{\delta,\varepsilon}(\varphi)$
5. $\vdash_{\mathrm{SL_a}} \lambda_\varepsilon(\varphi) \leftrightarrow \alpha'_{\overline{1},\varepsilon}(\varphi)$ *and* $\vdash_{\mathrm{SL_a}} \rho_\varepsilon(\varphi) \leftrightarrow \beta'_{\overline{1},\varepsilon}(\varphi)$
6. $\vdash_{\mathrm{SL_{ae}}} \varphi \to \lambda_\varepsilon(\varphi)$ *and* $\vdash_{\mathrm{SL_{ae}}} \varphi \to \rho_\varepsilon(\varphi)$

[8] In the literature on substructural logics, the names λ_ε and ρ_ε denote slightly more complicated terms, namely $\lambda_\varepsilon = (\varepsilon \to \star \,\&\, \varepsilon) \wedge \overline{1}$ and $\rho_\varepsilon = (\varepsilon \rightsquigarrow \varepsilon \,\&\, \star) \wedge \overline{1}$. In the theory of residuated lattices these terms are called respectively left and right conjugate and are useful for obtaining a bijective correspondence between the lattices of congruences and convex normal subalgebras; see e.g. (Galatos et al., 2007, Theorem 3.47).

In order to prove the main result of this section, almost (MP)-basedness of SL, we need the following syntactical lemmata.

LEMMA 3.11. *The following are provable in* SL:

(Aux1) $\vdash \alpha_{\chi,\varphi}(\varphi \to \psi) \to (\chi \,\&\, \varphi \to \chi \,\&\, \psi)$

(Aux2) $\vdash \alpha'_{\varphi,\chi}(\varphi \to \psi) \to (\varphi \,\&\, \chi \to \psi \,\&\, \chi)$

(Aux3) $\vdash \beta_{\chi\to\varphi,\chi}(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$

(Aux4) $\vdash \beta'_{\chi\rightsquigarrow\varphi,\chi}(\varphi \to \psi) \to ((\chi \rightsquigarrow \varphi) \to (\chi \rightsquigarrow \psi))$

Proof. SL proves (Aux1):

(a) $\vdash \varphi \,\&\, (\varphi \to \psi) \to \psi$ — ($P_{SL}2$)

(b) $\vdash \chi \,\&\, (\varphi \,\&\, (\varphi \to \psi)) \to \chi \,\&\, \psi$ — (a) and ($P_{SL}8$)

(c) $\vdash (\chi \,\&\, \varphi \to \chi \,\&\, (\varphi \,\&\, (\varphi \to \psi))) \to (\chi \,\&\, \varphi \to \chi \,\&\, \psi)$ — (b) and (Pf)

SL proves (Aux2):

(a) $\vdash \varphi \,\&\, (\varphi \to \psi) \to \psi$ — ($P_{SL}2$)

(b) $\vdash (\varphi \,\&\, (\varphi \to \psi)) \,\&\, \chi \to \psi \,\&\, \chi$ — (a) and ($P_{SL}9$)

(c) $\vdash (\varphi \,\&\, \chi \to (\varphi \,\&\, (\varphi \to \psi)) \,\&\, \chi) \to (\varphi \,\&\, \chi \to \psi \,\&\, \chi)$ — (b) and (Pf)

SL proves (Aux3):

(a) $\vdash \chi \,\&\, (\chi \to \varphi) \to \varphi$ — ($P_{SL}2$)

(b) $\vdash (\chi \,\&\, (\chi \to \varphi)) \,\&\, (\varphi \to \psi) \to \varphi \,\&\, (\varphi \to \psi)$ — (a) and ($P_{SL}9$)

(c) $\vdash \varphi \,\&\, (\varphi \to \psi) \to \psi$ — ($P_{SL}2$)

(d) $\vdash (\chi \,\&\, (\chi \to \varphi)) \,\&\, (\varphi \to \psi) \to \psi$ — (b), (c), and (T)

(e) $\vdash (\chi \to (\chi \,\&\, (\chi \to \varphi)) \,\&\, (\varphi \to \psi)) \to (\chi \to \psi)$ — (d) and (Pf)

(f) $\vdash [(\chi \to \varphi) \to (\chi \to (\chi \,\&\, (\chi \to \varphi)) \,\&\, (\varphi \to \psi))] \to [(\chi \to \varphi) \to (\chi \to \psi)]$ — (e) and (Pf)

SL proves (Aux4):

(a) $\vdash (\chi \rightsquigarrow \varphi) \,\&\, \chi \to \varphi$ — ($As_{\ell\ell}$) and (Res_1)

(b) $\vdash ((\chi \rightsquigarrow \varphi) \,\&\, \chi) \,\&\, (\varphi \to \psi) \to \varphi \,\&\, (\varphi \to \psi)$ — (a) and ($P_{SL}9$)

(c) $\vdash \varphi \,\&\, (\varphi \to \psi) \to \psi$ — ($P_{SL}2$)

(d) $\vdash ((\chi \rightsquigarrow \varphi) \,\&\, \chi) \,\&\, (\varphi \to \psi) \to \psi$ — (b), (c), and (T)

(e) $\vdash (\chi \rightsquigarrow ((\chi \rightsquigarrow \varphi) \,\&\, \chi) \,\&\, (\varphi \to \psi)) \to (\chi \rightsquigarrow \psi)$ — (d) and $(Pf)_{\rightsquigarrow}$

(f) $\vdash [(\chi \rightsquigarrow \varphi) \to (\chi \rightsquigarrow ((\chi \rightsquigarrow \varphi) \,\&\, \chi) \,\&\, (\varphi \to \psi))] \to [(\chi \rightsquigarrow \varphi) \to (\chi \rightsquigarrow \psi)]$ — (e) and (Pf) □

LEMMA 3.12. *For every* $\star$*-formula* $\gamma \in \{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon} \mid \delta,\varepsilon$ *formulae*$\}$ *and every pair of formulae* φ, ψ*, we have:* $\varphi \to \psi \vdash_{SL} \gamma(\varphi) \to \gamma(\psi)$.

Proof. All the cases are easily proved in a similar way. Let us show the case of $\alpha_{\delta,\varepsilon}$ as an example.

(a) $\varphi \to \psi \vdash \delta \,\&\, (\varepsilon \,\&\, \varphi) \to \delta \,\&\, (\varepsilon \,\&\, \psi)$ — ($P_{SL}8$) twice

(b) $\varphi \to \psi \vdash (\delta \,\&\, \varepsilon \to \delta \,\&\, (\varepsilon \,\&\, \varphi)) \to (\delta \,\&\, \varepsilon \to \delta \,\&\, (\varepsilon \,\&\, \psi))$ — (a) and (Pf) □

Theorem 3.13. SL *is almost* (MP)*-based*[9] *with respect to the set*

$$\mathrm{bDT_{SL}} = \{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon}, \star \wedge \overline{1} \mid \delta, \varepsilon \textit{ formulae}\}.$$

Proof. Theorem 3.9. shows that there is a presentation of SL with (MP) as the only binary rule and unary rules $\varphi \vdash \gamma(\varphi)$ for each $\gamma \in \mathrm{bDT_{SL}}$. We need to prove the final condition in the definition of almost (MP)-based axiomatic systems, in particular we show that for each $\gamma \in \mathrm{bDT_{SL}}$ and each formulae φ, ψ there is $\gamma' \in \mathrm{bDT^*_{SL}}$ such that

$$\vdash \gamma'(\varphi \to \psi) \to (\gamma(\varphi) \to \gamma(\psi)).$$

If γ is $\star \wedge \overline{1}$ we can set $\gamma' = \gamma$ due to ($\mathrm{P_{SL}}$24). Next we prove the claim for $\alpha'_{\delta,\varepsilon}$, the other cases are proved analogously:

(a) $\alpha_{\delta,\varphi}(\varphi \to \psi) \to [\delta \,\&\, \varphi \to \delta \,\&\, \psi]$ (Aux1)

(b) $\alpha'_{\delta\&\varphi,\varepsilon}(\delta \,\&\, \varphi \to \delta \,\&\, \psi) \to [(\delta \,\&\, \varphi) \,\&\, \varepsilon \to (\delta \,\&\, \psi) \,\&\, \varepsilon]$ (Aux2)

(c) $\beta_{\delta\&\varepsilon\to(\delta\&\varphi)\&\varepsilon,\delta\&\varepsilon}((\delta \,\&\, \varphi) \,\&\, \varepsilon \to (\delta \,\&\, \psi) \,\&\, \varepsilon) \to [\alpha'_{\delta,\varepsilon}(\varphi) \to \alpha'_{\delta,\varepsilon}(\psi)]$ (Aux3)

(d) $\alpha'_{\delta\&\varphi,\varepsilon}(\alpha_{\delta,\varphi}(\varphi \to \psi)) \to \alpha'_{\delta\&\varphi,\varepsilon}(\delta \,\&\, \varphi \to \delta \,\&\, \psi)$ (a) and Lemma 3.12.

(e) $\alpha'_{\delta\&\varphi,\varepsilon}(\alpha_{\delta,\varphi}(\varphi \to \psi)) \to [(\delta \,\&\, \varphi) \,\&\, \varepsilon \to (\delta \,\&\, \psi) \,\&\, \varepsilon]$ (b), (d), and (T)

(f) $\beta_{\delta\&\varepsilon\to(\delta\&\varphi)\&\varepsilon,\delta\&\varepsilon}(\alpha'_{\delta\&\varphi,\varepsilon}(\alpha_{\delta,\varphi}(\varphi \to \psi))) \to [\alpha'_{\delta,\varepsilon}(\varphi) \to \alpha'_{\delta,\varepsilon}(\psi)]$
(e), Lemma 3.12., and (c) □

At the end of this subsection we show how we can simplify the sets of basic deductive terms in prominent axiomatic extensions of SL. The results are summarized in Table 4; in the case of $\mathrm{SL_e}$ it follows from the second claim of Proposition 3.10., in case of logics with weakening we use the fact that the rule ($\mathrm{Adj_u}$) is redundant and the term $\star \wedge \overline{1}$ is not needed in the crucial step of the proof in Theorem 3.13.; for associative logics it implicitly follows from (Cintula & Noguera, 2011, Theorem 2.6.8), or from the following result which we add for the reader's convenience.

Corollary 3.14. $\mathrm{SL_a}$ *is almost* (MP)*-based with respect to the set*

$$\mathrm{bDT_{SL_a}} = \{\lambda_\varepsilon, \rho_\varepsilon, \star \wedge \overline{1} \mid \varepsilon \textit{ a formula}\}.$$

Proof. The fact that $\mathrm{SL_a}$ can be axiomatized by using the rules $\varphi \vdash \gamma(\varphi)$ for $\gamma \in \mathrm{bDT_{SL_a}}$ follows from claims 3, 4, and 5 of Proposition 3.10.

From the proof of the previous theorem and claim 5 of Proposition 3.10. we know that for each $\gamma \in \mathrm{bDT_{SL_a}}$ and each formulae φ, ψ there is $\gamma' \in \mathrm{bDT^*_{SL}}$ such that

$$\vdash_{\mathrm{SL}} \gamma'(\varphi \to \psi) \to (\gamma(\varphi) \to \gamma(\psi)).$$

We complete the proof by showing that for each $\gamma' \in \mathrm{bDT^*_{SL}}$ there is $\gamma_0 \in \mathrm{bDT^*_{SL_a}}$ such that for each formula χ holds:

$$\vdash_{\mathrm{SL_a}} \gamma_0(\chi) \to \gamma'(\chi).$$

The base case follows from claims 1, 3, and 4 of Proposition 3.10. The induction step then easily follows using Lemma 3.12. and claim 3 and 4 again. □

[9] During the review process we have learned (Nick Galatos, personal communication), that the three rules mentioned in Section 2.2.4 of Galatos & Ono (2010) could be used to obtain an alternative almost (MP)-based presentation of SL (for justification of our choice recall Footnote 1).

