PhD course on Mathematical Fuzzy Logic: 5th lesson

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Outline

Logic is the science that studies correct reasoning.

The are many kinds of correct reasoning, hence many logics.

In Mathematics we typically assume the Bivalence Principle.

Bivalence Principle

Every proposition is either true or false.

Therefore, the usual logic for mathematical reasoning is classical logic.

In classical logic every predicate yields a perfect division between those objects it applies to, and those it does not. We call them *crisp*.

Examples: prime number, even number, monotonic function, continuous function, divisible group, ... (any mathematical predicate)

Therefore, formulae of classical logic are evaluated at $\mathcal{B}_2$ (the two-element Boolean algebra).

Correct reasoning in classical logic

Definition

Given $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ we say that φ is a logical consequence of Γ, denoted $\Gamma \models_{\mathcal{B}_2} \varphi$, iff for every $\mathcal{B}_2$-evaluation e such that $e(\gamma) = 1$ for every $\gamma \in \Gamma$, we have $e(\varphi) = 1$.

Definition

Given $\psi_1, \ldots, \psi_n, \varphi \in \mathrm{Fm}_{\mathcal{L}}$ we say that $\langle \psi_1, \ldots, \psi_n, \varphi \rangle$ is a correct reasoning if $\{\psi_1, \ldots, \psi_n\} \models_{\mathcal{B}_2} \varphi$ (equivalently: $\{\psi_1, \ldots, \psi_n\} \vdash_{\mathrm{CPC}} \varphi$). In this case, $\psi_1, \ldots, \psi_n$ are the premises of the reasoning and φ is the conclusion.

Remark

$$\begin{array}{c} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \\ \hline \varphi \end{array}$$

is a correct reasoning iff there is no interpretation making the premises true and the conclusion false.

Sorites paradox [Eubulides of Miletus, IV century BC]

A man who has no money is poor. If a poor man earns one euro, he remains poor. Therefore, a man who has one million euros is poor.

Formalization:

p_n: *A man who has exactly* n *euros is poor*

$$\frac{\begin{array}{l} p_0 \\ p_0 \to p_1 \\ p_1 \to p_2 \\ p_2 \to p_3 \\ \vdots \\ p_{999999} \to p_{1000000} \end{array}}{p_{1000000}}$$

- There is no doubt that the premise p_0 is true.
- There is no doubt that the conclusion $p_{1000000}$ is false.
- For each i, the premise $p_i \rightarrow p_{i+1}$ seems to be true.
- The reasoning is logically correct (application of *modus ponens* one million times).
- We have a paradox!

The predicates that generate this kind of paradoxes are called *vague*.

Remark

A predicate is vague iff it has borderline cases, i.e. there are objects for which we cannot tell whether they fall under the scope of the predicate.

Example: Consider the predicate *tall*. Is a man measuring 1.78 meters tall?

- It is not a problem of ambiguity. Once we fix an unambiguous context, the problem remains.
- It is not a problem of uncertainty. Uncertainty typically appears when some relevant information is not known. Even if we assume that all relevant information is known, the problem remains.
- It cannot be solved by establishing a crisp definition of the predicate. The problem is: with the meaning that the predicate *tall* has in the natural language, whatever it might be, is a man measuring 1.78 meters tall?

Solutions in Analytical Philosophy

(1) Nihilist solution: *Vague predicates have no meaning*. If they would have, sorites paradox would lead to a contradiction.

(2) Epistemicist solution: *Vagueness is a problem of ignorance*. All predicates are crisp, but our epistemological constitution makes us unable to know the exact extension of a vague predicate. Some premise $p_i \to p_{i+1}$ is false.

(3) Supervaluationist solution: The meaning of vague predicate is the set of its precisifications (possible ways to make it crisp). *Truth is supertruth*, i.e. true under all precisifications. Some premise $p_i \to p_{i+1}$ is false.

Solutions in Analytical Philosophy (continuation)

(4) Pragmatist solution: *Vague predicates do not have a univocal meaning*. A vague language is a set of crisp languages. For every utterance of a sentence involving a vague predicate, pragmatical conventions endow it with some particular crisp meaning. Some premise $p_i \to p_{i+1}$ is false.

