PhD course on Mathematical Fuzzy Logic: 2nd lesson

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- $\mathrm{G} = \mathrm{IPC} + (p \to q) \vee (q \to p)$
- $\Gamma \vdash_{\mathrm{G}} \varphi$ iff $\Gamma \cup \{(p \to q) \vee (q \to p)\} \vdash_{\mathrm{IPC}} \varphi$
- $\mathbb{G} = Alg(\mathrm{G}) = \{\mathcal{A} \in \mathbb{HA} \mid \mathcal{A}$ satisfies $\tau[(p \to q) \vee (q \to p)]\}$.
- $\Pi \models_{\mathbb{G}} \alpha \approx \beta$ iff $\Pi \cup \tau[(p \to q) \vee (q \to p)] \models_{\mathbb{HA}} \alpha \approx \beta$
- We obtain the same relation between the logic and the algebraic semantics as before:
  1. $\Gamma \vdash_{\mathrm{G}} \varphi$ iff $\tau[\Gamma] \models_{\mathbb{G}} \tau(\varphi)$
  2. $\Pi \models_{\mathbb{G}} \varphi \approx \psi$ iff $\rho[\Pi] \vdash_{\mathrm{G}} \rho(\varphi \approx \psi)$
  3. $\varphi \vdash_{\mathrm{G}} \rho(\tau(\varphi))$ and $\rho(\tau(\varphi)) \vdash_{\mathrm{G}} \varphi$
  4. $\varphi \approx \psi \models_{\mathbb{G}} \tau(\rho(\varphi \approx \psi))$ and $\tau(\rho(\varphi \approx \psi)) \models_{\mathbb{G}} \varphi \approx \psi$

$\mathbb{G}$ is the equivalent algebraic semantics of G.

The elements of $\mathbb{G}$ are called G-*algebras*.

Theorem ((Global) Deduction Theorem)

For every set of formulae $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash_{\mathrm{G}} \psi \text{ iff } \Gamma \vdash_{\mathrm{G}} \varphi \to \psi$$

Lemma

Given $\mathcal{A} \in \mathbb{HA}$ and $a \in A$, let $Fi(a)$ be the filter generated by a. Then $Fi(a) = \{x \in A \mid a \leq x\}$.

Theorem

Let $\mathcal{A} \in \mathbb{G}$. Then: $\mathcal{A}$ is finitely subdirectly irreducible iff it is linearly ordered (a G-chain).

Corollary

Every G*-algebra is representable as a subdirect product of* G*-chains.*

Corollary

For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{G}} \varphi$ *iff* $\Gamma \models_{\{\mathrm{G}-chains\}} \varphi$.

Corollary

For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{G}} \varphi$ *iff* $\Gamma \models_{\{countable\ \mathrm{G}-chains\}} \varphi$.

Finite G-chains

For every $n \geq 1$ we define:
$\mathcal{G}_n = \langle \{0, \ldots, n-1\}, \wedge, \vee, \rightarrow, 0, n-1 \rangle$

$$a \wedge b = \min\{a, b\}$$

$$a \vee b = \max\{a, b\}$$

$$a \rightarrow b = \begin{cases} n-1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

Every finite G-chain with exactly n elements is isomorphic to $\mathcal{G}_n$.

Denumerable G-chain

$$\mathcal{G}_\omega = \langle \omega \cup \{\omega\}, \wedge, \vee, \rightarrow, 0, \omega \rangle$$

$$a \wedge b = \min\{a, b\}$$

$$a \vee b = \max\{a, b\}$$

$$a \rightarrow b = \begin{cases} \omega & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

Completeness theorems

Theorem

For every finite $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{G}} \varphi$ *iff* $\Gamma \models_{\{\mathcal{G}_n \mid n \geq 1\}} \varphi$.

Corollary

$\mathbb{G} = \mathbf{V}(\{\mathcal{G}_n \mid n \geq 1\}) = \mathbf{Q}(\{\mathcal{G}_n \mid n \geq 1\})$.

