PhD course on Mathematical Fuzzy Logic: 1st lesson

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Logic is the science that studies correct reasoning.

It is a part of Philosophy.

Mathematical Logic is the study of correct reasoning using mathematical tools (and the usage of Logic to give foundations to Mathematics).
It is a part of Mathematics.
It includes:

- Set theory
- Model theory
- Recursion theory or computability theory
- Proof theory
- Algebraic logic
- Constructive mathematics

The are many kinds of correct reasoning, hence many logics:

1. Classical logic
2. Non-classical logics:
    - Modal logics
    - Intuitionistic logic
    - Linear logics
    - Fuzzy logics
    - Relevance logics
    - Paraconsistent logics
    - Non-monotonic logics
    - ...

1. Propositional logics
2. Predicate logics
    - First-order logics
    - Higher-order logics

Fuzzy Logic (engineering, artificial intelligence, soft computing)

Mathematical Fuzzy Logic (mathematical logic)

Algebraic Logic is the subdiscipline of Mathematical Logic which studies logical systems (classical and non-classical) by using tools from Universal Algebra.

Logic	Algebraic counterpart
Classical logic	Boolean algebras
Modal logics	Modal algebras
Intuitionistic logic	Heyting algebras
Linear logics	Commutative residuated lattices
Fuzzy logics	Semilinear residuated lattices
Relevance logics	Commutative contractive residuated lattices
...	...

Universal Algebra is the field of Mathematics which studies algebraic structures.

- algebraic language or similarity type: $\mathcal{L} = \langle F, \tau \rangle$ (operational symbols with arity)
- countable set of variables: X
- terms: $\mathrm{Te}_{\mathcal{L}}$
  1. If $x \in X$, then $x \in \mathrm{Te}_{\mathcal{L}}$
  2. If $f \in \mathcal{L}$ n-ary and $t_1, \dots, t_n \in \mathrm{Te}_{\mathcal{L}}$, then $f(t_1, \dots, t_n) \in \mathrm{Te}_{\mathcal{L}}$
- equations: $\mathrm{Eq}_{\mathcal{L}} = \{t \approx s \mid t, s \in \mathrm{Te}_{\mathcal{L}}\}$
- quasiequations: $\mathrm{QEq}_{\mathcal{L}} = \{t_1 \approx s_1 \odot \dots \odot\ t_n \approx s_n \Rightarrow t \approx s \mid t_1, \dots, t_n, t, s_1, \dots, s_n, s \in \mathrm{Te}_{\mathcal{L}}\}$
- algebra: $\mathcal{A} = \langle A, \langle f^{\mathcal{A}} \mid f \in \mathcal{L} \rangle \rangle$, $A \neq \emptyset$ (universe), $f^{\mathcal{A}} : A^n \to A$, f n-ary.

- evaluation: $e : X \to A$
- interpretation: $t^{\mathcal{A}}(e)$, $t \in \mathrm{Te}_{\mathcal{L}}$
  1. If $x \in X$, then its interpretation is $x^{\mathcal{A}}(e) = e(x)$
  2. If $f \in \mathcal{L}$ n-ary and $t_1, \dots, t_n \in \mathrm{Te}_{\mathcal{L}}$, then the interpretation of $f(t_1, \dots, t_n) \in \mathrm{Te}_{\mathcal{L}}$ is $f(t_1, \dots, t_n)^{\mathcal{A}}(e) = f^{\mathcal{A}}(t_1^{\mathcal{A}}(e), \dots, t_n^{\mathcal{A}}(e))$.