Table 4. bDT*s of prominent substructural logics*

Logic L	$\mathrm{bDT_L}$
SL	$\{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon}, \star \wedge \overline{1} \mid \delta, \varepsilon \text{ formulae}\}$
$\mathrm{SL_w}$	$\{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon} \mid \delta, \varepsilon \text{ formulae}\}$
$\mathrm{SL_e}$	$\{\alpha_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \star \wedge \overline{1} \mid \delta, \varepsilon \text{ formulae}\}$
$\mathrm{SL_{ew}}$	$\{\alpha_{\delta,\varepsilon}, \beta_{\delta,\varepsilon} \mid \delta, \varepsilon \text{ formulae}\}$
$\mathrm{SL_a}$	$\{\lambda_\varepsilon, \rho_\varepsilon, \star \wedge \overline{1} \mid \varepsilon \text{ a formula}\}$
$\mathrm{SL_{ae}}$	$\{\star \wedge \overline{1}\}$
$\mathrm{SL_{aew}}$	$\{\star\}$

3.2. Deduction theorem and filter generation In this section we prove a general form of (parameterized) local deduction theorem for almost (MP)-based substructural logics and use it to obtain a description of generated filters. To this end, we need first a few additional syntactical properties of sets of (iterated) basic deduction terms and their closures under conjunction.

DEFINITION 3.15. *Given a set* Γ *of* $\star$*-formulae, an* SL*-algebra* $\boldsymbol{A}$*, and* $X \subseteq A$*, we define*

- $\Pi(\Gamma)$ *as the smallest set of* $\star$*-formulae containing* $\Gamma \cup \{\overline{1}\}$ *and closed under* $\&$.
- $\Gamma^{\boldsymbol{A}}$ *as the set of unary polynomials built using terms from* Γ *with coefficients from* A *and variable* $\star$*, i.e.,*

$$\Gamma^{\boldsymbol{A}} = \{\delta(\star, a_1, \dots, a_n) \mid \delta(\star, p_1, \dots, p_n) \in \Gamma \text{ and } a_1, \dots, a_n \in A\}.$$

- $\Gamma^{\boldsymbol{A}}(X)$ *as the set* $\{\delta^{\boldsymbol{A}}(x) \mid \delta(\star) \in \Gamma^{\boldsymbol{A}} \text{ and } x \in X\}$.

We omit the symbol $\boldsymbol{A}$ when known from the context.

LEMMA 3.16. *Let* L *be a substructural logic and assume that it is almost* (MP)*-based with a set of basic deduction terms* bDT*. Then*

1. *for each* $\gamma \in \mathrm{bDT}^*$ *and formulae* φ, ψ *there exists* $\gamma' \in \mathrm{bDT}^*$ *such that*
$$\varphi \to \psi \vdash_{\mathrm{L}} \gamma'(\varphi) \to \gamma(\psi),$$
2. *for each* $\gamma \in \mathrm{bDT}^*$ *and formulae* φ, ψ *there exist* $\gamma_1, \gamma_2 \in \mathrm{bDT}^*$ *such that*
$$\vdash_{\mathrm{L}} \gamma_1(\varphi \to \psi) \to (\gamma_2(\varphi) \to \gamma(\psi)),$$
3. *for each* $\gamma \in \mathrm{bDT}^*$ *and formulae* φ, ψ *there exist* $\gamma_1, \gamma_2 \in \mathrm{bDT}^*$ *such that*
$$\vdash_{\mathrm{L}} \gamma_1(\varphi) \,\&\, \gamma_2(\psi) \to \gamma(\varphi \,\&\, \psi),$$
4. *for each* $\gamma \in \mathrm{bDT}^*$*,* $\delta \in \Pi(\mathrm{bDT}^*)$*, and a formula* φ *there exists* $\hat{\delta} \in \Pi(\mathrm{bDT}^*)$ *such that*
$$\vdash_{\mathrm{L}} \hat{\delta}(\varphi) \to \gamma(\delta(\varphi)).$$

Proof. We prove the first two claims at once by induction. The base case $\gamma = \star$ is trivial in both claims. Assume that $\gamma = \beta(\delta)$ for some $\beta \in \mathrm{bDT}$ and $\delta \in \mathrm{bDT}^*$. The induction assumption of the first claim gives us $\delta' \in \mathrm{bDT}^*$ such that

$$\varphi \to \psi \vdash_{\mathrm{L}} \delta'(\varphi) \to \delta(\psi).$$

Now we use the definition of bDT for $\delta'(\varphi)$ and $\delta(\psi)$ and obtain $\beta_1, \beta_2 \in$ bDT* such that:

$$\vdash_{\mathrm{L}} \beta_1(\delta'(\varphi) \to \delta(\psi)) \to (\beta_2(\delta'(\varphi)) \to \beta(\delta(\psi))).$$

Thus if we set $\gamma' = \beta_2(\delta')$ the proof of the first claim is done (just observe that $\varphi \to \psi \vdash_{\mathrm{L}} \beta_1(\delta'(\varphi) \to \delta(\psi)))$.

In the second claim, assuming again that $\gamma = \beta(\delta)$ for some $\beta \in$ bDT and $\delta \in$ bDT*, the induction assumption gives us $\delta_1, \delta_2 \in$ bDT* such that

$$\vdash_{\mathrm{L}} \delta_1(\varphi \to \psi) \to (\delta_2(\varphi) \to \delta(\psi)),$$

Now we use the definition of bDT for $\delta_2(\varphi)$ and $\delta(\psi)$ and obtain $\beta_1, \beta_2 \in$ bDT* such that:

$$\vdash_{\mathrm{L}} \beta_1(\delta_2(\varphi) \to \delta(\psi)) \to (\beta_2(\delta_2(\varphi)) \to \beta(\delta(\psi))).$$

Now we apply the first claim for $\gamma = \beta_1$, $\varphi = \delta_1(\varphi \to \psi)$, $\psi = \delta_2(\varphi) \to \delta(\psi)$ and obtain $\beta_1' \in$ bDT* such that

$$\vdash_{\mathrm{L}} \beta_1'(\delta_1(\varphi \to \psi)) \to \beta_1(\delta_2(\varphi) \to \delta(\psi)).$$

Transitivity and setting $\gamma_1 = \beta_1'(\delta_1)$ and $\gamma_2 = \beta_2(\delta_2)$ completes the proof of the second claim.

To prove the third claim we use the second one for $\psi = \varphi \,\&\, \psi$ and obtain $\gamma_1, \gamma_2 \in$ bDT*

$$\vdash_{\mathrm{L}} \gamma_1(\varphi \to \varphi \,\&\, \psi) \to (\gamma_2(\varphi) \to \gamma(\varphi \,\&\, \psi)).$$

Since $\vdash_{\mathrm{L}} \psi \to (\varphi \to \varphi \,\&\, \psi)$ (Adj$_{\&}$) we can use the first claim for $\gamma = \gamma_1$ to obtain $\gamma_1' \in$ bDT* such that

$$\vdash_{\mathrm{L}} \gamma_1'(\psi) \to \gamma_1(\varphi \to \varphi \,\&\, \psi).$$

Claim 3 then simply follows by (T) and (Res$_1$).

To prove the last claim we proceed by induction via the depth of the tree representing δ. If $\delta \in$ bDT* or $\delta = \overline{1}$, the proof is done by setting $\hat{\delta} = \gamma(\delta)$ or $\hat{\delta} = \overline{1}$ respectively. Next assume that $\delta = \eta_1 \,\&\, \eta_2$ for some $\eta_1, \eta_2 \in \Pi(\mathrm{bDT}^*)$. By the third claim we obtain $\gamma_1, \gamma_2 \in$ bDT* such that $\vdash_{\mathrm{L}} \gamma_1(\eta_1(\varphi)) \,\&\, \gamma_2(\eta_2(\varphi)) \to \gamma(\eta_1(\varphi) \,\&\, \eta_2(\varphi))$. Then, by the induction assumption, we obtain $\hat{\delta}_1, \hat{\delta}_2 \in \Pi(\mathrm{bDT}^*)$ such that $\vdash_{\mathrm{L}} \hat{\delta}_1(\varphi) \to \gamma_1(\eta_1(\varphi))$ and $\vdash_{\mathrm{L}} \hat{\delta}_2(\varphi) \to \gamma_2(\eta_2(\varphi))$. Setting $\hat{\delta} = \hat{\delta}_1 \,\&\, \hat{\delta}_2$ completes the proof using (P$_{\mathrm{SL}}$10). □

We are ready now to prove a semantical (or transferred) version of (parameterized) local deduction theorem, cf. (Cintula & Noguera, 2011, Theorem 2.6.3).

Theorem 3.17. *Let* L *be an almost* (MP)*-based substructural logic with a set of basic deduction terms* bDT. *Let* $\boldsymbol{A}$ *be an* $\mathcal{L}_{\mathrm{SL}}$*-algebra and* $X \cup \{x\} \subseteq A$. *Then* $y \in \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X, x)$ *iff* $\gamma^{\boldsymbol{A}}(x)\backslash y \in \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X)$ *for some* $\gamma \in (\Pi(\mathrm{bDT}^*))^{\boldsymbol{A}}$.

Proof. Right-to-left direction: clearly $\gamma(x) \in \mathrm{Fi}(X, x)$ (because $\varphi \vdash \gamma_0(\varphi)$ for each $\gamma_0 \in$ bDT*, $\varphi, \psi \vdash \varphi \& \psi$ and $\mathrm{Fi}(X, x)$ is closed under the rules of L). Since $\mathrm{Fi}(X, x)$ is closed under *modus ponens* we obtain that $y \in \mathrm{Fi}(X, x)$.

To prove the other direction let us take $y \in \mathrm{Fi}(X, x)$, we show that for each a in a proof of y from the assumptions $X \cup \{x\}$ (recall Proposition 2.5.) there is $\gamma_a \in \Pi(\mathrm{bDT}^*)$ such that $\gamma_a(x)\backslash a \in \mathrm{Fi}(X)$. If $a = x$ we set $\gamma_a = \star$; if a is in X or is the value of some axiom we set $\gamma_a = \overline{1}$.

Assume that a is obtained by *modus ponens* from $b \in \mathrm{Fi}(X,x)$ and $b\backslash a \in \mathrm{Fi}(X,x)$. By induction hypothesis, we obtain $\gamma_b, \gamma_{b\backslash a} \in \Pi(\mathrm{bDT}^*)$ such that $\gamma_{b\backslash a}(x)\backslash(b\backslash a)$, $\gamma_b(x)\backslash b \in \mathrm{Fi}(X)$. Therefore (using (Sf)) we have $(b\backslash a)\backslash(\gamma_b(x)\backslash a) \in \mathrm{Fi}(X)$ and by transitivity $\gamma_{b\backslash a}(x)\backslash(\gamma_b(x)\backslash a) \in \mathrm{Fi}(X)$. The proof is done by setting $\gamma_a = \gamma_b \cdot \gamma_{b\backslash a}$ and using residuation.

Assume that $a = \beta(b)$ from some $\beta \in \mathrm{bDT}$ and is obtained from $b \in \mathrm{Fi}(X,x)$ by the rule $\varphi \vdash \beta(\varphi)$. By the induction hypothesis, we have $\gamma_b \in \Pi(\mathrm{bDT}^*)$ such that $\gamma_b(x)\backslash b \in \mathrm{Fi}(X)$. Using the first claim of Lemma 3.16. we obtain $\gamma \in \mathrm{bDT}^*$ such that $\gamma(\gamma_b(x))\backslash\beta(b) \in \mathrm{Fi}(X)$. Using the fourth claim of Lemma 3.16. we obtain $\hat{\gamma_b} \in \Pi(\mathrm{bDT}^*)$ such that $\hat{\gamma_b}(x)\backslash\gamma(\gamma_b(x)) \in \mathrm{Fi}(X)$ and so transitivity completes the proof. □

This theorem has two important consequences; the first one is a straightforward corollary in the particular case when $\boldsymbol{A}$ is the algebra of formulae and recalling that in this case $\varphi \in \mathrm{Fi}(\Gamma)$ iff $\Gamma \vdash_{\mathrm{L}} \varphi$.