Theorem

For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{G}} \varphi$ *iff* $\Gamma \models_{\mathcal{G}_\omega} \varphi$.

Corollary

$\mathbb{G} = \mathbf{V}(\mathcal{G}_\omega) = \mathbf{Q}(\mathcal{G}_\omega)$.

Rational G-chain

$$[0,1]_{\mathrm{G}}^{\mathbb{Q}} = \langle [0,1]^{\mathbb{Q}}, \wedge, \vee, \rightarrow, 0, 1 \rangle$$

$$a \wedge b = \min\{a, b\}$$

$$a \vee b = \max\{a, b\}$$

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

Real G-chain

$$[0,1]_{\mathrm{G}} = \langle [0,1], \wedge, \vee, \to, 0, 1 \rangle$$

$$a \wedge b = \min\{a, b\}$$

$$a \vee b = \max\{a, b\}$$

$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

Theorem

For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{G}} \varphi$ *iff* $\Gamma \models_{[0,1]_{\mathrm{G}}^{\mathbb{Q}}} \varphi$ *iff* $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$.

Theorem

$\mathbb{G}$ is a locally finite variety, i.e. every finite subset of a G-algebra generates a finite subalgebra.

Corollary

All varieties of G-algebras are generated by finite G-chains.

Theorem (Exercise 5)

The subvarieties of $\mathbb{G}$ are exactly:
$\mathbf{V}(\mathcal{G}_1) \subseteq \mathbf{V}(\mathcal{G}_2) \subseteq \mathbf{V}(\mathcal{G}_3) \subseteq \ldots \subseteq \mathbf{V}(\mathcal{G}_n) \subseteq \mathbf{V}(\mathcal{G}_{n+1}) \subseteq \ldots \mathbb{G}$. *$\mathbf{V}(\mathcal{G}_1)$ is the trivial variety and $\mathbf{V}(\mathcal{G}_2)$ is (termwise equivalent to) the variety of Boolean algebras. For every $n \geq 3$, an axiomatization (relative to $\mathbb{G}$) for $\mathbf{V}(\mathcal{G}_n)$ is given by the equation:*
$\bigvee_{i<n}(x_i \to x_{i+1}) \approx \overline{1}$. *The inclusions in the chain of varieties are strict.*

1913 L.E.J. Brouwer proposes intuitionism as a new (genuine) form of mathematics.

1920 Jan Łukasiewicz publishes the first work ever on many-valued logic (a three-valued logic to deal with future contingents).

1922 He generalizes it to an n-valued logic for each $n \geq 3$.

1928 Heyting considers the logic behind intuitionism and endowes it with a Hilbert-style calculus.

1930 Together with Alfred Tarski, Łukasiewicz generalizes his logics to a $[0, 1]$-valued logic. They also provide a Hilbert-style calculus with 5 axioms and *modus ponens* and conjecture that it is complete w.r.t. the infinitely-valued logic.

1932 Kurt Gödel studies an infinite family of finite linearly ordered matrices for intuitionistic logic. They are not a complete semantics.

1934 Gentzen introduces natural deduction and sequent calculus for intuitionistic logic.

1935 Mordchaj Wajsberg claims to have proved Łukasiewicz's conjecture, but he never shows the proof.

1937 Tarski and Stone develop topological interpretations of intuitionistic logic.

1958 Rose and Rosser publish a proof of completeness of Łukasiewicz logic based on syntactical methods.

1959 Meredith shows that the fifth axiom of Łukasiewicz logic is redundant.

1959 Chang publishes a proof of completeness of Łukasiewicz logic based on algebraic methods.

1959 Michael Dummett resumes Gödel's work from 1932 and proposes a denumerable linearly ordered matrix for intuitionism. He gives a correct and complete Hilbert-style calculus for this matrix which turns out to be an axiomatic extension of IPC: Gödel-Dummett logic.