(5) Degree-based solution: *Truth comes in degrees*. p_0 is completely true and $p_{1000000}$ is completely false. The premises $p_i \to p_{i+1}$ are very true, but not completely.

Many-valued logics to deal with vagueness

1949 Sören Halldén in *The Logic of Nonsense* proposes a three-valued logic to model vague predicates.

1955 Stephan Körner proposes an alternative three-valued treatment.

1965 Lotfi Zadeh proposes Fuzzy Set Theory (FST) as a mathematical treatment of vagueness and imprecision. FST becomes an extremely popular paradigm for engineering applications, known also as *Fuzzy Logic*.

1969 Goguen shows how to combine Zadeh's fuzzy sets and Łukasiewicz logic to solve sorites paradox.

Fuzzy sets as a model for vague predicates

Formally a fuzzy set is a pair $\langle X, \mu \rangle$ where X is a classical set and $\mu : X \to [0,1]$ is a function (called *membership function*) that maps every object $x \in X$ to its membership degree $\mu(x) \in [0,1]$.

Example: For the predicate *tall* take $X := [0.3, 2.4]$ (containing all possible heights) and

$$\mu(x) = \begin{cases} 0 & \text{if } x \leq 1.2, \\ \frac{5}{3}x - 2 & \text{if } 1.2 \leq x \leq 1.8, \\ 1 & \text{if } x \geq 1.8. \end{cases}$$

If fuzzy sets interpret atomic vague propositions, their set-theoretic operations correspond to logical connectives for vague propositions.

- Strong conjunction: t-norm.
- Strong disjunction: t-conorm.
- Negation: negation function.
- Weak conjunction: min.
- Weak disjunction: max.
- Implication: residuum of the t-norm.

Fuzzy Logic solution to sorites paradox

- $X = \{0, 1, 2, \ldots, 10^6\}$, $\mu : X \to [0, 1]$.
- The truth value of p_n will be $\mu(n)$.
- $\mu(0) = 1$ and $\mu(10^6) = 0$ (the first premise is completely true, the conclusion is completely false).
- Take $\varepsilon = 10^{-6}$ and $\mu(n) = 1 - n\varepsilon$.
- Compute the value of $p_n \to p_{n+1}$ by means of Łukasiewicz implication.
- $\mu(n) \to \mu(n+1) = 1 - \mu(n) + \mu(n+1) = 1 - (1 - n\varepsilon) + (1 - (n+1)\varepsilon) = 1 - \varepsilon < 1$ (all these premises have the same truth value: almost completely true).

Outline

- Each continuous t-norm is residuated.
- Notice that in the proof actually only left-continuity is used.

Proposition

Let $*$ be a t-norm. Then:
$*$ is residuated if, and only if, it is left-continuous.

Historical remarks on t-norms

1930 Łukasiewicz and Tarski: $[0,1]$-valued logic that uses a particular continuous t-norm to interpret (strong) conjunction.

1942 Menger: Statistical metrics. Triangular norms (t-norms) as functions to deal with triangular inequality.

1957 Mostert, Shields: Representation of continuous t-norms as ordinal sums of three basic components.

1959 Dummett resumes Gödel's work from 1932 and proposes a $[0,1]$-valued logic that uses $\min$ function (a continuous t-norm) to interpret conjunction.

1960 Schweizer and Sklar: Development of statistical metric spaces.

1965 Ling: Independent proof of the representation of continuous t-norms as ordinal sums of three basic components.

1965 Zadeh: Fuzzy Set Theory (FST) as a mathematical treatment of vagueness and imprecision. FST becomes a very popular paradigm for engineering applications, known also as *Fuzzy Logic*. He proposes $\min$ function for the intersection of fuzzy sets.

1969 Goguen: combined use of Zadeh's fuzzy sets and Łukasiewicz logic to solve sorites paradox.

late 70s Several researchers in FST propose the usage of t-norms to interpret intersection of fuzzy sets and conjunction in the logics. All known residuated t-norms are continuous.

1995 Fodor: Nilpotent Minimum, a left-continuous non-continuous t-norm.

1996 Esteva, Godo, Hájek: Product logic as a $[0,1]$-valued logic that uses product t-norm to interpret (strong) conjunction.