COROLLARY 3.18. (Local Deduction theorem) *Let* L *be an almost* (MP)*-based substructural logic with a set of basic deduction terms* bDT*. Then for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae the following holds:*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathrm{L}} \gamma(\varphi) \to \psi \textit{ for some } \gamma \in \Pi(\mathrm{bDT}^*).$$

Therefore, we obtain a (parameterized or non-parameterized, depending on the presence of parameters in the set bDT) local deduction theorem for SL and its axiomatic extensions (sometimes with a simplified set bDT; see Table 4). On the other hand, Theorem 3.17. can be used to obtain the following algebraic description of the filter generated by a set.

COROLLARY 3.19. (Filter generation) *Let* L *be an almost* (MP)*-based substructural logic with a set of basic deduction terms* bDT*. Let* $\boldsymbol{A}$ *be an* L*-algebra and* $X \subseteq A$*. Then* $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X) = \{a \in A \mid a \geq x \textit{ for some } x \in (\Pi(\mathrm{bDT}^*))^{\boldsymbol{A}}(X)\}$.

Proof. Clearly $\mathrm{bDT}^*(X) \subseteq \mathrm{Fi}(X)$ (because $\varphi \vdash \gamma(\varphi)$ for each $\gamma \in \mathrm{bDT}^*$ and $\mathrm{Fi}(X)$ is closed under the rules of L). Furthermore we obtain $(\Pi(\mathrm{bDT}^*))^{\boldsymbol{A}}(X) \subseteq \mathrm{Fi}(X)$ from $\varphi, \psi \vdash \varphi \,\&\, \psi$. Finally take $x \in (\Pi(\mathrm{bDT}^*))^{\boldsymbol{A}}(X)$. We know that $a \geq x$ implies that $x\backslash a \geq \overline{1}$ and so $x\backslash a \in \mathrm{Fi}(X)$. Thus the closedness of $\mathrm{Fi}(X)$ under *modus ponens* completes the proof of one direction.

To prove the other inclusion assume that $a \in \mathrm{Fi}(X)$. There has to be a finite set $\{x_1, \dots x_n\} = X' \subseteq X$ such that $a \in \mathrm{Fi}(X')$ (due to Proposition 2.5.). Repeated use of the previous theorem gives us $\gamma_1, \dots, \gamma_n \in (\Pi(\mathrm{bDT}^*))^{\boldsymbol{A}}$ such that

$$\gamma_n(x_n) \cdot (\dots \cdot \gamma_1(x_1))\dots)\backslash a = \gamma_1(x_1)\backslash(\gamma_2(x_2)\backslash \dots (\gamma_n(x_n)\backslash a)\dots) \in \mathrm{Fi}(\emptyset) = \\ = \{x \mid x \geq \overline{1}\}.$$

Therefore $a \geq x$ for $x = \gamma_n(x_n) \cdot (\dots \cdot \gamma_1(x_1))\dots) \in (\Pi(\mathrm{bDT}^*))^{\boldsymbol{A}}(X)$. □

3.3. Proof by Cases Property and its applications In Abstract Algebraic Logic, the classical Proof by Cases Property:

$$\frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, \varphi \vee \psi \vdash \chi}$$

has inspired a systematical study of disjunction connectives, by means of a generalized form of the meta-rule which leads to a generalized notion of disjunction.

Following the notation and terminology from Cintula & Noguera (2013), given an $\mathcal{L}_{\mathrm{SL}}$-algebra $\boldsymbol{A}$, sets $X, Y \subseteq A$, and a set of formulae $\nabla(p, q, \vec{r})$ in two variables p, q and possibly parameters $\vec{r}$ we define

$$X \nabla^{\boldsymbol{A}} Y = \{\delta^{\boldsymbol{A}}(x, y, a_1, \ldots, a_n) \mid \delta(p, q, p_1, \ldots, p_n) \in \nabla, x \in X, y \in Y, \text{ and } a_i \in A\}.$$

Again, we omit the symbol $\boldsymbol{A}$ when known from the context. Finally, we set one more convention: we write $\Gamma \vdash \Delta$ instead of $\Gamma \vdash \psi$ for each $\psi \in \Delta$.

DEFINITION 3.20. *Given a logic* L*, a set of formulae* $\nabla(p, q, \vec{r})$ *is called a* p-disjunction *(in* L*) whenever it satisfies the* p-protodisjunction *condition*

$$\text{(PD)} \qquad \varphi \vdash_{\mathrm{L}} \varphi \nabla \psi \quad \textit{and} \quad \psi \vdash_{\mathrm{L}} \varphi \nabla \psi.$$

and the Proof by Cases Property*,* PCP *for short:*

$$\frac{\Gamma, \varphi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \vdash_{\mathrm{L}} \chi}{\Gamma, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi}.$$

If ∇ *has no parameters we drop the prefix 'p-'. A logic* L *is called* (p-)disjunctional *if there is a (p-)disjunction in* L*.*

We know from (Czelakowski, 2001, Theorem 2.5.17) that every finitary protoalgebraic distributive logic is p-disjunctional. Therefore, from this result we could already obtain that SL and its axiomatic extensions are p-disjunctional. Indeed algebraizable logics form a subclass of protoalgebraic logics and, as we mentioned in the preliminaries, SL and its axiomatic extensions are distributive. However, here we can do better by providing an explicit, reasonably simple, description of the p-disjunction, which then can be used to obtain many consequences by applying general AAL theorems. Another advantage of our approach is that it is applicable to all substructural logics, not just to axiomatic extensions of SL.

Our approach is based on (Cintula & Noguera, 2011, Theorem 2.6.9) which shows that, under certain conditions, an almost (MP)-based presentation of a substructural logic can be used to obtain a p-disjunction. Here we prove a stronger version of that theorem by removing those conditions at the price of a (seemingly) slightly more complicated form of the resulting p-disjunction. By 'seemingly' we mean that in the majority of substructural logics we study in this paper this complication is actually nonexistent.

THEOREM 3.21. *Let* L *be an almost* (MP)*-based substructural logic with a set of basic deduction terms* bDT*. Then the following set is a p-disjunction in* L*:*

$$\nabla_{\mathrm{L}} = \{\gamma_1(p) \vee \gamma_2(q) \mid \gamma_1, \gamma_2 \in (\mathrm{bDT} \cup \{\star \wedge \overline{1}\})^*\}$$

Proof. Clearly the set $\mathrm{bDT} \cup \{\star \wedge \overline{1}\}$ is a set of basic deduction terms (because already the logic SL proves ($\mathrm{Adj_u}$) and ($\mathrm{P_{SL}}$24)). Therefore ∇_{L} obviously satisfies the condition (PD); we prove that it satisfies PCP as well.

Assume that $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{L}} \chi$. From Corollary 3.18. we obtain $\delta_\varphi, \delta_\psi \in \Pi((\mathrm{bDT} \cup \{\star \wedge \overline{1}\})^*)$ such that $\Gamma \vdash_{\mathrm{L}} \delta_\varphi(\varphi) \to \chi$ and $\Gamma \vdash_{\mathrm{L}} \delta_\psi(\psi) \to \chi$. Thus also $\Gamma \vdash_{\mathrm{L}} \delta_\varphi(\varphi) \wedge \overline{1} \to \chi$ and $\Gamma \vdash_{\mathrm{L}} \delta_\psi(\psi) \wedge \overline{1} \to \chi$ (due to ($\wedge$1) and (T)) and so, without a loss of generality, we might assume that the outmost term in δ_φ and δ_ψ is $\star \wedge \overline{1}$ and so we have $\vdash_{\mathrm{L}} \delta(\varphi) \,\&\, \psi \to \psi$ and $\vdash_{\mathrm{L}} \psi \,\&\, \delta(\varphi) \to \psi$ (due to ($\mathrm{P_{SL}}$22) and ($\mathrm{P_{SL}}$23)) for both $\delta = \delta_\varphi$ and $\delta = \delta_\psi$.

Table 5. *(p-)disjunctions in prominent substructural logics*

Logic L	$\mathrm{bDT_L}$	(p-)disjunction ∇_L in L
SL	$\{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon}, \star \wedge \overline{1}\} \mid \delta, \varepsilon$ formulae$\}$	$\{\gamma_1(p) \vee \gamma_2(q) \mid \gamma_1, \gamma_2 \in \mathrm{bDT_{SL}}\}$
$\mathrm{SL_a}$	$\{\lambda_\varepsilon, \rho_\varepsilon, \star \wedge \overline{1} \mid \varepsilon$ a formula$\}$	$\{\gamma_1(p) \vee \gamma_2(q) \mid \gamma_1, \gamma_2 \in \mathrm{bDT_{SL_a}}\}$
$\mathrm{SL_{ae}}$	$\{\star \wedge \overline{1}\}$	$\{(\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1})\}$
$\mathrm{SL_{aew}}$	$\{\star\}$	$\{\varphi \vee \psi\}$

We also know that $\Gamma \vdash_L \delta_\varphi(\varphi) \vee \delta_\psi(\psi) \to \chi$ by ($\vee$3). The proof is done by showing by induction over the sum of the depths of the trees representing $\delta_\varphi, \delta_\psi$ that:

$$\varphi \nabla_L \psi \vdash_L \delta_\varphi(\varphi) \vee \delta_\psi(\psi).$$

The base of induction (when $\delta_\varphi, \delta_\psi \in \mathrm{bDT} \cup \{\star \wedge \overline{1}\}$) is trivial. For the induction step assume that $\delta_\psi = \delta_1 \mathbin{\&} \delta_2$. Using ($\mathrm{P_{SL}}$20), ($\mathrm{P_{SL}}$21), ($\vee$1), ($\vee$2), and ($\vee$3) we obtain the following chain of implications:

$$(\delta_\varphi(\varphi) \vee \delta_1(\psi)) \mathbin{\&} (\delta_\varphi(\varphi) \vee \delta_2(\psi)) \to$$

$$\to [\delta_\varphi(\varphi) \mathbin{\&} \delta_\varphi(\varphi)] \vee [\delta_\varphi(\varphi) \mathbin{\&} \delta_2(\psi)] \vee [\delta_1(\psi) \mathbin{\&} \delta_\varphi(\varphi)] \vee [\delta_1(\psi) \mathbin{\&} \delta_2(\psi)] \to$$

$$\to \delta_\varphi(\varphi) \vee \delta_\varphi(\varphi) \vee \delta_\varphi(\varphi) \vee [\delta_1(\psi) \mathbin{\&} \delta_2(\psi)] \to \delta_\varphi(\varphi) \vee \delta_\psi(\psi).$$

The induction assumption used for $\delta_\varphi(\varphi) \vee \delta_1(\psi)$ and $\delta_\varphi(\varphi) \vee \delta_2(\psi)$ together with ($\mathrm{Adj}_\&$) completes the proof. □

If bDT^* contains a formula δ such that $\vdash_L \delta \leftrightarrow \star \wedge \overline{1}$ (which is the case in all axiomatic extensions of SL) we can omit the extra formula $\star \wedge \overline{1}$ from the formulation of the above theorem. Therefore we can simplify the description of p-disjunctions in these logics; see Table 5 (also note that for each $\gamma \in \mathrm{bDT_{SL_{ae}}}$ we have, using ($\mathrm{P_{SL}}$28), $\vdash_{\mathrm{SL_{ae}}} \gamma \leftrightarrow \star \wedge \overline{1}$).

Let us now present the promised applications of having a p-disjunction in a logic. We start with the description of intersections of filters.