1963 Hay shows the finite strong completeness of Łukasiewicz logic.

1965 Saul Kripke introduces his relational semantics for intuitionistic logic.

1965 Lotfi Zadeh proposes Fuzzy Set Theory (FST) as a mathematical treatment of vagueness and imprecision. FST becomes an extremely popular paradigm for engineering applications, known also as *Fuzzy Logic*.

1969 Goguen shows how to combine Zadeh's fuzzy sets and Łukasiewicz logic to solve some vagueness logical paradoxes.

1996 Hájek, Godo and Esteva define and axiomatize a new $[0, 1]$-valued system: Product logic.

1998 In the foundational monograph of Mathematical Fuzzy Logic (*Metamathematics of Fuzzy Logic*) Petr Hájek deals with many particular fuzzy logics including: Łukasiewicz, Gödel-Dummett and Product logics.

3-valued logic

Truth-values = $\{0, \frac{1}{2}, 1\}$ (false, possible and true)

$\to$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

	$\neg$
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

n-valued logic Ł$_n$

For each $n \geq 4$, truth-values = $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$

$$a \to b = \min\{1, 1 - a + b\}$$

$$\neg a = 1 - a$$

Algebra: $\mathbf{Ł}_n = \langle \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}, \to, \neg \rangle$

$\Gamma \models_{Ł_n} \varphi$ iff for every $\mathbf{Ł}_n$-evaluation e such that $e[\Gamma] \subseteq \{1\}$, $e(\varphi) = 1$.

Infinitely-valued logic Ł

Truth-values: $[0,1]$

$$a \to b = \min\{1, 1 - a + b\}$$

$$\neg a = 1 - a$$

Algebra: $[0,1]_{Ł} = \langle [0,1], \to, \neg \rangle$

$\Gamma \models_{Ł} \varphi$ iff for every $[0,1]_{Ł}$-evaluation e such that $e[\Gamma] \subseteq \{1\}$, $e(\varphi) = 1$.

Defined connectives:

$$a\&b = \neg(a \to \neg b)$$

$$a \vee b = (a \to b) \to b$$

$$a \wedge b = \neg(\neg a \vee \neg b)$$

Exercise 6

$$a\&b = \max\{0, a + b - 1\}$$

$$a \vee b = \max\{a, b\}$$

$$a \wedge b = \min\{a, b\}$$

Splitting of conjunction properties

1. $a\&b \leq c$ iff $b \leq a \to c$ (residuation)
2. $a \to b = 1$ iff $a \wedge b = a$ iff $a \leq b$ ($\wedge = \min$)

Axiomatic system

(Ł1) $\varphi \to (\psi \to \varphi)$

(Ł2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(Ł3) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

(Ł4) $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$

(Ł5) $((\varphi \to \psi) \to (\psi \to \varphi)) \to (\psi \to \varphi)$

Inference rule: *modus ponens*

Hypothesis

For every $\{\varphi\} \in \mathrm{Fm}_{\mathcal{L}}$, $\vdash_{\text{Ł}} \varphi$ iff $\models_{\text{Ł}} \varphi$

Theorem (Local Deduction Theorem)

For every set of formulae $\Gamma \cup \{\varphi, \psi\}$, *there is* $n \geq 1$ *such that:*

$$\Gamma, \varphi \vdash_{Ł} \psi \text{ iff } \Gamma \vdash_{Ł} \varphi \& \stackrel{n}{\ldots} \& \varphi \to \psi$$

Alternative axiomatics [Hájek 1998]

(B1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(B2)	$\varphi \& \psi \to \psi$
(B3)	$\varphi \& \psi \to \psi \& \varphi$
(B4)	$\varphi \& (\varphi \to \psi) \to \psi \& (\psi \to \varphi)$
(B5a)	$\varphi \to (\psi \to \chi) \to (\varphi \& \psi \to \chi)$
(B5b)	$(\varphi \& \psi \to \chi) \to \varphi \to (\psi \to \chi)$
(B6)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(B7)	$\overline{0} \to \varphi$
(B8)	$((\varphi \to \overline{0}) \to \overline{0}) \to \varphi$

Inference rule: *modus ponens*

Defined connectives: $\neg\varphi = \varphi \to \overline{0}$, $\overline{1} = \neg\overline{0}$,
$\varphi \vee \psi = (\varphi \to \psi) \to \psi$, $\varphi \wedge \psi = \varphi \& (\varphi \to \psi)$.