1998 Hájek introduces the logic BL and conjectures that it is the logic of all continuous t-norms. He shows that Ł, G, and Π are axiomatic extensions of BL.

2000 Hájek, Cignoli, Esteva, Godo and Torrens: BL is the logic of all continuous t-norms.

2001 Esteva and Godo introduce the logic MTL and conjecture that it is the logic of all left-continuous t-norms. They show that the logic NM of Fodor's t-norm and BL are axiomatic extensions of MTL.

2002 Jenei and Montagna: MTL is the logic of all left-continuous t-norms.

$\mathcal{L} = \{\&, \to, \wedge, \overline{0}\}$

(MTL1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(MTL2)	$\varphi \& \psi \to \varphi$
(MTL3)	$\varphi \& \psi \to \psi \& \varphi$
(MTL4)	$\varphi \wedge \psi \to \varphi$
(MTL5)	$\varphi \wedge \psi \to \psi \wedge \varphi$
(MTL6)	$\varphi \& (\varphi \to \psi) \to \varphi \wedge \psi$
(MTL7a)	$(\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi)$
(MTL7b)	$(\varphi \& \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(MTL8)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(MTL9)	$\overline{0} \to \varphi$

Inference rule: *modus ponens*

Defined connectives: $\varphi \vee \psi = ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$, $\varphi \leftrightarrow \psi = (\varphi \to \psi) \& (\psi \to \varphi)$, $\neg \varphi = \varphi \to \overline{0}$, $\overline{1} = \neg \overline{0}$

Theorem (Local Deduction Theorem)

For every set of formulae $\Gamma \cup \{\varphi, \psi\}$, *there is* $n \geq 1$ *such that:*

$$\Gamma, \varphi \vdash_{\mathrm{MTL}} \psi \text{ iff } \Gamma \vdash_{\mathrm{MTL}} \varphi \& \ldots^{n} \& \varphi \to \psi$$

Definition

An algebra $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ is an MTL-algebra if

1. $\langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ is a bounded lattice
2. $\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ is an ordered commutative monoid
3. $\rightarrow^{\mathcal{A}}$ is the residuum of $\&^{\mathcal{A}}$:
$$a \&^{\mathcal{A}} c \leq b \text{ iff } c \leq a \rightarrow^{\mathcal{A}} b$$
4. $\mathcal{A}$ satisfies the prelinearity equation: $(x \rightarrow y) \vee (y \rightarrow x) \approx \overline{1}$.

where the relation $\leq$ is the lattice ordering. The negation operation is defined as $\neg^{\mathcal{A}} a = a \rightarrow^{\mathcal{A}} \overline{0}^{\mathcal{A}}$. If the order is linear, we call it MTL-*chain*.

Let $*$ be a left-continuous t-norm and $\Rightarrow_*$ its residuum. Then $[0,1]_* = \langle [0,1], *, \Rightarrow_*, \min, \max, 0, 1 \rangle$ is an MTL-chain.

All MTL-chains over $[0,1]$ are of this form and are called *standard chains*.

Theorem

The class of all MTL*-algebras,* $\mathbb{MTL}$*, is a variety.*

Important subvarieties of $\mathbb{MTL}$:

1. $\mathbb{BL}$: $x\&(x \to y) \approx x \wedge y$ (divisibility equation).
2. $\mathbb{MV}$: divisibility + $\neg\neg x \approx x$ (involution equation).
3. $\mathbb{P}$: divisibility + $\neg x \vee ((x \to x\&y) \to y) \approx \overline{1}$ (cancellativity equation).
4. $\mathbb{G}$: $x \approx x\&x$ (contraction equation).
5. $\mathbb{SBL}$: divisibility + $x \wedge \neg x \approx \overline{0}$ (pseudocomplementation equation).
6. $\mathbb{BA}$: $x \vee \neg x \approx \overline{1}$ (excluded middle).