THEOREM 3.22. ((Cintula & Noguera, 2013, Theorem 4.7)) *For each* SL-*algebra* $\boldsymbol{A}$ *and each* $X, Y \subseteq A$ *we have* $\mathrm{Fi}(X) \cap \mathrm{Fi}(Y) = \mathrm{Fi}(X \nabla_{\mathrm{SL}}^{\boldsymbol{A}} Y)$.

Of course, if $\boldsymbol{A}$ is in a subquasivariety of some substructural logic with a simpler p-disjunction ∇, this result can be accordingly simplified.

The second application concerns the axiomatization of substructural logics given by special classes of SL-algebras. Recall that in first-order logic a positive clause C is a disjunction of finitely-many atomic formulae. We define *a positive equational clause* as a disjunction of finitely-many equations $C = \bigvee_{i \in \mathcal{I}_C} \delta_i \approx \varepsilon_i$. A set of positive equational clauses $\mathcal{C}$ is said to be valid in an SL-algebra $\boldsymbol{A}$, written as $\boldsymbol{A} \models \mathcal{C}$, if for each $C \in \mathcal{C}$ and each $\boldsymbol{A}$-evaluation e there is $i \in \mathcal{I}_C$ such that $e(\delta_i) = e(\varepsilon_i)$; a set of algebras satisfying a certain set of positive equational clauses is called a *positive universal class*. Theorem 3.23. shows how to axiomatize substructural logics given by positive equational classes of SL-algebras.

THEOREM 3.23. *Let* L *be a substructural logic with a p-disjunction* ∇ *and* $\mathcal{C}$ *a set of positive equational clauses. Then:*

$$\models_{\{\boldsymbol{A}\in\mathbb{L}\mid\boldsymbol{A}\models\mathcal{C}\}} = \mathrm{L} + \bigcup\{\nabla_{i\in\mathcal{I}_C}(\delta_i \leftrightarrow \varepsilon_i) \mid C \in \mathcal{C}\}.$$

Proof. Direct application of (Cintula & Noguera, 2013, Theorem 5.7). □

Note that if the set of positive equational clauses is recursive, so it is the axiomatization of its corresponding logic. As a corollary we obtain a way to axiomatize intersections of axiomatic extensions of a given logic (again, see Cintula & Noguera (2013) for the detailed general formulation).

COROLLARY 3.24. *Let* L *be a substructural logic with a p-disjunction* ∇*, and let* L_1*,* L_2 *be axiomatic extensions of* L *by sets of axioms* AX_1 *and* AX_2*, respectively. Without loss of generality we can assume that* AX_1 *and* AX_2 *are written in disjoint sets of variables. Then:*

$$\mathrm{L}_1 \cap \mathrm{L}_2 = \mathrm{L} + \bigcup\{\varphi\nabla\psi \mid \varphi \in AX_1 \textit{ and } \psi \in AX_2\}.$$

Equivalently, the theorem and its corollary can be dualized as a description of the variety of SL-algebras generated by a positive universal class and as an effective method to compute equational bases for joins of relative subvarieties of a given quasivariety of SL-algebras. The following corollaries generalize the results by Galatos (2004) for classes of residuated lattices.

COROLLARY 3.25. *Let* $\mathcal{C}$ *be a set of positive equational clauses. Then an equational base for the variety of* SL*-algebras generated by those satisfying* $\mathcal{C}$ *can be obtained by adding the following:*

$$\overline{1} \approx \overline{1} \wedge [\nabla_{i\in\mathcal{I}_C}(\delta_i \leftrightarrow \varepsilon_i)] \textit{ for each } C \in \mathcal{C}.$$

COROLLARY 3.26. *Let* $\mathbb{L}$ *be a quasivariety of* SL*-algebras,* ∇ *a p-disjunction for the corresponding logic, and let* $\mathbb{L}_1$*,* $\mathbb{L}_2$ *be relative subvarieties of* $\mathbb{L}$ *given by sets of equations* $\mathcal{E}_1$ *and* $\mathcal{E}_2$*, respectively. Without loss of generality we can assume that* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *are written in disjoint sets of variables. Then:*

$$\mathbb{L}_1 \vee \mathbb{L}_2 = \mathbb{L} + \bigcup\{((\delta_1 \leftrightarrow \varepsilon_1)\nabla(\delta_2 \leftrightarrow \varepsilon_2)) \wedge \overline{1} \approx \overline{1} \mid \delta_1 \approx \varepsilon_1 \in \mathcal{E}_1 \textit{ and } \delta_2 \approx \varepsilon_2 \in \mathcal{E}_2\}.$$

Observe that this result can be generalized to joins of finitely-many relative subvarieties (as well as the previous one extends to intersection of finitely-many axiomatic extensions). In particular, we obtain that the join of finitely-many recursively based relative subvarieties is recursively based.

§4. Semilinear substructural logics This section is devoted to semilinear extensions of substructural logics. The notion of semilinear logic has been introduced in the very general setting of weakly p-implicational logics by Cintula & Noguera (2010) and systematically used as a general framework for the study of mathematical fuzzy logic by Cintula & Noguera (2011). Let us first recall four equivalent (in the present context) definitions of semilinear logic: the first one is the original definition (Cintula & Noguera, 2011, Definition 3.1.2), the second one is purely semantical and stands behind the name 'semilinear' as explained in the introduction, the third one is a purely syntactical characterization called Semilinear

Property SLP, and the last one is also syntactical and it is based on the well-known prelinearity axiom and the behavior of lattice disjunction as a proper disjunction (the equivalence of the first three follows from (*Ibid.*, Theorem 3.1.8) while the equivalence with the last condition follows from (*Ibid.*, Proposition 3.2.9)).

DEFINITION 4.27. *Let* L *be a substructural logic and* $\mathbb{K}$ *the class of all* L*-chains. We say that* L *is* semilinear *if one of the following equivalent conditions is met:*

1. *For every set of formulae* Γ *and every formula* φ *we have:*
$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \textit{if, and only if,} \quad \Gamma \models_{\mathbb{K}} \varphi.$$
2. $\mathbb{K}$ *is the class of all relatively finitely subdirectly irreducible* L*-algebras.*
3. *For every set of formulae* Γ *and every formulae* φ, ψ, χ *we have:*
$$\frac{\Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \to \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}$$
4. $\vee$ *satisfies prelinearity and the Proof by Cases Property in* L*, i.e.* L *proves* $(\varphi \to \psi) \vee (\psi \to \varphi)$ *and for every set of formulae* Γ *and every formulae* φ, ψ, χ *we have*
$$\frac{\Gamma, \varphi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \vdash_{\mathrm{L}} \chi}{\Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi}$$

In the first subsection we show several equivalent ways to axiomatize the minimum semilinear logic extending a given almost (MP)-based substructural logic. Generalizing the work done by Cintula & Noguera (2011) we make a heavy use of the p-disjunction to produce the axiomatization, namely we write it in terms of the corresponding set bDT. In the second subsection we prove that these semilinear extensions are also complete with respect to distinguished classes of chains, namely those over the real and the rational unit interval.

4.1. Axiomatization of semilinear extensions Although the logic L^{ℓ} is primarily defined by Cintula & Noguera (2010) as the weakest semilinear logic extending L, the next definition formalizes this notion in the form suitable for this paper by using an 'implicit' Hilbert-style axiomatic system.

DEFINITION 4.28. *Let* L *be a substructural logic and* $\mathbb{K}$ *the class of* L*-chains. We define the logic* L^{ℓ} *as the extension of* L *by*

- *axioms* $\{\varphi \mid \emptyset \models_{\mathbb{K}} \varphi\}$
- *rules* $\{\langle \Gamma, \varphi \rangle \mid \Gamma \models_{\mathbb{K}} \varphi\}$.

The general theory explained by Cintula & Noguera (2011) gives us two immediate ways how to axiomatize L^{ℓ} in some better/simpler way (assuming that L is almost (MP)-based). They appear in the next theorem as alternatives A and B. Both these alternatives have some advantages but are unnecessary complicated: the first one adds only axioms but needs to use all iterated deductive terms, whereas the other one uses only basic terms but adds new rules. We show that in the case of substructural logics we can obtain a third and a fourth alternative combining the advantages of the first two (we present these two variants because they generalize two different usual formulations appearing in the literature).

Theorem 4.29. *Let* L *be an almost* (MP)*-based substructural logic with a set* bDT *of basic deductive terms. Then* L^ℓ *is axiomatized, relatively to* L*, by any of the following four sets of axioms/rules:*

A	$\gamma_1(\varphi \to \psi) \vee \gamma_2(\psi \to \varphi)$, *for every* $\gamma_1, \gamma_2 \in (\mathrm{bDT} \cup \{\star \wedge \overline{1}\})^*$
B	$(\varphi \to \psi) \vee (\psi \to \varphi)$
	$(\varphi \to \psi) \vee \chi, \varphi \vee \chi \vdash \psi \vee \chi$
	$\varphi \vee \psi \vdash \gamma(\varphi) \vee \psi$, *for every* $\gamma \in \mathrm{bDT}$
C	$((\varphi \to \psi) \wedge \overline{1}) \vee \gamma((\psi \to \varphi) \wedge \overline{1})$, *for every* $\gamma \in \mathrm{bDT} \cup \{\star\}$
D	$(\varphi \vee \psi \to \psi) \vee \gamma(\varphi \vee \psi \to \varphi)$, *for every* $\gamma \in \mathrm{bDT} \cup \{\star \wedge \overline{1}\}$.

Proof. Let L_X (for $X \in \{A, B, C, D\}$) denote the corresponding extension of L. Using Theorem 3.21. we know that $\{\gamma_1(p) \vee \gamma_2(q) \mid \gamma_1, \gamma_2 \in (\mathrm{bDT} \cup \{\star \wedge \overline{1}\})^*\}$ is a p-disjunction in L. Therefore $L_A = L^\ell$ due to (Cintula & Noguera, 2011, Theorem 3.2.1). To show that $L_B = L^\ell$ just use (Cintula & Noguera, 2011, Proposition 3.2.9) and (Cintula & Noguera, 2011, Theorem 2.7.27).

To complete the proof we will show the following chain of inclusions: $L^\ell \supseteq L_C \supseteq L_D \supseteq L_B$. For the first one take $\gamma \in \mathrm{bDT} \cup \{\star\}$; then we have:

(a) $\varphi \to \psi \vdash_{L^\ell} ((\varphi \to \psi) \wedge \overline{1}) \vee \gamma((\psi \to \varphi) \wedge \overline{1})$ — $(\mathrm{Adj_u})$, $(\vee 1)$, and (MP)
(b) $\psi \to \varphi \vdash_{L^\ell} \gamma((\psi \to \varphi) \wedge \overline{1})$ — $(\mathrm{Adj_u})$ and $\varphi \vdash \gamma(\varphi)$
(c) $\psi \to \varphi \vdash_{L^\ell} ((\varphi \to \psi) \wedge \overline{1}) \vee \gamma((\psi \to \varphi) \wedge \overline{1})$ — (b), $(\vee 2)$, and (MP)
(d) $\vdash_{L^\ell} ((\varphi \to \psi) \wedge \overline{1}) \vee \gamma((\psi \to \varphi) \wedge \overline{1})$ — (a), (c), and SLP[10]

Next we prove the second inclusion, let us first assume that $\gamma \in \mathrm{bDT}$:

(a) $\vdash_{L_C} (\varphi \to \psi) \wedge \overline{1} \to (\varphi \vee \psi \to \psi) \vee \gamma(\varphi \vee \psi \to \varphi)$ — $(\mathrm{P_{SL}}26)$, $(\vee 1)$, and (T)
(b) $\vdash_{L_C} \gamma'((\psi \to \varphi) \wedge \overline{1}) \to \gamma(\varphi \vee \psi \to \varphi)$ — $(\mathrm{P_{SL}}27)$ and Lemma 3.16.
(c) $\vdash_{L_C} \gamma'((\psi \to \varphi) \wedge \overline{1}) \to (\varphi \vee \psi \to \psi) \vee \gamma(\varphi \vee \psi \to \varphi)$ — (b), $(\vee 2)$, and (T)
(d) $\vdash_{L_C} ((\varphi \to \psi) \wedge \overline{1}) \vee \gamma'((\psi \to \varphi) \wedge \overline{1}) \to (\varphi \vee \psi \to \psi) \vee \gamma(\varphi \vee \psi \to \varphi)$ — (a), (c), and $(\vee 3)$
(e) $\vdash_{L_C} (\varphi \vee \psi \to \psi) \vee \gamma(\varphi \vee \psi \to \varphi)$ — (d) and (MP)

The proof for $\gamma = \star \wedge \overline{1}$ is analogous: in step (b) we would set $\gamma' = \star$ and prove it using $(\mathrm{P_{SL}}27)$, $(\mathrm{Adj_u})$, $(\mathrm{P_{SL}}24)$, (MP), $(\mathrm{P_{SL}}28)$, and (T). To prove the last inclusion we first show that L_D proves prelinearity:

(a) $\vdash_{L_D} (\varphi \vee \psi \to \psi) \to (\varphi \to \psi)$ — $(\vee 1)$ and (Sf)
(b) $\vdash_{L_D} (\varphi \vee \psi \to \psi) \to (\varphi \to \psi) \vee (\psi \to \varphi)$ — (a), $(\vee 1)$, and (T)
(c) $\vdash_{L_D} (\varphi \vee \psi \to \varphi) \to (\varphi \to \psi) \vee (\psi \to \varphi)$ — analogously
(d) $\vdash_{L_D} (\varphi \vee \psi \to \varphi) \wedge \overline{1} \to (\varphi \to \psi) \vee (\psi \to \varphi)$ — (c), $(\wedge 1)$, and (T)
(e) $\vdash_{L_D} (\varphi \vee \psi \to \psi) \vee ((\varphi \vee \psi \to \varphi) \wedge \overline{1}) \to (\varphi \to \psi) \vee (\psi \to \varphi)$ — (b), (d), and $(\vee 3)$
(f) $\vdash_{L_D} (\varphi \to \psi) \vee (\psi \to \varphi)$ — (e) and (MP)

[10] Clearly, as L^ℓ is a semilinear logic we know it satisfies the Semilinear Property; see Definition 4.27.

Next we show $\varphi \vee \psi \vdash_{\mathrm{L}_D} \gamma(\varphi) \vee \psi$ for each $\gamma \in \mathrm{bDT}$:

(a) $\varphi \vee \psi \vdash_{\mathrm{L}_D} (\varphi \vee \psi \to \psi) \to \psi$ (As)
(b) $\varphi \vee \psi \vdash_{\mathrm{L}_D} (\varphi \vee \psi \to \psi) \to \gamma(\varphi) \vee \psi$ (a), ($\vee$2), and (T)
(c) $\varphi \vee \psi \vdash_{\mathrm{L}_D} (\varphi \vee \psi \to \varphi) \to \varphi$ (As)
(d) $\varphi \vee \psi \vdash_{\mathrm{L}_D} \gamma'(\varphi \vee \psi \to \varphi) \to \gamma(\varphi)$ (c) and Lemma 3.16.
(e) $\varphi \vee \psi \vdash_{\mathrm{L}_D} \gamma'(\varphi \vee \psi \to \varphi) \to \gamma(\varphi) \vee \psi$ (d), ($\vee$1), and (T)
(f) $\varphi \vee \psi \vdash_{\mathrm{L}_D} (\varphi \vee \psi \to \psi) \vee \gamma'(\varphi \vee \psi \to \varphi) \to \gamma(\varphi) \vee \psi$ (b), (e), and ($\vee$3)
(g) $\varphi \vee \psi \vdash_{\mathrm{L}_D} \gamma(\varphi) \vee \psi$ (f) and (MP)

Note that the same proof would work for $\gamma = \star \wedge \overline{1}$; only in step (d) we would set $\gamma' = \star \wedge \overline{1}$ and prove it from (c) using (Adj$_{\mathrm{u}}$), ($\mathrm{P_{SL}}$24), and (MP). Thus we know that $\varphi \vee \psi \vdash_{\mathrm{L}_D} (\varphi \wedge \overline{1}) \vee \psi$ which we use to prove $(\varphi \to \psi) \vee \chi, \varphi \vee \chi \vdash \psi \vee \chi$:

(a) $\varphi \vee \chi \vdash \chi \to \psi \vee \chi$ ($\vee$2)
(b) $\vdash_{\mathrm{L}_D} (\varphi \to \psi) \wedge \overline{1} \to (\varphi \vee \chi \to \psi \vee \chi)$ ($\mathrm{P_{SL}}$25)
(c) $\vdash_{\mathrm{L}_D} \varphi \vee \chi \to ((\varphi \to \psi) \wedge \overline{1} \rightsquigarrow \psi \vee \chi)$ ($\mathrm{E}_{\rightsquigarrow 1}$)
(d) $\varphi \vee \chi \vdash_{\mathrm{L}_D} (\varphi \to \psi) \wedge \overline{1} \to \psi \vee \chi$ (c), (MP), and (Symm$_1$)
(e) $\varphi \vee \chi \vdash_{\mathrm{L}_D} ((\varphi \to \psi) \wedge \overline{1}) \vee \chi \to \psi \vee \chi$ (a), (d), and ($\vee$3)
(f) $(\varphi \to \psi) \vee \chi \vdash_{\mathrm{L}_D} ((\varphi \to \psi) \wedge \overline{1}) \vee \chi$ see the previous paragraph
(g) $(\varphi \to \psi) \vee \chi, \varphi \vee \chi \vdash_{\mathrm{L}_D} \psi \vee \chi$ (e), (f), and (MP) □

Table 6 collects axiomatizations of important semilinear substructural logics obtained as axiomatization C from Theorem 4.29. We present them in the form of axiom schemata, sometimes altered a little for simplicity or to obtain some form known from the literature. These simplifications follow from the following few simple observations:

- In logics with weakening we use the fact that $\vdash_{\mathrm{SL_w}} \varphi \leftrightarrow \varphi \wedge \overline{1}$ to work with the axiomatization C' $(\varphi \to \psi) \vee \gamma(\psi \to \varphi)$, for every $\gamma \in \mathrm{bDT} \cup \{\star\}$.
- The axiom for $\gamma = \star \wedge \overline{1}$ can be omitted from all axiomatizations because it follows from the one for $\gamma = \star$ using ($\mathrm{P_{SL}}$28).
- The axiom for $\gamma = \star$ can be omitted from all but the last two axiomatizations because it follows from the one for $\alpha_{\overline{1},\overline{1}}$ (or $\lambda_{\overline{1}}$) using the first (or also the fifth) claim of Proposition 3.10.
- In the case of $\mathrm{SL_e}$, we first note that the proposed single formula to axiomatize $\mathrm{SL}^{\ell}_{\mathrm{e}}$ is an instance of formulae from axiomatization A. On the other hand, setting $\delta = \varepsilon = \overline{1}$ or respectively $\delta' = \varepsilon' = \overline{1}$ and using the first claim of Proposition 3.10., we obtain the remaining two axioms from axiomatization C.
- In $\mathrm{SL_a}$ we proceed analogously to the previous case.

4.2. Completeness properties Next we prove that the non-associative semilinear logics axiomatized above are not only complete with respect to the semantics of all their chains, but also with respect to some distinguished classes of chains, namely those defined over the rational and real unit interval (*standard completeness*). In fact, we will prove completeness in the following strong sense.

Table 6. *Axiomatization of* L^ℓ *for prominent substructural logics*

Logic L	additional axioms needed to axiomatize L^ℓ
SL	$((\varphi \to \psi) \wedge \overline{1}) \vee \gamma((\psi \to \varphi) \wedge \overline{1})$, for every $\gamma \in \{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon}\}$
$\mathrm{SL_w}$	$(\varphi \to \psi) \vee \gamma(\psi \to \varphi)$, for every $\gamma \in \{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon}\}$
$\mathrm{SL_e}$	$\alpha_{\delta,\varepsilon}((\varphi \to \psi) \wedge \overline{1}) \vee \beta_{\delta',\varepsilon'}((\psi \to \varphi) \wedge \overline{1})$
$\mathrm{SL_{ew}}$	$\alpha_{\delta,\varepsilon}(\varphi \to \psi) \vee \beta_{\delta',\varepsilon'}(\psi \to \varphi)$
$\mathrm{SL_a}$	$(\lambda_\varepsilon(\varphi \to \psi) \wedge \overline{1}) \vee (\rho_{\varepsilon'}(\psi \to \varphi) \wedge \overline{1})$
$\mathrm{SL_{ae}}$	$((\varphi \to \psi) \wedge \overline{1}) \vee ((\psi \to \varphi) \wedge \overline{1})$
$\mathrm{SL_{aew}}$	$(\varphi \to \psi) \vee (\psi \to \varphi)$

DEFINITION 4.30. *Let* L *be a substructural semilinear logic and let* $\mathbb{K}$ *be a class of* L*-chains. We say that* L *has the property of* strong $\mathbb{K}$-completeness*,* S$\mathbb{K}$C *for short, when for every set of formulae* $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{\mathrm{L}} \varphi$ *if, and only if,* $\Gamma \models_{\mathbb{K}} \varphi$*.*

We will need the following characterization of S$\mathbb{K}$C (given in general by (Cintula & Noguera, 2011, Theorem 3.4.6)).

THEOREM 4.31. *Let* L *be a substructural semilinear logic and let* $\mathbb{K}$ *be a class of* L*-chains. Then* L *has the* S$\mathbb{K}$C *if, and only if, every countable nontrivial* L*-chain is embeddable into a member of* $\mathbb{K}$*.*

Let $\mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$. In the light of Definition 4.30. we define the class $\mathcal{Q}$ (resp. $\mathcal{R}$) of all $\mathrm{SL_S}$-chains whose universe is the rational unit interval $\mathsf{Q} \cap [0,1]$ (resp. the real unit interval $[0,1]$). Note that if $\boldsymbol{A}$ is in $\mathcal{Q}$ or $\mathcal{R}$ then $\overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}$ need not coincide with the real numbers $0, 1$ which play the role of $\bot$ and $\top$. They coincide iff $\{\mathrm{i}, \mathrm{o}\} \subseteq \mathrm{S}$. The remaining part of the paper is devoted to the proof of the following theorem.

THEOREM 4.32. *Let* $\mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$*. Then the logic* $\mathrm{SL}^\ell_{\mathrm{S}}$ *has the* S$\mathcal{Q}$C *and* S$\mathcal{R}$C*.*

Before we prove Theorem 4.32., we introduce several auxiliary constructions which we will need in its proof. Let $\langle A, \leq\rangle$ be a chain and $a, b \in A$. We denote the fact that a is a *subcover* of b as $a \prec b$, i.e., $a \prec b$ holds iff $a < b$ and there is no $c \in A$ such that $a < c < b$. A chain $\langle A, \leq\rangle$ is said to be *dense* if $a \prec b$ does not hold for any $a, b \in A$.