Theorem

$\vdash_{Ł} (\varphi \to \psi) \vee (\psi \to \varphi)$

Algebraization of Łukasiewicz logic

Definition

An algebra $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ is an MV-algebra if

1. $\langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ is a bounded lattice
2. $\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ is an ordered commutative monoid
3. $\rightarrow^{\mathcal{A}}$ is the residuum of $\&^{\mathcal{A}}$:
$$a \&^{\mathcal{A}} c \leq b \text{ iff } c \leq a \rightarrow^{\mathcal{A}} b$$
4. $\mathcal{A}$ satisfies the prelinearity equation: $(x \rightarrow y) \vee (y \rightarrow x) \approx \overline{1}$.
5. $\mathcal{A}$ satisfies the divisibility equation: $x \& (x \rightarrow y) \approx x \wedge y$.
6. $\mathcal{A}$ satisfies the involution equation: $(x \rightarrow \overline{0}) \rightarrow \overline{0} \approx x$.

where the relation $\leq$ is the lattice ordering. The negation operation is defined as $\neg^{\mathcal{A}} a = a \rightarrow^{\mathcal{A}} \overline{0}^{\mathcal{A}}$. If the order is linear, we call it MV-*chain*.

- For every $n \geq 2$, $\mathbf{Ł}_n = \langle \{0, \frac{1}{n-1}, \dots, \frac{n-2}{n-1}, 1\}, \&, \rightarrow, \wedge, \vee, 0, 1 \rangle$ is an MV-chain.
- $[0,1]_{Ł} = \langle [0,1], \&, \rightarrow, \wedge, \vee, 0, 1 \rangle$ is an MV-chain.

Theorem

The class of all MV*-algebras,* $\mathbb{MV}$*, is a variety.*

Definition

Given $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, we define the consequence relation $\Gamma \models_{\mathbb{MV}} \varphi$ iff for every $\mathcal{A} \in \mathbb{MV}$ and every $\mathcal{A}$-evaluation e: if $e[\Gamma] \subseteq \{\overline{1}^{\mathcal{A}}\}$, then $e(\varphi) = \overline{1}^{\mathcal{A}}$.

$\langle \mathcal{L}, \models_{\mathbb{MV}} \rangle$ is a finitary logic:

1. $\varphi \models_{\mathbb{MV}} \varphi$ (Reflexivity)
2. If $\Gamma \subseteq \Delta$ and $\Gamma \models_{\mathbb{MV}} \varphi$, then $\Delta \models_{\mathbb{MV}} \varphi$ (Monotonicity)
3. If $\Gamma \models_{\mathbb{MV}} \varphi$ and for all $\psi \in \Gamma$, $\Delta \models_{\mathbb{MV}} \psi$, then $\Delta \models_{\mathbb{MV}} \varphi$ (Cut)
4. If $\Gamma \models_{\mathbb{MV}} \varphi$ and σ is a substitution, then $\sigma[\Gamma] \models_{\mathbb{MV}} \sigma(\varphi)$ (Structurality)
5. If $\Gamma \models_{\mathbb{MV}} \varphi$, then there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \models_{\mathbb{MV}} \varphi$ (Finitarity)

Algebraic completeness

Theorem (Exercise 7)

For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{Ł} \varphi$ *iff* $\Gamma \models_{\mathbb{MV}} \varphi$.