  $e : \mathrm{Te}_{\mathcal{L}} \to A$
- satisfaction:
  1. $\mathcal{A} \models t \approx s$: for every evaluation e, $t^{\mathcal{A}}(e) = s^{\mathcal{A}}(e)$.
  2. $\mathcal{A} \models t_1 \approx s_1 \odot \dots \odot\ t_n \approx s_n \Rightarrow t \approx s$: for every evaluation e, if $t_1^{\mathcal{A}}(e) = s_1^{\mathcal{A}}(e), \dots, t_n^{\mathcal{A}}(e) = s_n^{\mathcal{A}}(e)$, then $t^{\mathcal{A}}(e) = s^{\mathcal{A}}(e)$.
- equational consequence: given a class of algebras in the same language $\mathbb{K}$ and $\Pi \cup \{t \approx s\} \subseteq \mathrm{Eq}_{\mathcal{L}}$, $\Pi \models_{\mathbb{K}} t \approx s$: for every $\mathcal{A} \in \mathbb{K}$ and every evaluation e in $\mathcal{A}$ if $p^{\mathcal{A}}(e) = q^{\mathcal{A}}(e)$ for every $p \approx q \in \Pi$, then $t^{\mathcal{A}}(e) = s^{\mathcal{A}}(e)$.

Example

$\mathcal{L} = \{\wedge, \vee\}$ binary operations. $\mathcal{A} = \langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}} \rangle$ is a lattice iff it satisfies:

- L1 $x \wedge y \approx y \wedge x$, $x \vee y \approx y \vee x$
- L2 $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$, $x \vee (y \vee z) \approx (x \vee y) \vee z$
- L3 $x \wedge x \approx x$, $x \vee x \approx x$
- L4 $x \approx x \wedge (x \vee y)$, $x \approx x \vee (x \wedge y)$

They correspond to lattice-ordered sets ($a \leq b$ iff $a \wedge^{\mathcal{A}} b = a$, $\wedge = \inf$, $\vee = \sup$).
$\mathcal{A}$ is complete iff every non-empty $X \subseteq A$ has inf and sup.
$\mathcal{A}$ is distributive iff it further satisfies $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$.
$\mathcal{A}$ is bounded iff $\mathcal{L}$ has two 0-ary symbols $\overline{1}, \overline{0}$ and $\mathcal{A}$ satisfies $x \wedge \overline{0} \approx \overline{0}$, $x \vee \overline{1} \approx \overline{1}$.

- subuniverse: non-empty $B \subseteq A$ closed under the operations of $\mathcal{A}$.
- subalgebra: $\mathcal{B} \subseteq \mathcal{A}$ if B is a subuniverse of $\mathcal{A}$, and $f^{\mathcal{B}}(b_1, \ldots, b_n) = f^{\mathcal{A}}(b_1, \ldots, b_n)$ for every $b_1, \ldots, b_n \in B$. Operator: $\mathbf{S}(\mathcal{A}) = \{$subalgebras of $\mathcal{A}\}$.
- homomorphism: $h : \mathcal{A} \to \mathcal{B}$ such that $h(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(ha_1, \ldots, ha_n)$. h is an endomorphism if $\mathcal{A} = \mathcal{B}$.
- embedding: $h : \mathcal{A} \hookrightarrow \mathcal{B}$ injective homomorphism.
- homomorphic image: $h : \mathcal{A} \twoheadrightarrow \mathcal{B}$, $\mathcal{B}$ is a homomorphic image of $\mathcal{A}$. Operator: $\mathbf{H}$.
- isomorphism: Injective and surjective homomorphism. Operator: $\mathbf{I}$.