Let $\mathrm{T} \subseteq \{\mathrm{i}, \mathrm{o}\}$. Suppose that we have a dpb_{T}-chain $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}, \bot, \top\rangle$ which is countable and nontrivial (i.e., it has at least two elements). We show that it is possible to extend $\boldsymbol{A}$ to a dense dpb_{T}-chain $\boldsymbol{D}$. If $\boldsymbol{A}$ is not dense then there is at least one element a which has a subcover a'. As we want to extend $\boldsymbol{A}$ so that it becomes dense, we have to fill for each such element a the gap between a and a' by a countable dense chain. This can be done by pasting a copy of rational numbers (namely $\mathsf{Q} \cap (0,1)$) into the gap between a and a' (see Figure 1). Formally we can define the set D as the following subset of $A \times (\mathsf{Q} \cap (0,1])$:

$$D = \{\langle a, 1\rangle \mid a \in A\} \cup \{\langle a, q\rangle \mid q \in \mathsf{Q} \cap (0,1) \text{ and } (\exists a' \in A) \text{ such that } a' \prec a\}\,.$$

Then the lexicographic order $\leq_{\mathrm{lex}}$ on D is a dense linear order, $\langle \top, 1\rangle$ is a top element, and $\langle \bot, 1\rangle$ is a bottom element. Thus the algebra

$$\boldsymbol{D} = \langle D, \wedge^{\boldsymbol{D}}, \vee^{\boldsymbol{D}}, \langle \overline{0}^{\boldsymbol{A}}, 1\rangle, \langle \overline{1}^{\boldsymbol{A}}, 1\rangle, \langle \bot, 1\rangle, \langle \top, 1\rangle\rangle\,,$$

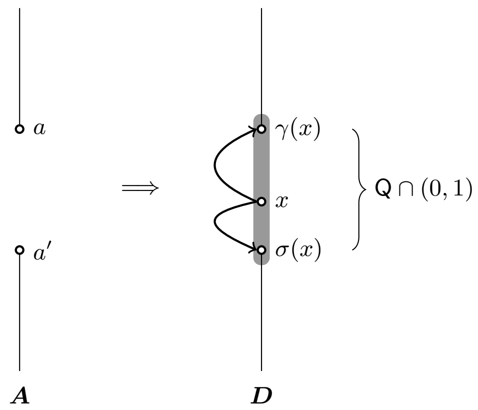

Fig. 1. The behaviour of closure and interior operators γ and σ on newly added elements.

where $\wedge^{\boldsymbol{D}}$ and $\vee^{\boldsymbol{D}}$ are defined by $\leq_{\text{lex}}$, is a *dpb*-chain. Moreover, if $\overline{1}^{\boldsymbol{A}} = \top$ then $\langle \overline{1}^{\boldsymbol{A}}, 1\rangle = \langle \top, 1\rangle$. Similarly, $\langle \overline{0}^{\boldsymbol{A}}, 1\rangle = \langle \bot, 1\rangle$ if $\overline{0}^{\boldsymbol{A}} = \bot$. Hence $\boldsymbol{D}$ is even a dpb_{T}-chain. Finally, it is clear that the subset $A \times \{1\} \subseteq D$ forms a dpb_{T}-chain isomorphic to $\boldsymbol{A}$.

Observe that we can define two operators on the chain $\boldsymbol{D}$ whose image is $A \times \{1\}$, namely a closure operator γ and an interior operator σ defined as follows:

$$\begin{aligned}\gamma(a, q) &= \langle a, 1\rangle\,,\\ \sigma(a, q) &= \begin{cases}\langle a, 1\rangle & \text{if } q = 1,\\ \langle a', 1\rangle & \text{if } q < 1 \text{ and } a' \prec a.\end{cases}\end{aligned}$$

Note that $A \times \{1\}$ is the set of γ-closed and σ-open elements. Summing up, if we identify A with $A \times \{1\}$, we obtain the following general lemma.

LEMMA 4.33. *Let* $\mathrm{T} \subseteq \{\mathrm{i}, \mathrm{o}\}$ *and let* $\boldsymbol{A}$ *be a countable nontrivial* dpb_{T}*-chain. Then* $\boldsymbol{A}$ *can be extended to a countably infinite dense* dpb_{T}*-chain* $\boldsymbol{D}$*. Moreover, there are a closure operator* γ *and an interior operator* σ *on* $\boldsymbol{D}$ *such that* $A = \gamma[D] = \sigma[D]$.

Further we introduce a sort of extension construction. Let $\mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$ and $\mathrm{T} = \mathrm{S} \setminus \{\mathrm{e}, \mathrm{c}\}$. Suppose we have a dpb_{T}-chain $\langle B, \wedge, \vee, \overline{0}, \overline{1}, \bot, \top\rangle$, a subset $A \subseteq B$ such that $\{\overline{0}, \overline{1}, \bot, \top\} \subseteq A$, and an rt_{S}-groupoid

$$\boldsymbol{A} = \langle A, \wedge, \vee, \circ^{\boldsymbol{A}}, \backslash^{\boldsymbol{A}}, /^{\boldsymbol{A}}, \overline{0}, \overline{1}, \bot, \top\rangle\,.$$

Further, assume that there are a closure operator γ and an interior operator σ on $\langle B, \wedge, \vee\rangle$ such that $\gamma[B] = \sigma[B] = A$. This means that for every $b \in B$ we can find the least $a \in A$ such that $b \leq a$ (namely $\gamma(b)$) and the greatest $a' \in A$ such that $a' \leq b$ (namely $\sigma(b)$). We define an algebra $\boldsymbol{B} = \langle B, \wedge, \vee, \circ^{\boldsymbol{B}}, \backslash^{\boldsymbol{B}}, /^{\boldsymbol{B}}, \overline{0}, \overline{1}, \bot, \top\rangle$ as follows:

$$x \circ^{\boldsymbol{B}} y = \gamma(x) \circ^{\boldsymbol{A}} \gamma(y)\,, \qquad x /^{\boldsymbol{B}} y = \sigma(x) /^{\boldsymbol{A}} \gamma(y)\,, \qquad x \backslash^{\boldsymbol{B}} y = \gamma(x) \backslash^{\boldsymbol{A}} \sigma(y)\,.$$

LEMMA 4.34. *The algebra* $\boldsymbol{B}$ *is an* rt_{S}*-groupoid.*

Proof. First, we prove that $\boldsymbol{B}$ is residuated. Suppose that $x \circ^{\boldsymbol{B}} y = \gamma(x) \circ^{\boldsymbol{A}} \gamma(y) \leq z$. Since $\gamma(x) \circ^{\boldsymbol{A}} \gamma(y)$ is σ-open, we have $\gamma(x) \circ^{\boldsymbol{A}} \gamma(y) = \sigma(\gamma(x) \circ^{\boldsymbol{A}} \gamma(y)) \leq \sigma(z)$. Consequently, $x \leq \gamma(x) \leq \sigma(z)/^{\boldsymbol{A}}\gamma(y) = z/^{\boldsymbol{B}}y$. Conversely, suppose that $x \leq z/^{\boldsymbol{B}}y = \sigma(z)/^{\boldsymbol{A}}\gamma(y)$. Since $\sigma(z)/^{\boldsymbol{A}}\gamma(y)$ is γ-closed, we obtain $\gamma(x) \leq \gamma(\sigma(z)/^{\boldsymbol{A}}\gamma(y)) = \sigma(z)/^{\boldsymbol{A}}\gamma(y)$. Consequently, $x \circ^{\boldsymbol{B}} y = \gamma(x) \circ^{\boldsymbol{A}} \gamma(y) \leq \sigma(z) \leq z$. Analogously for the left division. Finally, note that

$$\overline{1} \circ^{\boldsymbol{B}} x = \gamma(\overline{1}) \circ^{\boldsymbol{A}} \gamma(x) = \overline{1} \circ^{\boldsymbol{A}} \gamma(x) \geq \gamma(x) \geq x\,.$$

Similarly, $x \circ^{\boldsymbol{B}} \overline{1} \geq x$. Thus $\boldsymbol{B}$ is an *rt*-groupoid.

Next we have to show that $\boldsymbol{B}$ is in fact an rt_{S}-groupoid. To see this, note that $\boldsymbol{B}$ is commutative if $\boldsymbol{A}$ is. If $\boldsymbol{A}$ is contractive then we have $x \circ^{\boldsymbol{B}} x = \gamma(x) \circ^{\boldsymbol{A}} \gamma(x) \geq \gamma(x) \geq x$ for any $x \in B$. □

Note that this extension construction does not preserve unitality of $\overline{1}$. Namely, if $\boldsymbol{A}$ is unital, the algebra $\boldsymbol{B}$ is in general only semiunital. In order to fix this, we introduce the following construction allowing us to combine two rt_{S}-groupoids on the same chain together. Let $\mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$ and $\mathrm{T} = \mathrm{S} \setminus \{\mathrm{e}, \mathrm{c}\}$. Assume that we have two different $r\ell_{\mathrm{S}}$-groupoid structures on a dpb_{T}-chain $\langle A, \wedge, \vee, \overline{0}, \overline{1}, \bot, \top\rangle$, i.e., we have two rt_{S}-groupoids

$$\boldsymbol{A}_1 = \langle A, \wedge, \vee, \circ^{\boldsymbol{A}_1}, \backslash^{\boldsymbol{A}_1}, /^{\boldsymbol{A}_1}, \overline{0}, \overline{1}, \bot, \top\rangle \quad \boldsymbol{A}_2 = \langle A, \wedge, \vee, \circ^{\boldsymbol{A}_2}, \backslash^{\boldsymbol{A}_2}, /^{\boldsymbol{A}_2}, \overline{0}, \overline{1}, \bot, \top\rangle.$$

Then we define an algebra $\boldsymbol{A}_1 \wedge \boldsymbol{A}_2 = \langle A, \wedge, \vee, \circ, \backslash, /, \overline{0}, \overline{1}, \bot, \top\rangle$ on the same dpb_{T}-chain as follows:

$$a \circ b = (a \circ^{\boldsymbol{A}_1} b) \wedge (a \circ^{\boldsymbol{A}_2} b)\,, \quad a\backslash b = (a\backslash^{\boldsymbol{A}_1} b) \vee (a\backslash^{\boldsymbol{A}_2} b)\,, \quad a/b = (a/^{\boldsymbol{A}_1} b) \vee (a/^{\boldsymbol{A}_2} b)\,.$$

Lemma 4.35. *The algebra* $\boldsymbol{A}_1 \wedge \boldsymbol{A}_2$ *is an* rt_{S}*-groupoid. In addition, if one of* $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ *is an* SL_{S}*-chain then* $\boldsymbol{A}_1 \wedge \boldsymbol{A}_2$ *is an* SL_{S}*-chain as well.*

Proof. First, $\overline{1} \circ a = (\overline{1} \circ^{\boldsymbol{A}_1} a) \wedge (\overline{1} \circ^{\boldsymbol{A}_2} a) \geq a \wedge a = a$. Similarly, $a \leq a \circ \overline{1}$. Further we have the following chain of equivalences:

$$\begin{aligned} a \circ b = (a \circ^{\boldsymbol{A}_1} b) \wedge (a \circ^{\boldsymbol{A}_2} b) \leq c \quad &\text{iff} \quad a \circ^{\boldsymbol{A}_1} b \leq c \text{ or } a \circ^{\boldsymbol{A}_2} b \leq c \\ &\text{iff} \quad b \leq a\backslash^{\boldsymbol{A}_1} c \text{ or } b \leq a\backslash^{\boldsymbol{A}_2} c \\ &\text{iff} \quad b \leq (a\backslash^{\boldsymbol{A}_1} c) \vee (a\backslash^{\boldsymbol{A}_2} c) = a\backslash c\,. \end{aligned}$$

Similarly we can prove $a \circ b \leq c$ iff $a \leq c/b$.

It is easy to see that commutativity is preserved by the construction of $\boldsymbol{A}_1 \wedge \boldsymbol{A}_2$. To see that contraction is preserved, note that $a \circ a = (a \circ^{\boldsymbol{A}_1} a) \wedge (a \circ^{\boldsymbol{A}_2} a) \geq a \wedge a = a$. Thus $\boldsymbol{A}_1 \wedge \boldsymbol{A}_2$ is an $r\ell_{\mathrm{S}}$-groupoid.