1. For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,
   $\Gamma \vdash_{Ł} \varphi$ iff $\{\psi \approx \overline{1} \mid \psi \in \Gamma\} \models_{\mathbb{MV}} \varphi \approx \overline{1}$
2. For every $\Pi \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}$,
   $\Pi \models_{\mathbb{MV}} \varphi \approx \psi$ iff $\{\alpha \leftrightarrow \beta \mid \alpha \approx \beta \in \Pi\} \vdash_{Ł} \varphi \leftrightarrow \psi$
3. For every $\varphi \in \mathrm{Fm}_{\mathcal{L}}$,
   $\varphi \vdash_{Ł} \varphi \leftrightarrow \overline{1}$ and $\varphi \leftrightarrow \overline{1} \vdash_{Ł} \varphi$
4. For every $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$,
   $\varphi \approx \psi \models_{\mathbb{MV}} \varphi \leftrightarrow \psi \approx \overline{1}$ and $\varphi \leftrightarrow \psi \approx \overline{1} \models_{\mathbb{MV}} \varphi \approx \psi$

Translations:

- $\tau : \varphi \mapsto \varphi \approx \overline{1}$
- $\rho : \alpha \approx \beta \mapsto \alpha \leftrightarrow \beta$

MV-algebras are the equivalent algebraic semantics of Ł.

Filters

Definition

Let $\mathcal{A}$ be an MV-algebra. A set $F \subseteq A$ is a filter iff:

1. $\overline{1} \in F$
2. if $a, b \in F$, then $a \& b \in F$
3. if $a \in F$ and $a \leq b$, then $b \in F$

Definition

Let $\mathcal{A}$ be an MV-algebra. A set $F \subseteq A$ is an implicative filter iff:

1. $\overline{1} \in F$
2. if $a, a \to b \in F$, then $b \in F$

Definition

Let $\mathcal{A}$ be an MV-algebra. A set $F \subseteq A$ is a logical filter iff for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ for every e $\mathcal{A}$-evaluation, if $\Gamma \vdash_{\text{Ł}} \varphi$ and $e[\Gamma] \subseteq F$, then $e(\varphi) \in F$.

filters = implicative filters = logical filters

$\mathcal{A} \in \mathbb{MV}$, $F \subseteq A$ filter. A congruence θ of $\mathcal{A}$ is compatible with F iff:

$$\text{if } \langle a, b\rangle \in \theta \text{ and } a \in F, \text{ then } b \in F$$

Proposition

Given $\mathcal{A} \in \mathbb{MV}$ and a filter $F \subseteq A$, we define a relation:

$$\Omega_{\mathcal{A}}(F) = \{\langle a, b\rangle \in A^2 \mid a \to b, b \to a \in F\}$$

Then $\Omega_{\mathcal{A}}(F)$ is a congruence of $\mathcal{A}$, $F = \overline{1}/\Omega_{\mathcal{A}}(F)$. Moreover $\Omega_{\mathcal{A}}(F)$ is compatible with F and it is the greatest one with this property.

Proposition

Let $\mathcal{A}$ be an MV-algebra and θ a congruence of $\mathcal{A}$. Then $\overline{1}/\theta$ is a filter of $\mathcal{A}$.

Proposition

Let $\mathcal{A}$ be an MV-algebra and θ a congruence of $\mathcal{A}$. Then $\Omega_{\mathcal{A}}(\overline{1}/\theta) = \theta$.

Theorem

$\mathcal{A} \in \mathbb{MV}$. $\Omega_{\mathcal{A}}$ is an isomorphism between the lattice of filters and the lattice of congruences of $\mathcal{A}$.

Theorem

$\mathcal{A} \in \mathbb{MV}$, $F \subseteq A$ filter. Then for every $a, b \in A$: $\langle a, b\rangle \in \Omega_{\mathcal{A}}(F)$ iff for every formula $\varphi(x, \overrightarrow{z})$, and $\overrightarrow{c} \in A^{<\omega}$ we have $\varphi^{\mathcal{A}}(a, \overrightarrow{c}) \in F$ iff $\varphi^{\mathcal{A}}(b, \overrightarrow{c}) \in F$.