- congruence: $\theta \subseteq A \times A$ equivalence relation such that, if $\langle a_1, b_1\rangle, \dots \langle a_n, b_n\rangle \in \theta$, then $\langle f^{\mathcal{A}}(a_1, \dots, a_n), f^{\mathcal{A}}(b_1, \dots, b_n)\rangle \in \theta$.
  Set of all congruences: $Co(\mathcal{A})$.
- quotient algebra: $\mathcal{A}/\theta$, $\pi_\theta : \mathcal{A} \twoheadrightarrow \mathcal{A}/\theta$.
- direct product: $\mathcal{A} = \prod_{i \in I} \mathcal{A}_i$,
  $f^{\mathcal{A}}(a_1, \dots, a_n)(i) = f^{\mathcal{A}_i}(a_1(i), \dots, a_n(i))$. $\pi_j : \mathcal{A} \twoheadrightarrow \mathcal{A}_j$.
  Operator: $\mathbf{P}$.
- subdirect product: $\mathcal{A}$ is a subdirect product of $\{\mathcal{A}_i \mid i \in I\}$ if there is $h : \mathcal{A} \hookrightarrow \prod_{i \in I} \mathcal{A}_i$ such that for every $i \in I$ $\pi_i \circ h : \mathcal{A} \twoheadrightarrow \mathcal{A}_i$. h is called a *subdirect representation* of $\mathcal{A}$.
  Operator: $\mathbf{P}_{SD}$.

Theorem

$\mathcal{A}$ is a subdirect product of $\{\mathcal{A}_i \mid i \in I\}$ iff there exists $\{\theta_i \mid i \in I\} \subseteq Co(\mathcal{A})$ such that $\bigcap_{i \in I} \theta_i = Id_{\mathcal{A}}$ and for every $i \in I$, $\mathcal{A}_i \cong \mathcal{A}/\theta_i$.

- (finitely) subdirectly irreducible algebra: $\mathcal{A}$ is (finitely) subdirectly irreducible iff for every (finite) subdirect representation $h : \mathcal{A} \hookrightarrow \prod_{i \in I} \mathcal{A}_i$, there is $i \in I$ such that $\pi_i \circ h$ is an isomorphism.

Corollary

$\mathcal{A}$ is (finitely) subdirectly irreducible iff for every (finite) $\{\theta_i \mid i \in I\} \subseteq Co(\mathcal{A})$ such that $\bigcap_{i \in I} \theta_i = Id_{\mathcal{A}}$, there is $i \in I$ such that $\theta_i = Id_{\mathcal{A}}$ ($Id_{\mathcal{A}}$ is (finitely) meet-irreducible).

Theorem

Every algebra is subdirectly irreducible or representable as a subdirect product of subdirectly irreducible algebras.

- reduced product: $\{\mathcal{A}_i \mid i \in I\}$, $\mathcal{F}$ filter over I ($\mathcal{F} \subseteq \mathcal{P}(I)$, $\mathcal{F} \neq \emptyset$, closed under intersection, if $X \in \mathcal{F}$ and $X \subseteq Y$ then $Y \in \mathcal{F}$). Given $a, b \in \prod_{i \in I} A_i$, $\langle a, b \rangle \in \theta_{\mathcal{F}}$ iff $\{i \in I \mid a(i) = b(i)\} \in \mathcal{F}$. $\theta_{\mathcal{F}} \in Co(\prod_{i \in I} \mathcal{A}_i)$. The reduced product modulo $\mathcal{F}$ is $\prod_{i \in I} \mathcal{A}_i / \theta_{\mathcal{F}}$. Operator: $\mathbf{P}_R$.
- ultraproduct: Reduced product where $\mathcal{F}$ is an ultrafilter ($X \in \mathcal{F}$ iff $I \setminus X \notin \mathcal{F}$). Operator: $\mathbf{P}_U$.
- variety: Class of algebras closed under $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$ (Th. Birkhoff: equational class). Generated variety: $\mathbf{V}(\mathbb{K}) = \mathbf{HSP}(\mathbb{K})$.
- quasivariety: Class of algebras closed under $\mathbf{I}$, $\mathbf{S}$ and $\mathbf{P}_R$ (Th. Mal'cev: quasiequational class). Generated quasivariety: $\mathbf{Q}(\mathbb{K}) = \mathbf{ISP}_R(\mathbb{K}) = \mathbf{ISPP}_U(\mathbb{K})$.

Moving to Logic

Basic idea of Algebraic Logic: Logical formulae can be seen as terms in an algebraic language.