To see the additional part, assume without any loss of generality that $\boldsymbol{A}_2$ is an SL_{S}-chain. Then $a \circ^{\boldsymbol{A}_2} \overline{1} = a = \overline{1} \circ^{\boldsymbol{A}_2} a$. Thus we have $\overline{1} \circ a = (\overline{1} \circ^{\boldsymbol{A}_1} a) \wedge (\overline{1} \circ^{\boldsymbol{A}_2} a) = (\overline{1} \circ^{\boldsymbol{A}_1} a) \wedge a = a$ because $\overline{1} \circ^{\boldsymbol{A}_1} a \geq a$. Similarly, $a \circ \overline{1} = a$. □

Finally, let $\mathrm{T} \subseteq \{\mathrm{i}, \mathrm{o}\}$, $\mathrm{S} = \mathrm{T} \cup \{\mathrm{e}, \mathrm{c}\}$, and $\boldsymbol{C} = \langle C, \wedge, \vee, \overline{0}, \overline{1}, \top, \bot\rangle$ a dpb_{T}-chain. We will show that there is a greatest groupoid operation on $\boldsymbol{C}$ making $\boldsymbol{C}$ into an SL_{S}-chain $\boldsymbol{M}(\boldsymbol{C}) = \langle C, \wedge, \vee, \odot, \to, \overline{0}, \overline{1}, \bot, \top\rangle$. We define

$$x \odot y = \begin{cases} \top & \text{if } x, y > \overline{1}, \\ \bot & \text{if } x = \bot \text{ or } y = \bot, \\ x \wedge y & \text{if } x, y \leq \overline{1}, \\ x \vee y & \text{otherwise.} \end{cases}$$

LEMMA 4.36. *The algebra* $\boldsymbol{M}(\boldsymbol{C}) = \langle C, \wedge, \vee, \odot, \to, \overline{0}, \overline{1}, \bot, \top\rangle$ *is an* $\mathrm{SL_S}$*-chain, where* $\to$ *is the uniquely determined residual of* $\odot$*. Moreover,* $\odot$ *is the maximum among all groupoid operations* $\circ$ *on* $\boldsymbol{C}$ *making it into an* $\mathrm{SL_S}$*-chain w.r.t. the pointwise order, i.e.,* $x \circ y \leq x \odot y$ *for all* $x, y \in C$*.*

Proof. $\boldsymbol{M}(\boldsymbol{C})$ is clearly a dpb_{T}-chain. It is also easy to see that $\odot$ is commutative and contractive. Further, we have $\overline{1} \odot x = \overline{1} \wedge x = x$ if $x \leq \overline{1}$ and $\overline{1} \odot x = \overline{1} \vee x = x$ if $x > \overline{1}$. Thus $\overline{1}$ is a neutral element for $\odot$. In order to show that $\odot$ is residuated, it suffices (due to commutativity of $\odot$) to show that for all $a \in C$ the map $f_a \colon C \to C$ defined by $f_a(x) = a \odot x$ is residuated. Depending on a, the map f_a could be of a different shape. If $a \leq \overline{1}$ then

$$f_a(x) = \begin{cases} a & \text{if } x \in [a, \overline{1}], \\ x & \text{otherwise.} \end{cases}$$

If $a > \overline{1}$ then

$$f_a(x) = \begin{cases} \bot & \text{if } x = \bot, \\ a & \text{if } \bot < x \leq \overline{1}, \\ \top & \text{if } x > \overline{1}. \end{cases}$$

In both cases it is easy to see that f_a is residuated (it is monotone and the inverse image of any principal downset is a principal downset).

Let $\circ$ be a groupoid operation on $\boldsymbol{C}$ making it into an $\mathrm{SL_S}$-chain. Since $\overline{1}$ is a neutral element for $\circ$, we must have $x \circ y \leq x \wedge y = x \odot y$ for $x, y \leq \overline{1}$. Further, for $\bot < x \leq \overline{1}$ and $y > \overline{1}$ we must have $x \circ y \leq y = x \vee y = x \odot y$ and similarly $x \circ y \leq x \odot y$ for $x > \overline{1}$ and $\bot < y \leq \overline{1}$. Since $\circ$ is residuated, it has to satisfy $\bot \circ x = x \circ \bot = \bot = \bot \odot x = x \odot \bot$. Finally, $x \circ y \leq \top = x \odot y$ for $x, y > \overline{1}$. Thus $x \circ y \leq x \odot y$ for all $x, y \in C$. □

Proof of Theorem 4.32. Let $\mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$, $\mathrm{T} = \mathrm{S} \setminus \{\mathrm{e}, \mathrm{c}\}$. By Theorem 4.31. in order to show S$\mathcal{Q}$C for $\mathrm{SL}^{\ell}_{\mathrm{S}}$, it is sufficient to prove that each countable nontrivial $\mathrm{SL_S}$-chain $\boldsymbol{A}$ can be embedded into a countably infinite dense $\mathrm{SL_S}$-chain $\boldsymbol{D}$ because every countable infinite dense chain having a minimum $\bot$ and a maximum $\top$ is order-isomorphic to $\mathsf{Q} \cap [0, 1]$.

Suppose that we have an $\mathrm{SL_S}$-chain $\boldsymbol{A} = \langle A, \wedge, \vee, \circ^{\boldsymbol{A}}, \backslash^{\boldsymbol{A}}, /^{\boldsymbol{A}}, \overline{0}, \overline{1}, \bot, \top\rangle$ which is countable and nontrivial. Then its reduct $\langle A, \wedge, \vee, \overline{0}, \overline{1}, \bot, \top\rangle$ forming a dpb_{T}-chain can be extended to a countably infinite dense dpb_{T}-chain $\langle D, \wedge, \vee, \overline{0}, \overline{1}, \bot, \top\rangle$ by Lemma 4.33. in such a way that there are a closure operator γ and an interior operator σ on $\langle D, \wedge, \vee, \overline{0}, \overline{1}, \bot, \top\rangle$ such that $\gamma[D] = \sigma[D] = A$. The next step is to extend the multiplication on the $\mathrm{SL_S}$-chain $\boldsymbol{A}$ to $\boldsymbol{D}$. This can be done by applying Lemma 4.34. to our dense dpb_{T}-chain $\langle D, \wedge, \vee, \overline{0}, \overline{1}, \bot, \top\rangle$. Consequently, we obtain an rt_{S}-groupoid $\boldsymbol{D} = \langle D, \wedge, \vee, \circ^{\boldsymbol{D}}, \backslash^{\boldsymbol{D}}, /^{\boldsymbol{D}}, \overline{0}, \overline{1}, \bot, \top\rangle$. However, $\boldsymbol{D}$ is in general only semiunital because $\overline{1}$ need not be a neutral element. In particular, $\overline{1} \circ^{\boldsymbol{D}} x = \gamma(x) \geq x$, i.e., the result of $\overline{1} \circ^{\boldsymbol{D}} x$ could be greater than we need. Thus we have to further modify $\circ^{\boldsymbol{D}}$. By Lemma 4.36. the dpb_{T}-chain $\langle D, \wedge, \vee, \overline{0}, \overline{1}, \bot, \top\rangle$ also forms an $\mathrm{SL}_{\mathrm{T}\cup\{\mathrm{e},\mathrm{c}\}}$-chain $\boldsymbol{M}(\boldsymbol{D}) = \langle D, \wedge, \vee, \odot, \to, \overline{0}, \overline{1}, \bot, \top\rangle$ such that $\odot$ is the maximum among all residuated groupoid operations on $\boldsymbol{D}$ having $\overline{1}$ as a neutral element. Thus it seems to be natural to lessen the values of $\circ^{\boldsymbol{D}}$ by a combination with $\odot$. Namely, $\boldsymbol{D} \wedge \boldsymbol{M}(\boldsymbol{D}) = \langle D, \wedge, \vee, \circ, \backslash, /, \overline{0}, \overline{1}, \bot, \top\rangle$ is an $\mathrm{SL_S}$-chain by Lemma 4.35.

Finally, we show that $\boldsymbol{A}$ can be embedded into $\boldsymbol{D} \wedge \boldsymbol{M}(\boldsymbol{D})$. We claim that the identity map from A to D is the desired embedding. Given $x, y \in A$, $\gamma(x) = \sigma(x) = x$ and $\gamma(y) = \sigma(y) = y$. By Lemma 4.36. we have $x \circ^{\boldsymbol{A}} y \leq x \odot y$. Thus

$$x \circ y = (\gamma(x) \circ^{\boldsymbol{A}} \gamma(y)) \wedge (x \odot y) = (x \circ^{\boldsymbol{A}} y) \wedge (x \odot y) = x \circ^{\boldsymbol{A}} y\,.$$

For the right division we have

$$x/y = (\sigma(x)/^{\boldsymbol{A}}\gamma(y)) \vee (y \to x) = (x/^{\boldsymbol{A}}y) \vee (y \to x)\,.$$

Using again Lemma 4.36. together with the commutativity of $\odot$, we obtain

$$(y \to x) \circ^{\boldsymbol{A}} y \leq (y \to x) \odot y = y \odot (y \to x) \leq x\,.$$

Thus by residuation $y \to x \leq x/^{\boldsymbol{A}}y$. Consequently, we have $x/y = x/^{\boldsymbol{A}}y$. Similarly, we can prove $x\backslash y = x\backslash^{\boldsymbol{A}}y$ which finishes the proof of the S$\mathcal{Q}$C for $\mathrm{SL}^{\ell}_{\mathrm{S}}$.

Now it is easy to extend this result to S$\mathcal{R}$C using the Dedekind–MacNeille completion and again Theorem 4.31. Let $\boldsymbol{A}$ be an $\mathrm{SL_S}$-chain from $\mathcal{Q}$ and $\boldsymbol{A}'$ its lattice reduct. Then, as was shown by Galatos & Jipsen (2013), $\boldsymbol{A}$ can be embedded into an $\mathrm{SL_S}$-algebra $\boldsymbol{B}$ whose lattice reduct is the Dedekind–MacNeille completion of $\boldsymbol{A}'$. Since the Dedekind–MacNeille completion of the chain $\mathsf{Q} \cap [0,1]$ is order-isomorphic to $[0,1]$, we are done. □

A The proof of Theorem 3.9. To prove one direction we only need to know the derivability of the new rules of $\mathcal{AS}$ in SL (all its axioms are either shown to be theorems of SL in the preliminaries or can be proved easily e.g. in the Gentzen calculus for SL). Conversely, we show that $\mathcal{AS}$ proves all axioms and rules of SL.