$\Omega_{\mathcal{A}}(F)$ is called the Leibniz congruence.

Lemma

Given $\mathcal{A} \in \mathbb{MV}$ and $a \in A$, let $Fi(a)$ be the filter generated by a. Then $Fi(a) = \{x \in A \mid a^n \leq x$ for some $n \geq 1\}$.

Theorem

Let $\mathcal{A} \in \mathbb{MV}$. Then: $\mathcal{A}$ is finitely subdirectly irreducible iff it is linearly ordered (an MV-chain).

Corollary

Every MV*-algebra is representable as a subdirect product of* MV*-chains.*

Corollary

For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{Ł} \varphi$ *iff* $\Gamma \models_{\{\mathrm{MV}-chains\}} \varphi$.

Corollary

For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{Ł} \varphi$ *iff* $\Gamma \models_{\{countable\ \mathrm{MV}-chains\}} \varphi$.

Definition

An *Abelian* ℓ*-group* is a first-order structure $\mathcal{G} = \langle G, +, -, 0, \leq \rangle$ where $\langle G, +, -, 0 \rangle$ is an Abelian group and $\leq$ is a lattice order such that $+$ is monotonic.

Theorem (Gurevich - Kokorin)

Let $\mathcal{R} = \langle \mathbb{R}, +, -, 0, \leq \rangle$ *be the Abelian* ℓ*-group of reals with the sum and let* $\varphi(x_0, \ldots, x_n)$ *be a quantifier-free formula in the language of* ℓ*-groups. If* $\mathcal{R} \models \forall x_0 \ldots \forall x_n \varphi(x_0, \ldots, x_n)$*, then* $\mathcal{G} \models \forall x_0 \ldots \forall x_n \varphi(x_0, \ldots, x_n)$ *for every linearly ordered Abelian* ℓ*-group* $\mathcal{G}$*.*

Definition

Let $\mathcal{G} = \langle G, +, -, 0, \leq \rangle$ be an Abelian linearly ordered ℓ-group and $e \in G$ such that $e > 0$. We define an algebra $MV(\mathcal{G}, e) = \langle [0, e], \rightarrow, 0 \rangle$ where: $a \rightarrow b = 1$ if $a \leq b$, and $a \rightarrow b = e - a + b$ otherwise.

Theorem

For every Abelian linearly ordered ℓ-group $\mathcal{G}$ and every $e > 0$, $MV(\mathcal{G}, e)$ is an MV-chain. Moreover, for every MV*-chain $\mathcal{A}$ there is an Abelian linearly ordered ℓ-group $\mathcal{G}$ and $e > 0$ such that $\mathcal{A} = MV(\mathcal{G}, e)$.*

Finite strong standard completeness

Theorem

For every finite $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{Ł} \varphi$ *iff* $\Gamma \models_{[0,1]_{Ł}} \varphi$.

The same result holds for $[0, 1]_{Ł}^{\mathbb{Q}}$.

Corollary

$$\mathbb{MV} = \mathbf{V}([0, 1]_{Ł}) = \mathbf{Q}([0, 1]_{Ł}) = \mathbf{V}([0, 1]_{Ł}^{\mathbb{Q}}) = \mathbf{Q}([0, 1]_{Ł}^{\mathbb{Q}})$$

Failure of strong standard completeness

- $\varphi \oplus \psi := \neg(\neg\varphi \& \neg\psi)$
- $a \oplus b = \min\{a + b, 1\}$
- $\Sigma = \{p \oplus \stackrel{n}{\ldots} \oplus p \to q \mid n \geq 1\} \cup \{\neg p \to q\}$
- $\Sigma \models_{[0,1]_{Ł}} q$
- For every finite $\Sigma_0 \subseteq \Sigma$, $\Sigma \not\models_{[0,1]_{Ł}} q$.
- $\Sigma \nvdash_{Ł} q$
- The same holds for $[0, 1]_{Ł}^{\mathbb{Q}}$.