- propositional language: an algebraic language $\mathcal{L}$.
- logical connectives: operational symbols in $\mathcal{L}$.
- set of formulae: $\mathrm{Fm}_{\mathcal{L}} = \mathrm{Te}_{\mathcal{L}}$.
- algebra of formulae: $\mathbf{Fm} = \langle \mathrm{Fm}_{\mathcal{L}}, \langle f^{\mathbf{Fm}} \mid f \in \mathcal{L} \rangle \rangle$.
- substitutions: Endomorphisms of $\mathbf{Fm}(X)$.
- evaluation: An $\mathcal{A}$-evaluation is any homomorphism $e : \mathbf{Fm} \to \mathcal{A}$, where $\mathcal{A}$ is any algebra of type $\mathcal{L}$.

A propositional logic is a pair $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ where $\mathcal{L}$ is a propositional language and $\vdash_{\mathrm{L}} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}) \times \mathrm{Fm}_{\mathcal{L}}$ satisfies:

1. Consequence relation:
   For every $\Gamma \cup \Delta \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,
   (a) $\varphi \vdash_{\mathrm{L}} \varphi$ (Reflexivity).
   (b) If $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathrm{L}} \varphi$ (Monotonicity).
   (c) If $\Gamma \vdash_{\mathrm{L}} \varphi$ and for every $\psi \in \Gamma$, $\Delta \vdash_{\mathrm{L}} \psi$, then $\Delta \vdash_{\mathrm{L}} \varphi$ (Cut).
2. Structural:
   For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ and $\sigma \in \mathrm{Sub}_{\mathcal{L}}$ if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$.

Basic syntactical elements

- Language: $\mathcal{L} = \{\wedge, \vee, \rightarrow, \overline{1}, \overline{0}\}$ (primitive connectives).
- Defined connectives: $\neg\varphi = \varphi \rightarrow \overline{0}$,
  $\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$

Hilbert-style calculus IPC

Axioms:

A0. $\overline{1}$
A1. $\varphi \to (\psi \to \varphi)$
A2. $\varphi \to (\psi \to \varphi \wedge \psi)$
A3. $\varphi \wedge \psi \to \varphi$
A4. $\varphi \wedge \psi \to \psi$
A5. $\varphi \to \varphi \vee \psi$
A6. $\psi \to \varphi \vee \psi$
A7. $\varphi \vee \psi \to ((\varphi \to \delta) \to ((\psi \to \delta) \to \delta))$
A8. $(\varphi \to \psi) \to ((\varphi \to (\psi \to \delta)) \to (\varphi \to \delta))$
A9. $\overline{0} \to \varphi$

Rule of inference: *modus ponens*

$$\frac{\varphi, \varphi \to \psi}{\psi}$$

Given $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,
$\Gamma \vdash_{\mathrm{IPC}} \varphi$ iff there is a proof of φ from Γ in the calculus IPC.

1. $\varphi \vdash_{\mathrm{IPC}} \varphi$ (Reflexivity)
2. If $\Gamma \subseteq \Delta$ and $\Gamma \vdash_{\mathrm{IPC}} \varphi$, then $\Delta \vdash_{\mathrm{IPC}} \varphi$ (Monotonicity)
3. If $\Gamma \vdash_{\mathrm{IPC}} \varphi$ and for all $\psi \in \Gamma$, $\Delta \vdash_{\mathrm{IPC}} \psi$, then $\Delta \vdash_{\mathrm{IPC}} \varphi$ (Cut)
4. If $\Gamma \vdash_{\mathrm{IPC}} \varphi$ and σ is a substitution, then $\sigma[\Gamma] \vdash_{\mathrm{IPC}} \sigma(\varphi)$ (Structurality)
5. If $\Gamma \vdash_{\mathrm{IPC}} \varphi$, then there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\mathrm{IPC}} \varphi$ (Finitarity)

Theorem ((Global) Deduction Theorem)