1. $\mathbb{IMTL}$: $\neg\neg x \approx x$ (involution equation).
2. $\mathbb{PMTL}$: $\neg x \vee ((x \to x\&y) \to y) \approx \overline{1}$ (cancellativity equation).
3. $\mathbb{SMTL}$: $x \wedge \neg x \approx \overline{0}$ (pseudocomplementation equation).
4. $\mathbb{WCMTL}$: $\neg(x\&y) \vee ((x \to x\&y) \to y) \approx \overline{1}$ (weak cancellativity equation).
5. $\mathbb{WNM}$: $\neg(x\&y) \vee (x \wedge y \to x\&y) \approx \overline{1}$ (weak nilpotent minimum equation).
6. $\mathbb{NM}$: involution + $\neg(x\&y) \vee (x \wedge y \to x\&y) \approx \overline{1}$ (weak nilpotent minimum equation).
7. $\mathbb{C}_n\mathbb{MTL}$: $x^{n-1} \approx x^n$ (n-contraction equation).
8. $\mathbb{C}_n\mathbb{IMTL}$: involution + $x^{n-1} \approx x^n$ (n-contraction equation).

Definition

Given $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, we define the consequence relation $\Gamma \models_{\mathbb{MTL}} \varphi$ iff for every $\mathcal{A} \in \mathbb{MTL}$ and every $\mathcal{A}$-evaluation e: if $e[\Gamma] \subseteq \{\overline{1}^{\mathcal{A}}\}$, then $e(\varphi) = \overline{1}^{\mathcal{A}}$.

$\langle \mathcal{L}, \models_{\mathrm{MTL}} \rangle$ is a finitary logic:

1. $\varphi \models_{\mathrm{MTL}} \varphi$ (Reflexivity)
2. If $\Gamma \subseteq \Delta$ and $\Gamma \models_{\mathrm{MTL}} \varphi$, then $\Delta \models_{\mathrm{MTL}} \varphi$ (Monotonicity)
3. If $\Gamma \models_{\mathrm{MTL}} \varphi$ and for all $\psi \in \Gamma$, $\Delta \models_{\mathrm{MTL}} \psi$, then $\Delta \models_{\mathrm{MTL}} \varphi$ (Cut)
4. If $\Gamma \models_{\mathrm{MTL}} \varphi$ and σ is a substitution, then $\sigma[\Gamma] \models_{\mathrm{MTL}} \sigma(\varphi)$ (Structurality)
5. If $\Gamma \models_{\mathrm{MTL}} \varphi$, then there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \models_{\mathrm{MTL}} \varphi$ (Finitarity)

Algebraic completeness

Theorem

For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{MTL}} \varphi$ *iff* $\Gamma \models_{\mathbb{MTL}} \varphi$.

1. For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,
   $$\Gamma \vdash_{\mathrm{MTL}} \varphi \text{ iff } \{\psi \approx \overline{1} \mid \psi \in \Gamma\} \models_{\mathbb{MTL}} \varphi \approx \overline{1}$$
2. For every $\Pi \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}$,
   $$\Pi \models_{\mathbb{MTL}} \varphi \approx \psi \text{ iff } \{\alpha \leftrightarrow \beta \mid \alpha \approx \beta \in \Pi\} \vdash_{\mathrm{MTL}} \varphi \leftrightarrow \psi$$
3. For every $\varphi \in \mathrm{Fm}_{\mathcal{L}}$,
   $$\varphi \vdash_{\mathrm{MTL}} \varphi \leftrightarrow \overline{1} \text{ and } \varphi \leftrightarrow \overline{1} \vdash_{\mathrm{MTL}} \varphi$$
4. For every $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$,
   $$\varphi \approx \psi \models_{\mathbb{MTL}} \varphi \leftrightarrow \psi \approx \overline{1} \text{ and } \varphi \leftrightarrow \psi \approx \overline{1} \models_{\mathbb{MTL}} \varphi \approx \psi$$

Translations:

- $\tau : \varphi \mapsto \varphi \approx \overline{1}$
- $\rho : \alpha \approx \beta \mapsto \alpha \leftrightarrow \beta$

MTL-algebras are the equivalent algebraic semantics of MTL.

