

# PhD course on Mathematical Fuzzy Logic: 5th lesson

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# Outline

- 1 The vagueness problem
- 2 The logic of left-continuous t-norms

Logic is the science that studies **correct reasoning**.

There are many kinds of correct reasoning, hence many logics.

In Mathematics we typically assume the **Bivalence Principle**.

### Bivalence Principle

Every proposition is either true or false.

Therefore, the usual logic for mathematical reasoning is classical logic.

In classical logic every predicate yields a perfect division between those objects it applies to, and those it does not. We call them *crisp*.

Examples: prime number, even number, monotonic function, continuous function, divisible group, ... (any mathematical predicate)

Therefore, formulae of classical logic are evaluated at  $\mathcal{B}_2$  (the two-element Boolean algebra).

# Correct reasoning in classical logic

## Definition

Given  $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$  we say that  $\varphi$  is a **logical consequence** of  $\Gamma$ , denoted  $\Gamma \models_{\mathcal{B}_2} \varphi$ , iff for every  $\mathcal{B}_2$ -evaluation  $e$  such that  $e(\gamma) = 1$  for every  $\gamma \in \Gamma$ , we have  $e(\varphi) = 1$ .

## Definition

Given  $\psi_1, \dots, \psi_n, \varphi \in \text{Fm}_{\mathcal{L}}$  we say that  $\langle \psi_1, \dots, \psi_n, \varphi \rangle$  is a **correct reasoning** if  $\{\psi_1, \dots, \psi_n\} \models_{\mathcal{B}_2} \varphi$  (equivalently:  $\{\psi_1, \dots, \psi_n\} \vdash_{\text{CPC}} \varphi$ ). In this case,  $\psi_1, \dots, \psi_n$  are the **premises** of the reasoning and  $\varphi$  is the **conclusion**.

## Remark

$$\begin{array}{c} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \\ \hline \varphi \end{array}$$

is a correct reasoning iff **there is no interpretation making the premises true and the conclusion false.**

# Sorites paradox [Eubulides of Miletus, IV century BC]

*A man who has no money is poor. If a poor man earns one euro, he remains poor. Therefore, a man who has one million euros is poor.*

**Formalization:**

$p_n$ : *A man who has exactly  $n$  euros is poor*

$p_0$

$p_0 \rightarrow p_1$

$p_1 \rightarrow p_2$

$p_2 \rightarrow p_3$

$\vdots$

$p_{999999} \rightarrow p_{1000000}$

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$p_{1000000}$

- There is no doubt that the premise  $p_0$  is true.
- There is no doubt that the conclusion  $p_{1000000}$  is false.
- For each  $i$ , the premise  $p_i \rightarrow p_{i+1}$  seems to be true.
- The reasoning is logically correct (application of *modus ponens* one million times).
- **We have a paradox!**

The predicates that generate this kind of paradoxes are called *vague*.

### Remark

A predicate is vague iff it has **borderline cases**, i.e. there are objects for which we cannot tell whether they fall under the scope of the predicate.

**Example:** Consider the predicate *tall*. Is a man measuring 1.78 meters tall?

- It is not a problem of **ambiguity**. Once we fix an unambiguous context, the problem remains.
- It is not a problem of **uncertainty**. Uncertainty typically appears when some relevant information is not known. Even if we assume that all relevant information is known, the problem remains.
- It cannot be solved by **establishing a crisp definition of the predicate**. The problem is: with the meaning that the predicate *tall* has in the natural language, whatever it might be, is a man measuring 1.78 meters tall?

# Solutions in Analytical Philosophy

- (1) **Nihilist solution:** *Vague predicates have no meaning.* If they would have, sorites paradox would lead to a contradiction.
- (2) **Epistemicist solution:** *Vagueness is a problem of ignorance.* All predicates are crisp, but our epistemological constitution makes us unable to know the exact extension of a vague predicate. Some premise  $p_i \rightarrow p_{i+1}$  is false.
- (3) **Supervaluationist solution:** The meaning of vague predicate is the set of its precisifications (possible ways to make it crisp). *Truth is supertruth*, i.e. true under all precisifications. Some premise  $p_i \rightarrow p_{i+1}$  is false.

## Solutions in Analytical Philosophy (continuation)

- (4) **Pragmatist solution:** *Vague predicates do not have a univocal meaning.* A vague language is a set of crisp languages. For every utterance of a sentence involving a vague predicate, pragmatical conventions endow it with some particular crisp meaning. Some premise  $p_i \rightarrow p_{i+1}$  is false.
- (5) **Degree-based solution:** *Truth comes in degrees.*  $p_0$  is completely true and  $p_{1000000}$  is completely false. The premises  $p_i \rightarrow p_{i+1}$  are very true, but not completely.