For every set of formulae $\Gamma \cup \{\varphi, \psi\}$*,*

$$\Gamma, \varphi \vdash_{\mathrm{IPC}} \psi \text{ iff } \Gamma \vdash_{\mathrm{IPC}} \varphi \to \psi$$

Semantical interpretations:

- Brower-Heyting-Kolmogorov interpretation
- Kripke frames
- Topological interpretation
- Category-theoretical interpretation
- Game semantics
- Algebraic semantics

Definition

An algebra $\mathcal{A} = \langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \to^{\mathcal{A}}, \overline{1}^{\mathcal{A}}, \overline{0}^{\mathcal{A}} \rangle$ is a Heyting algebra if

1. $\langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{1}^{\mathcal{A}}, \overline{0}^{\mathcal{A}} \rangle$ is a bounded distributive lattice
2. for every $a, b \in A$, $a \to^{\mathcal{A}} b$ is a relative pseudo-complement of a and b, i.e. for every $c \in A$
$$a \wedge^{\mathcal{A}} c \leq b \text{ iff } c \leq a \to^{\mathcal{A}} b$$

where the relation $\leq$ is the lattice ordering. $\to^{\mathcal{A}}$ is the residuum of $\wedge^{\mathcal{A}}$.

Let $\mathbb{HA}$ be the class of all Heyting algebras.

Exercise 1

Show that any complete bounded distributive lattice (in particular any finite bounded distributive lattice) is a Heyting algebra.

Exercise 2

Given a Heyting algebra $\mathcal{A}$ prove that for every $a, b, c \in A$:

1. $a \leq b$ iff $a \rightarrow^{\mathcal{A}} b = \overline{1}^{\mathcal{A}}$
2. $a = b$ iff $a \leftrightarrow^{\mathcal{A}} b = \overline{1}^{\mathcal{A}}$
3. $a \leq b \rightarrow^{\mathcal{A}} a$
4. $a \wedge^{\mathcal{A}} (a \rightarrow^{\mathcal{A}} b) \leq b$
5. $a \rightarrow^{\mathcal{A}} (b \rightarrow^{\mathcal{A}} c) \leq (a \rightarrow^{\mathcal{A}} b) \rightarrow^{\mathcal{A}} (a \rightarrow^{\mathcal{A}} c)$
6. $a \rightarrow^{\mathcal{A}} \overline{1}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$
7. $(a \rightarrow^{\mathcal{A}} c) \wedge^{\mathcal{A}} (b \rightarrow^{\mathcal{A}} c) \leq (a \vee^{\mathcal{A}} b) \rightarrow^{\mathcal{A}} c$
8. $(a \rightarrow^{\mathcal{A}} b) \wedge^{\mathcal{A}} (a \rightarrow^{\mathcal{A}} c) \leq a \rightarrow^{\mathcal{A}} (b \wedge^{\mathcal{A}} c)$
9. $a \rightarrow^{\mathcal{A}} (a \vee^{\mathcal{A}} b) = \overline{1}^{\mathcal{A}}$

Exercise 2 (cont.)

10. $\overline{0}^{\mathcal{A}} \rightarrow^{\mathcal{A}} a = \overline{1}^{\mathcal{A}}$
11. $(a \wedge^{\mathcal{A}} b) \rightarrow^{\mathcal{A}} c = a \rightarrow^{\mathcal{A}} (b \rightarrow^{\mathcal{A}} c)$
12. $\neg^{\mathcal{A}}(a \vee^{\mathcal{A}} b) = \neg^{\mathcal{A}} a \wedge^{\mathcal{A}} \neg^{\mathcal{A}} b$

Theorem

Let $\mathcal{A}$ be an $\mathcal{L}$-algebra. $\mathcal{A} \in \mathbb{HA}$ iff it satisfies:

- E1 $x \to x \approx \overline{1}$
- E2 $\overline{1} \to x \approx x$
- E3 $x \to (y \to z) \approx (x \to y) \to (x \to z)$
- E4 $(x \to y) \to ((y \to x) \to y) \approx (y \to x) \to ((x \to y) \to x)$
- E5 $x \to x \vee y \approx \overline{1}, y \to x \vee y \approx \overline{1}$
- E6 $(x \to y) \to ((y \to z) \to (x \vee y \to z)) \approx \overline{1}$
- E7 $x \wedge y \to x \approx \overline{1}, x \wedge y \to y \approx \overline{1}$
- E8 $(x \to y) \to ((x \to z) \to (x \to y \wedge z)) \approx \overline{1}$
- E9 $\overline{0} \to x \approx \overline{1}$

$\mathbb{HA}$ is a variety.

Definition

Given $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, we define the consequence relation $\Gamma \models_{\mathbb{HA}} \varphi$ iff for every $\mathcal{A} \in \mathbb{HA}$ and every $\mathcal{A}$-evaluation e: if $e[\Gamma] \subseteq \{\overline{1}^{\mathcal{A}}\}$, then $e(\varphi) = \overline{1}^{\mathcal{A}}$.

$\langle \mathcal{L}, \models_{\mathbb{HA}} \rangle$ is a finitary logic:

1. $\varphi \models_{\mathbb{HA}} \varphi$ (Reflexivity)
2. If $\Gamma \subseteq \Delta$ and $\Gamma \models_{\mathbb{HA}} \varphi$, then $\Delta \models_{\mathbb{HA}} \varphi$ (Monotonicity)
3. If $\Gamma \models_{\mathbb{HA}} \varphi$ and for all $\psi \in \Gamma$, $\Delta \models_{\mathbb{HA}} \psi$, then $\Delta \models_{\mathbb{HA}} \varphi$ (Cut)
4. If $\Gamma \models_{\mathbb{HA}} \varphi$ and σ is a substitution, then $\sigma[\Gamma] \models_{\mathbb{HA}} \sigma(\varphi)$ (Structurality)
5. If $\Gamma \models_{\mathbb{HA}} \varphi$, then there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \models_{\mathbb{HA}} \varphi$ (Finitarity)

Algebraic completeness

Theorem

For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{IPC}} \varphi$ *iff* $\Gamma \models_{\mathbb{HA}} \varphi$.

Soundness: If $\Gamma \vdash_{\mathrm{IPC}} \varphi$, then $\Gamma \models_{\mathbb{HA}} \varphi$.

- Heyting algebras satisfy the axioms of IPC.
- Heyting algebras satisfy *modus ponens*: $\mathcal{A} \in \mathbb{HA}$, e $\mathcal{A}$-evaluation. If $e(\varphi) = \overline{1}^{\mathcal{A}}$ and $e(\varphi \to \psi) = \overline{1}^{\mathcal{A}}$, then $e(\psi) = \overline{1}^{\mathcal{A}}$ (because of $\overline{1} \to x \approx x$).

Completeness: If $\Gamma \models_{\mathbb{HA}} \varphi$, then $\Gamma \vdash_{\mathrm{IPC}} \varphi$.

- A theory is a set $T \subseteq \mathrm{Fm}_{\mathcal{L}}$ closed under $\vdash_{\mathrm{IPC}}$ (if $T \vdash_{\mathrm{IPC}} \varphi$, then $\varphi \in T$). $Th(\mathrm{IPC})$: set of all theories of the logic IPC.
- Given a theory T, we define the relation $\Omega(T)$ by: $\langle \varphi, \psi \rangle \in \Omega(T)$ iff $T \vdash_{\mathrm{IPC}} \varphi \leftrightarrow \psi$.
- $\Omega(T)$ is a congruence of $\mathbf{Fm}$.
- $\Omega(T)$ is compatible with T, i.e. for every φ and ψ

  if $\langle \varphi, \psi \rangle \in \Omega(T)$ and $\varphi \in T$, then $\psi \in T$.
- Exercise 3: $\Omega(T)$ is the greatest congruence of $\mathbf{Fm}$ compatible with T.