Filters

Definition

Let $\mathcal{A}$ be an MTL-algebra. A set $F \subseteq A$ is a filter iff:

1. $\overline{1} \in F$
2. if $a, b \in F$, then $a \& b \in F$
3. if $a \in F$ and $a \leq b$, then $b \in F$

Definition

Let $\mathcal{A}$ be an MTL-algebra. A set $F \subseteq A$ is an implicative filter iff:

1. $\overline{1} \in F$
2. if $a, a \to b \in F$, then $b \in F$

Definition

Let $\mathcal{A}$ be an MTL-algebra. A set $F \subseteq A$ is a logical filter iff for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ for every e $\mathcal{A}$-evaluation, if $\Gamma \vdash_{\mathrm{MTL}} \varphi$ and $e[\Gamma] \subseteq F$, then $e(\varphi) \in F$.

filters = implicative filters = logical filters

$\mathcal{A} \in \mathbb{MTL}$, $F \subseteq A$ filter. A congruence θ of $\mathcal{A}$ is compatible with F iff:

$$\text{if } \langle a, b\rangle \in \theta \text{ and } a \in F, \text{ then } b \in F$$

Proposition

Given $\mathcal{A} \in \mathbb{MTL}$ and a filter $F \subseteq A$, we define a relation:

$$\Omega_{\mathcal{A}}(F) = \{\langle a, b\rangle \in A^2 \mid a \to b, b \to a \in F\}$$

Then $\Omega_{\mathcal{A}}(F)$ is a congruence of $\mathcal{A}$, $F = \overline{1}/\Omega_{\mathcal{A}}(F)$. Moreover $\Omega_{\mathcal{A}}(F)$ is compatible with F and it is the greatest one with this property.

Proposition

Let $\mathcal{A}$ be an MTL-algebra and θ a congruence of $\mathcal{A}$. Then $\overline{1}/\theta$ is a filter of $\mathcal{A}$.

Proposition

Let $\mathcal{A}$ be an MTL-algebra and θ a congruence of $\mathcal{A}$. Then $\Omega_{\mathcal{A}}(\overline{1}/\theta) = \theta$.

Theorem

$\mathcal{A} \in \mathbb{MTL}$. $\Omega_{\mathcal{A}}$ *is an isomorphism between the lattice of filters and the lattice of congruences of* $\mathcal{A}$.

Lemma

Given $\mathcal{A} \in \mathbb{MTL}$ and $a \in A$, let $Fi(a)$ be the filter generated by a. Then $Fi(a) = \{x \in A \mid a^n \leq x \text{ for some } n \geq 1\}$.

Theorem

Let $\mathcal{A} \in \mathbb{MTL}$. Then: $\mathcal{A}$ is finitely subdirectly irreducible iff it is linearly ordered (an MTL-chain).

Corollary

Every MTL*-algebra is representable as a subdirect product of* MTL*-chains.*

Corollary

For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{MTL}} \varphi$ *iff* $\Gamma \models_{\{\mathrm{MTL}-chains\}} \varphi$.

Corollary

For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{MTL}} \varphi$ *iff* $\Gamma \models_{\{countable\ \mathrm{MTL}-chains\}} \varphi$.

Remark

All these properties remain valid for axiomatic extensions as well.

Some axiomatic extensions of MTL

BL	=	MTL	+	$\varphi \wedge \psi \to \varphi \& (\varphi \to \psi)$	(Div)
Ł	=	BL	+	$\neg\neg\varphi \to \varphi$	(Inv)
G	=	BL	+	$\varphi \to \varphi \& \varphi$	(C)
Π	=	BL	+	$\neg\psi \vee ((\psi \to \varphi \& \psi) \to \varphi)$	(Can)
SBL	=	BL	+	$\varphi \wedge \neg\varphi \to \overline{0}$	(PC)
IMTL	=	MTL	+	$\neg\neg\varphi \to \varphi$	(Inv)
SMTL	=	MTL	+	$\varphi \wedge \neg\varphi \to \overline{0}$	(PC)
ΠMTL	=	MTL	+	$\neg\psi \vee ((\psi \to \varphi \& \psi) \to \varphi)$	(Can)
WCMTL	=	MTL	+	$\neg(\varphi \& \psi) \vee ((\psi \to \varphi \& \psi) \to \varphi)$	(WCan)
WNM	=	MTL	+	$(\varphi \& \psi \to \overline{0}) \vee (\varphi \wedge \psi \to \varphi \& \psi)$	(WNM)
NM	=	IMTL	+	$(\varphi \& \psi \to \overline{0}) \vee (\varphi \wedge \psi \to \varphi \& \psi)$	(WNM)
C_nMTL	=	MTL	+	$\varphi^{n-1} \to \varphi^n$	(C_n)
C_nIMTL	=	IMTL	+	$\varphi^{n-1} \to \varphi^n$	(C_n)
CPC	=	MTL	+	$\varphi \vee \neg\varphi$	(EM)

Completion of countable MTL-chains [Jenei-Montagna-Ono]

Let $\mathcal{A}$ be a non-trivial countable MTL-chain.