SL proves (α):

(a) $\vdash \chi \to (\psi \to \psi \,\&\, \chi)$ (Adj$_\&$)
(b) $\chi \vdash \psi \to \psi \,\&\, \chi$ (a) and (MP)
(c) $\chi \vdash \varphi \,\&\, \psi \to \varphi \,\&\, (\psi \,\&\, \chi)$ ($\mathrm{P_{SL}}$8), (b), and (MP)

SL proves (α'):

(a) $\vdash \chi \to (\varphi \to \varphi \,\&\, \chi)$ (Adj$_\&$)
(b) $\chi \vdash \varphi \to \varphi \,\&\, \chi$ (a) and (MP)
(c) $\chi \vdash \varphi \,\&\, \psi \to (\varphi \,\&\, \chi) \,\&\, \psi$ ($\mathrm{P_{SL}}$9), (b), and (MP)

SL proves (β):

(a) $\vdash \chi \to (\varphi \,\&\, \psi \to (\varphi \,\&\, \psi) \,\&\, \chi)$ (Adj$_\&$)
(b) $\chi \vdash \varphi \,\&\, \psi \to (\varphi \,\&\, \psi) \,\&\, \chi$ (a) and (MP)
(c) $\chi \vdash \psi \to (\varphi \to (\varphi \,\&\, \psi) \,\&\, \chi)$ (b) and (Res)

SL proves (β'):

(a) $\chi \vdash \varphi \to (\psi \to (\psi \,\&\, \varphi) \,\&\, \chi)$ (β)
(b) $\chi \vdash \psi \to (\varphi \rightsquigarrow (\psi \,\&\, \varphi) \,\&\, \chi)$ (a) and ($\mathrm{E}_{\rightsquigarrow 1}$)

$\mathcal{AS}$ proves $\chi \to \varphi, \varphi \to \psi \vdash \chi \to \psi$ (T):

(a) $\vdash (\chi \to (\chi \,\&\, (\chi \to \varphi)) \,\&\, (\varphi \to \psi)) \to (\chi \to \psi)$ (T$'$)
(b) $\varphi \to \psi \vdash (\chi \to \varphi) \to (\chi \to (\chi \,\&\, (\chi \to \varphi)) \,\&\, (\varphi \to \psi))$ (β)

(c) $\chi \to \varphi, \varphi \to \psi \vdash (\chi \to (\chi \,\&\, (\chi \to \varphi)) \,\&\, (\varphi \to \psi))$ (b) and (MP)

(d) $\chi \to \varphi, \varphi \to \psi \vdash \chi \to \psi$ (a), (c), and (MP)

$\mathcal{AS}$ proves $\varphi \to \psi \vdash (\chi \to \varphi) \to (\chi \to \psi)$ (Pf):

(a) $\vdash (\chi \to (\chi \,\&\, (\chi \to \varphi)) \,\&\, (\varphi \to \psi)) \to (\chi \to \psi)$ (T′)

(b) $\varphi \to \psi \vdash (\chi \to \varphi) \to (\chi \to (\chi \,\&\, (\chi \to \varphi)) \,\&\, (\varphi \to \psi))$ (β)

(c) $\varphi \to \psi \vdash (\chi \to \varphi) \to (\chi \to \psi)$ (a), (b), and (T)

$\mathcal{AS}$ proves $\varphi \to \psi \vdash (\chi \rightsquigarrow \varphi) \to (\chi \rightsquigarrow \psi)$ ($\mathrm{Pf}_{\rightsquigarrow}$):

(a) $\vdash (\chi \rightsquigarrow ((\chi \rightsquigarrow \varphi) \,\&\, \chi) \,\&\, (\varphi \to \psi)) \to (\chi \rightsquigarrow \psi)$ ($\mathrm{T}'_{\rightsquigarrow}$)

(b) $\varphi \to \psi \vdash (\chi \rightsquigarrow \varphi) \to (\chi \rightsquigarrow ((\chi \rightsquigarrow \varphi) \,\&\, \chi) \,\&\, (\varphi \to \psi))$ (β')

(c) $\varphi \to \psi \vdash (\chi \rightsquigarrow \varphi) \to (\chi \rightsquigarrow \psi)$ (a), (b), and (T)

$\mathcal{AS}$ proves $\varphi \to (\psi \to \chi) \vdash \psi \,\&\, \varphi \to \chi$ (Res_1):

(a) $\vdash \psi \,\&\, (\varphi \,\&\, (\varphi \to (\psi \to \chi))) \to \chi$ (Res′)

(b) $\varphi \to (\psi \to \chi) \vdash \psi \,\&\, \varphi \to \psi \,\&\, (\varphi \,\&\, (\varphi \to (\psi \to \chi)))$ (α)

(c) $\varphi \to (\psi \to \chi) \vdash \psi \,\&\, \varphi \to \chi$ (a), (b), and (T)

$\mathcal{AS}$ proves $\varphi \to (\psi \rightsquigarrow \chi) \vdash \varphi \,\&\, \psi \to \chi$ ($\mathrm{Res}_{\rightsquigarrow 1}$):

(a) $\vdash (\varphi \,\&\, (\varphi \to (\psi \rightsquigarrow \chi))) \,\&\, \psi \to \chi$ ($\mathrm{Res}'_{\rightsquigarrow}$)

(b) $\varphi \to (\psi \rightsquigarrow \chi) \vdash \varphi \,\&\, \psi \to (\varphi \,\&\, (\varphi \to (\psi \rightsquigarrow \chi))) \,\&\, \psi$ (α')

(c) $\varphi \to (\psi \rightsquigarrow \chi) \vdash \varphi \,\&\, \psi \to \chi$ (a), (b), and (T)

$\mathcal{AS}$ proves $\psi \,\&\, \varphi \to \chi \vdash \varphi \to (\psi \to \chi)$ (Res_2):

(a) $\psi \,\&\, \varphi \to \chi \vdash (\psi \to \psi \,\&\, \varphi) \to (\psi \to \chi)$ (Pf)

(b) $\psi \,\&\, \varphi \to \chi \vdash (\varphi \to (\psi \to \psi \,\&\, \varphi)) \to (\varphi \to (\psi \to \chi))$ (a), (Pf), and (MP)

(c) $\vdash \varphi \to (\psi \to \psi \,\&\, \varphi)$ ($\mathrm{Adj}_{\&}$)

(d) $\psi \,\&\, \varphi \to \chi \vdash \varphi \to (\psi \to \chi)$ (b), (c), and (MP)

$\mathcal{AS}$ proves $\psi \,\&\, \varphi \to \chi \vdash \psi \to (\varphi \rightsquigarrow \chi)$ ($\mathrm{Res}_{\rightsquigarrow 2}$):

(a) $\psi \,\&\, \varphi \to \chi \vdash (\varphi \rightsquigarrow \psi \,\&\, \varphi) \to (\varphi \rightsquigarrow \chi)$ ($\mathrm{Pf}_{\rightsquigarrow}$)

(b) $\psi \,\&\, \varphi \to \chi \vdash (\psi \to (\varphi \rightsquigarrow \psi \,\&\, \varphi)) \to (\psi \to (\varphi \rightsquigarrow \chi))$ (a), (Pf), and (MP)

(c) $\vdash \psi \to (\varphi \rightsquigarrow \psi \,\&\, \varphi)$ ($\mathrm{Adj}_{\&\rightsquigarrow}$)

(d) $\psi \,\&\, \varphi \to \chi \vdash \psi \to (\varphi \rightsquigarrow \chi)$ (b), (c), and (MP)

$\mathcal{AS}$ proves $\psi \to (\varphi \to \chi) \vdash \varphi \to (\psi \rightsquigarrow \chi)$ ($\mathrm{E}_{\rightsquigarrow 1}$):

(a) $\psi \to (\varphi \to \chi) \vdash \varphi \,\&\, \psi \to \chi$ (Res_1)

(b) $\psi \to (\varphi \to \chi) \vdash \varphi \to (\psi \rightsquigarrow \chi)$ (a) and ($\mathrm{Res}_{\rightsquigarrow 2}$)

$\mathcal{AS}$ proves $\varphi \to (\psi \rightsquigarrow \chi) \vdash \psi \to (\varphi \to \chi)$ ($\mathrm{E}_{\rightsquigarrow 2}$):

(a) $\varphi \to (\psi \rightsquigarrow \chi) \vdash \varphi \,\&\, \psi \to \chi$ ($\mathrm{Res}_{\rightsquigarrow 1}$)

(b) $\varphi \to (\psi \rightsquigarrow \chi) \vdash \psi \to (\varphi \to \chi)$ (b) and (Res_2)

$\mathcal{AS}$ proves $\varphi \to \varphi$ (R): (Push), (Pop), and (T).

$\mathcal{AS}$ proves $\overline{1} \to (\varphi \to \varphi)$ (R′):

(a) $\varphi \to \varphi \vdash \overline{1} \to (\varphi \to \varphi)$ (Push) and (MP)

(b) $\vdash \overline{1} \to (\varphi \to \varphi)$ (R) and (a)

$\mathcal{AS}$ proves $\overline{1}$ ($\overline{1}$):

(a) $\vdash (\overline{1} \to \overline{1}) \to \overline{1}$ (Pop)

(b) $\vdash \overline{1}$ (R), (a), and (MP)

$\mathcal{AS}$ proves $\varphi \to ((\varphi \rightsquigarrow \psi) \to \psi)$ ($\mathrm{As}_{\ell\ell}$):

(a) $\vdash (\varphi \rightsquigarrow \psi) \to (\varphi \rightsquigarrow \psi)$ (R)

(b) $\vdash \varphi \to ((\varphi \rightsquigarrow \psi) \to \psi)$ (a) and ($\mathrm{E}_{\rightsquigarrow 2}$)

$\mathcal{AS}$ proves $\varphi \to \psi \vdash (\psi \to \chi) \to (\varphi \to \chi)$ (Sf):

(a) $\vdash (\psi \to \chi) \to (\psi \to \chi)$ (R)

(b) $\vdash \psi \to ((\psi \to \chi) \rightsquigarrow \chi)$ (a) and ($\mathrm{E}_{\rightsquigarrow 1}$)

(c) $\varphi \to \psi \vdash \varphi \to ((\psi \to \chi) \rightsquigarrow \chi)$ (Pf), (b), and (T)

(d) $\varphi \to \psi \vdash (\psi \to \chi) \to (\varphi \to \chi)$ (c) and ($\mathrm{E}_{\rightsquigarrow 2}$)

$\mathcal{AS}$ proves $\varphi \vdash (\varphi \to \psi) \to \psi$ (As):

(a) $\varphi \vdash \overline{1} \to \varphi$ (Push) and (MP)

(b) $\varphi \vdash (\varphi \to \psi) \to (\overline{1} \to \psi)$ (a) and (Sf)

(c) $\vdash (\overline{1} \to \psi) \to \psi$ (Pop)

(d) $\varphi \vdash (\varphi \to \psi) \to \psi$ (b), (c), and (T)

$\mathcal{AS}$ proves $\varphi, \psi \vdash \varphi \wedge \psi$ (Adj):

(a) $\varphi \vdash \varphi \wedge \overline{1}$ ($\mathrm{Adj_u}$)

(b) $\psi \vdash \psi \wedge \overline{1}$ ($\mathrm{Adj_u}$)

(c) $\vdash \psi \wedge \overline{1} \to (\varphi \wedge \overline{1} \to (\varphi \wedge \overline{1}) \,\&\, (\psi \wedge \overline{1}))$ ($\mathrm{Adj}_{\&}$)

(d) $\varphi, \psi \vdash (\varphi \wedge \overline{1}) \,\&\, (\psi \wedge \overline{1})$ (a), (b), (c), and (MP)

(e) $\vdash (\varphi \wedge \overline{1}) \,\&\, (\psi \wedge \overline{1}) \to \varphi \wedge \psi$ (&∧)

(f) $\varphi, \psi \vdash \varphi \wedge \psi$ (d), (e), and (MP)

$\mathcal{AS}$ proves $\varphi \rightsquigarrow \psi \vdash \varphi \to \psi$ (Symm_1):

(a) $\varphi \rightsquigarrow \psi \vdash \overline{1} \to (\varphi \rightsquigarrow \psi)$ (Push) and (MP)

(b) $\varphi \rightsquigarrow \psi \vdash \varphi \to (\overline{1} \to \psi)$ (a) and ($\mathrm{E}_{\rightsquigarrow 2}$)

(c) $\vdash (\overline{1} \to \psi) \to \psi$ (Pop)

(d) $\varphi \rightsquigarrow \psi \vdash \varphi \to \psi$ (b), (c), and (T)

$\mathcal{AS}$ proves $(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \to (\varphi \vee \psi \rightsquigarrow \chi)$ ($\vee 3_{\rightsquigarrow}$):

(a) $\vdash (\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \to (\varphi \rightsquigarrow \chi)$ (∧1)

(b) $\vdash \varphi \to ((\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \to \chi)$ ($\mathrm{E}_{\rightsquigarrow 2}$)

(c) $\vdash \psi \to ((\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \to \chi)$ analogously

(d) $\vdash \varphi \vee \psi \to ((\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \to \chi)$ (Adj), (∨3), (MP)

(e) $\vdash (\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \to (\varphi \vee \psi \rightsquigarrow \chi)$ ($\mathrm{E}_{\rightsquigarrow 1}$)

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