- For every formula φ, $\langle \varphi, \overline{1} \rangle \in \Omega(T)$ iff $\varphi \in T$
- $\Omega(T)$ is the interderivability relation modulo T, i.e.:

  $$\langle \varphi, \psi \rangle \in \Omega(T) \text{ iff } T, \varphi \vdash_{\mathrm{IPC}} \psi \text{ and } T, \psi \vdash_{\mathrm{IPC}} \varphi$$
- Lindenbaum-Tarski algebra: $\mathbf{Fm}/\Omega(T)$.

Exercise 4

Check that $\mathbf{Fm}/\Omega(T)$ is a Heyting algebra.

- Assume that $\Gamma \nvdash_{\mathrm{IPC}} \varphi$
- Let T be the theory generated by Γ.
- $\varphi \notin T$.
- Consider the algebra $\mathbf{Fm}/\Omega(T)$ and the $\mathbf{Fm}/\Omega(T)$-evaluation $e(p) = p/\Omega(T)$.
- For every formula ψ, $e(\psi) = \psi/\Omega(T)$.
- $\psi/\Omega(T) = \overline{1}/\Omega(T)$ iff $\psi \in T$.
- $e[\Gamma] \subseteq \{\overline{1}^{\mathbf{Fm}/\Omega(T)}\}$ and $e(\varphi) \neq \overline{1}^{\mathbf{Fm}/\Omega(T)}$.
- $\Gamma \not\models_{\mathbb{HA}} \varphi$.

1. For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,
$$\Gamma \vdash_{\mathrm{IPC}} \varphi \text{ iff } \{\psi \approx \overline{1} \mid \psi \in \Gamma\} \models_{\mathbb{HA}} \varphi \approx \overline{1}$$
2. For every $\Pi \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}$,
$$\Pi \models_{\mathbb{HA}} \varphi \approx \psi \text{ iff } \{\alpha \leftrightarrow \beta \mid \alpha \approx \beta \in \Pi\} \vdash_{\mathrm{IPC}} \varphi \leftrightarrow \psi$$
3. For every $\varphi \in \mathrm{Fm}_{\mathcal{L}}$,
$$\varphi \vdash_{\mathrm{IPC}} \varphi \leftrightarrow \overline{1} \text{ and } \varphi \leftrightarrow \overline{1} \vdash_{\mathrm{IPC}} \varphi$$
4. For every $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$,
$$\varphi \approx \psi \models_{\mathbb{HA}} \varphi \leftrightarrow \psi \approx \overline{1} \text{ and } \varphi \leftrightarrow \psi \approx \overline{1} \models_{\mathbb{HA}} \varphi \approx \psi$$

Translations:

- $\tau : \varphi \mapsto \varphi \approx \overline{1}$
- $\rho : \alpha \approx \beta \mapsto \alpha \leftrightarrow \beta$

Heyting algebras are the equivalent algebraic semantics of IPC.

Filters

Definition

Let $\mathcal{A}$ be a Heyting algebra. A set $F \subseteq A$ is a filter iff:

1. $\overline{1} \in F$
2. if $a, b \in F$, then $a \wedge b \in F$
3. if $a \in F$ and $a \leq b$, then $b \in F$

Definition

Let $\mathcal{A}$ be a Heyting algebra. A set $F \subseteq A$ is an implicative filter iff:

1. $\overline{1} \in F$
2. if $a, a \to b \in F$, then $b \in F$

Definition

Let $\mathcal{A}$ be a Heyting algebra. A set $F \subseteq A$ is a logical filter iff for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ for every e $\mathcal{A}$-evaluation, if $\Gamma \vdash_{\mathrm{IPC}} \varphi$ and $e[\Gamma] \subseteq F$, then $e(\varphi) \in F$.