- For every $a \in A$, $suc(a)$ is either the successor of a in the order of $\mathcal{A}$ if it exists or $suc(a) = a$ otherwise.
- $B = \{\langle a, 1\rangle \mid a \in A\} \cup \{\langle a, q\rangle \mid \exists a' \in A$ such that $a \neq a'$ and $suc(a') = a,\ q \in \mathbb{Q} \cap (0, 1)\}$.
- Consider the lexicographical order $\preceq$ on B.
- Define the following monoidal operation on B:

$$\langle a, q\rangle \circ \langle b, r\rangle = \begin{cases} \min_{\preceq}\{\langle a, q\rangle, \langle b, r\rangle\} & \text{if } a \&^{\mathcal{A}} b = \min\{a, b\} \\ \langle a \&^{\mathcal{A}} b, 1\rangle & \text{otherwise.} \end{cases}$$

- The ordered monoid $\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}}, \leq\rangle$ is embeddable into $\langle B, \circ, \langle \overline{1}^{\mathcal{A}}, 1\rangle, \preceq\rangle$ by mapping every $a \in A$ to $\langle a, 1\rangle$.

Completion of countable MTL-chains (cont.)

- $\mathcal{B} = \langle B, \circ, \langle \overline{1}^{\mathcal{A}}, 1\rangle, \preceq\rangle$ is a densely ordered countable monoid with maximum and minimum, so it is isomorphic to a monoid $\mathcal{B}' = \langle [0,1]^{\mathbb{Q}}, \circ', 1, \preceq'\rangle$. Obviously, $\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}}, \leq\rangle$ is also embeddable into $\mathcal{B}'$. Let h be such embedding. Moreover, restricted to $h[A]$, the residuum of $\circ'$ exists, call it $\Rightarrow$, and $h(a) \Rightarrow h(b) = h(a \rightarrow^{\mathcal{A}} b)$.
- $\mathcal{B}'$ is completed to $[0,1]$ by defining:

$$\forall \alpha, \beta \in [0,1] \quad \alpha * \beta = \sup\{x \circ' y \mid x \leq \alpha,\ y \leq \beta,\ x, y \in [0,1]^{\mathbb{Q}}\}.$$

- $*$ is a left-continuous t-norm, so it defines a standard MTL-algebra $[0,1]_*$, and $h : \mathcal{A} \hookrightarrow [0,1]_*$ is an embedding.

Strong standard completeness of MTL

Theorem

For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,
$\Gamma \vdash_{\mathrm{MTL}} \varphi$ *iff* $\Gamma \models_{\{[0,1]_* | * \text{ left-continuous t-norm}\}} \varphi$.

Other logics

- The same construction allows to prove strong standard completeness for G, WNM, SMTL and C_nMTL.
- However, the method does not preserve divisibility, cancellation, weak cancellation and involution in general.

Completion of countable IMTL-chains [Esteva-Godo-Gispert-Montagna]

Let $\mathcal{A}$ be a non-trivial countable IMTL-chain.

- Define the ordered monoid $\langle B, \circ, \langle \overline{1}^{\mathcal{A}}, 1\rangle, \preceq\rangle$ as before.
- Modify the monoidal operation in the following way:

$$\langle a, q\rangle \otimes \langle b, r\rangle = \begin{cases} \langle \overline{0}^{\mathcal{A}}, 1\rangle & \text{if } a = suc(\neg b), q + r \leq 1 \\ \langle a, q\rangle \circ \langle b, r\rangle & \text{otherwise.} \end{cases}$$

Completion of countable IMTL-chains (cont.)

- As before, the ordered monoid $\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}}, \leq \rangle$ is embeddable into $\langle B, \otimes, \langle \overline{1}^{\mathcal{A}}, 1 \rangle, \preceq \rangle$ which is isomorphic to a monoid $\mathcal{B}' = \langle [0,1]^{\mathbb{Q}}, \circ', \preceq' \rangle$. Again, $\mathcal{B}'$ is completed to $[0,1]$ by defining:

$$\forall \alpha, \beta \in [0,1] \quad \alpha * \beta = \sup\{x \circ' y \mid x \leq \alpha,\ y \leq \beta,\ x, y \in [0,1]^{\mathbb{Q}}\}.$$

- $*$ is a left-continuous t-norm with an involutive negation, so it defines a standard IMTL-algebra $[0,1]_*$, and $h : \mathcal{A} \hookrightarrow [0,1]_*$ is an embedding.

Strong standard completeness of IMTL

Theorem

For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,
$\Gamma \vdash_{\mathrm{IMTL}} \varphi$ *iff* $\Gamma \models_{\{[0,1]_* | * \text{ involutive left-continuous t-norm}\}} \varphi$.

Other logics

- The same construction allows to prove strong standard completeness for NM and C_nIMTL.
- However, the method does not preserve divisibility, cancellation and weak cancellation in general.

A partial embedding method [Horčík]

- Let $\mathcal{A} = \langle A, \&, \to, \wedge, \vee, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ be a ΠMTL-chain and $G \subseteq A$ a finite subset.
- Let $\mathcal{S}$ be the submonoid of $\mathcal{A}$ generated by G.
- By using Dickson's lemma it is proved that $\mathcal{S}$ is residuated and the residuum is given by: $a \to b = \max\{z \in S \mid a \& z \leq b\}$.
- $\mathcal{S} = \langle S, \&, \to, \wedge, \vee, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ is a countable ΠMTL-chain.
- Define a new chain over the set $S' = \{\langle s, r \rangle \mid s \in S \setminus \{\overline{0}^{\mathcal{A}}\}, r \in (0, 1]\} \cup \{\langle \overline{0}^{\mathcal{A}}, 1 \rangle\}$, with the lexicographical order $\leq_{lex}$ and the following operations:

$$\langle a, x \rangle \&' \langle b, y \rangle = \langle a \& b, xy \rangle$$

$$\langle a, x \rangle \to' \langle b, y \rangle = \begin{cases} \langle a \to b, 1 \rangle & \text{if } a \& (a \to b) < b, \\ \langle a \to b, \min\{1, y/x\} \rangle & \text{otherwise.} \end{cases}$$

A partial embedding method (cont.)

- $\mathcal{S}' = \langle S', \&', \rightarrow', \leq_{lex}, \langle \overline{0}^{\mathcal{A}}, 1\rangle, \langle \overline{1}^{\mathcal{A}}, 1\rangle\rangle$ is an MTL-chain and there is an embedding $\Psi : \mathcal{S} \rightarrow \mathcal{S}'$ defined by $\Psi(a) = \langle a, 1\rangle$. Moreover $\mathcal{S}'$ is cancellative.
- The set S' is order-isomorphic to the real unit interval $[0, 1]$, so there is a standard ΠMTL-chain $\mathcal{B}$ and an isomorphism $\Phi : \mathcal{S}' \rightarrow \mathcal{B}$. The function $\Phi \circ \Psi$ is a partial embedding of G into $\mathcal{B}$.

Finite strong standard completeness of ΠMTL

Theorem

For every finite $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$*,*
$\Gamma \vdash_{\Pi\mathrm{MTL}} \varphi$ *iff* $\Gamma \models_{\{[0,1]_* | * \text{ cancellative left-continuous t-norm}\}} \varphi$*.*

Other logics

- The same construction allows to prove finite strong standard completeness for WCMTL.

Logic	$\mathcal{RC}$	F$\mathcal{SRC}$	$\mathcal{SRC}$
MTL, IMTL, SMTL, G, WNM, NM, C_nMTL, C_nIMTL	Yes	Yes	Yes
WCMTL, ΠMTL, BL, SBL, Ł, Π	Yes	Yes	No
CPC	No	No	No