filters = implicative filters = logical filters

$\mathcal{A} \in \mathbb{HA}$, $F \subseteq A$ filter. A congruence θ of $\mathcal{A}$ is compatible with F iff:

$$\text{if } \langle a, b \rangle \in \theta \text{ and } a \in F, \text{ then } b \in F$$

Proposition

Given $\mathcal{A} \in \mathbb{HA}$ and a filter $F \subseteq A$, we define a relation:

$$\Omega_{\mathcal{A}}(F) = \{\langle a, b \rangle \in A^2 \mid a \to b, b \to a \in F\}$$

Then $\Omega_{\mathcal{A}}(F)$ is a congruence of $\mathcal{A}$, $F = \overline{1}/\Omega_{\mathcal{A}}(F)$. Moreover $\Omega_{\mathcal{A}}(F)$ is compatible with F and it is the greatest one with this property.

Proposition

Let $\mathcal{A}$ be a Heyting algebra and θ a congruence of $\mathcal{A}$. Then $\overline{1}/\theta$ is a filter of $\mathcal{A}$.

Proposition

Let $\mathcal{A}$ be a Heyting algebra and θ a congruence of $\mathcal{A}$. Then $\Omega_{\mathcal{A}}(\overline{1}/\theta) = \theta$.

Theorem

$\mathcal{A} \in \mathbb{HA}$. $\Omega_{\mathcal{A}}$ is an isomorphism between the lattice of filters and the lattice of congruences of $\mathcal{A}$.

Theorem

$\mathcal{A} \in \mathbb{HA}$, $F \subseteq A$ filter. Then for every $a, b \in A$: $\langle a, b\rangle \in \Omega_{\mathcal{A}}(F)$ iff for every formula $\varphi(x, \overrightarrow{z})$, and $\overrightarrow{c} \in A^{<\omega}$ we have $\varphi^{\mathcal{A}}(a, \overrightarrow{c}) \in F$ iff $\varphi^{\mathcal{A}}(b, \overrightarrow{c}) \in F$.

$\Omega_{\mathcal{A}}(F)$ is called the Leibniz congruence.

Axiomatic extensions of IPC: Superintuitionistic logics

- $\mathrm{S} = \mathrm{IPC} + Ax$
- $\Gamma \vdash_{\mathrm{S}} \varphi$ iff $\Gamma \cup Ax \vdash_{\mathrm{IPC}} \varphi$
- $Alg(\mathrm{S}) = \{\mathcal{A} \in \mathbb{HA} \mid \mathcal{A}$ satisfies $\tau[Ax]\}$.
- $\Pi \models_{Alg(\mathrm{S})} \alpha \approx \beta$ iff $\Pi \cup \tau[Ax] \models_{\mathbb{HA}} \alpha \approx \beta$
- We obtain the same relation between the logic and the algebraic semantics as before:
  1. $\Gamma \vdash_{\mathrm{S}} \varphi$ iff $\tau[\Gamma] \models_{Alg(\mathrm{S})} \tau(\varphi)$
  2. $\Pi \models_{Alg(\mathrm{S})} \varphi \approx \psi$ iff $\rho[\Pi] \vdash_{\mathrm{S}} \rho(\varphi \approx \psi)$
  3. $\varphi \vdash_{\mathrm{S}} \rho(\tau(\varphi))$ and $\rho(\tau(\varphi)) \vdash_{\mathrm{S}} \varphi$
  4. $\varphi \approx \psi \models_{Alg(\mathrm{S})} \tau(\rho(\varphi \approx \psi))$ and $\tau(\rho(\varphi \approx \psi)) \models_{Alg(\mathrm{S})} \varphi \approx \psi$

$Alg(\mathrm{S})$ is the equivalent algebraic semantics of S.

Classical Logic

- CPC = IPC + $\varphi \vee \neg\varphi$
- $Alg(\mathrm{CPC}) = \mathbb{BA}$ (Boolean algebras)

Boolean algebras are the equivalent algebraic semantics of classical logic.