## Many-valued logics to deal with vagueness

- 1949 Sören Halldén in *The Logic of Nonsense* proposes a three-valued logic to model vague predicates.
- 1955 Stephan Körner proposes an alternative three-valued treatment.
- 1965 Lotfi Zadeh proposes Fuzzy Set Theory (FST) as a mathematical treatment of vagueness and imprecision. FST becomes an extremely popular paradigm for engineering applications, known also as *Fuzzy Logic*.
- 1969 Goguen shows how to combine Zadeh's fuzzy sets and Łukasiewicz logic to solve sorites paradox.

# Fuzzy sets as a model for vague predicates

Formally a **fuzzy set** is a pair  $\langle X, \mu \rangle$  where  $X$  is a classical set and  $\mu : X \rightarrow [0, 1]$  is a function (called **membership function**) that maps every object  $x \in X$  to its membership degree  $\mu(x) \in [0, 1]$ .

**Example:** For the predicate *tall* take  $X := [0.3, 2.4]$  (containing all possible heights) and

$$\mu(x) = \begin{cases} 0 & \text{if } x \leq 1.2, \\ \frac{5}{3}x - 2 & \text{if } 1.2 \leq x \leq 1.8, \\ 1 & \text{if } x \geq 1.8. \end{cases}$$

If fuzzy sets interpret atomic vague propositions, their set-theoretic operations correspond to logical connectives for vague propositions.

- Strong conjunction: t-norm.
- Strong disjunction: t-conorm.
- Negation: negation function.
- Weak conjunction: min.
- Weak disjunction: max.
- Implication: residuum of the t-norm.

# Fuzzy Logic solution to sorites paradox

- $X = \{0, 1, 2, \dots, 10^6\}$ ,  $\mu : X \rightarrow [0, 1]$ .
- The truth value of  $p_n$  will be  $\mu(n)$ .
- $\mu(0) = 1$  and  $\mu(10^6) = 0$  (the first premise is completely true, the conclusion is completely false).
- Take  $\varepsilon = 10^{-6}$  and  $\mu(n) = 1 - n\varepsilon$ .
- Compute the value of  $p_n \rightarrow p_{n+1}$  by means of Łukasiewicz implication.
- $\mu(n) \rightarrow \mu(n+1) = 1 - \mu(n) + \mu(n+1) = 1 - (1 - n\varepsilon) + (1 - (n+1)\varepsilon) = 1 - \varepsilon < 1$  (all these premises have the same truth value: almost completely true).

# Outline

- 1 The vagueness problem
- 2 The logic of left-continuous t-norms

- Each continuous t-norm is residuated.
- Notice that in the proof actually only left-continuity is used.

### Proposition

Let  $*$  be a t-norm. Then:

$*$  is residuated if, and only if, it is left-continuous.

## Historical remarks on t-norms

- 1930 Łukasiewicz and Tarski:  $[0, 1]$ -valued logic that uses a particular continuous t-norm to interpret (strong) conjunction.
- 1942 Menger: Statistical metrics. Triangular norms (t-norms) as functions to deal with triangular inequality.
- 1957 Mostert, Shields: Representation of continuous t-norms as ordinal sums of three basic components.
- 1959 Dummett resumes Gödel's work from 1932 and proposes a  $[0, 1]$ -valued logic that uses  $\min$  function (a continuous t-norm) to interpret conjunction.
- 1960 Schweizer and Sklar: Development of statistical metric spaces.
- 1965 Ling: Independent proof of the representation of continuous t-norms as ordinal sums of three basic components.

- 1965 Zadeh: Fuzzy Set Theory (FST) as a mathematical treatment of vagueness and imprecision. FST becomes a very popular paradigm for engineering applications, known also as *Fuzzy Logic*. He proposes min function for the intersection of fuzzy sets.
- 1969 Goguen: combined use of Zadeh's fuzzy sets and Łukasiewicz logic to solve sorites paradox.
- late 70s Several researchers in FST propose the usage of t-norms to interpret intersection of fuzzy sets and conjunction in the logics. All known residuated t-norms are continuous.
- 1995 Fodor: Nilpotent Minimum, a left-continuous non-continuous t-norm.
- 1996 Esteva, Godo, Hájek: Product logic as a  $[0, 1]$ -valued logic that uses product t-norm to interpret (strong) conjunction.

- 1998 Hájek introduces the logic BL and conjectures that it is the logic of all continuous t-norms. He shows that  $\mathcal{L}$ , G, and  $\Pi$  are axiomatic extensions of BL.
- 2000 Hájek, Cignoli, Esteva, Godo and Torrens: BL is the logic of all continuous t-norms.
- 2001 Esteva and Godo introduce the logic MTL and conjecture that it is the logic of all left-continuous t-norms. They show that the logic NM of Fodor's t-norm and BL are axiomatic extensions of MTL.
- 2002 Jenei and Montagna: MTL is the logic of all left-continuous t-norms.

$$\mathcal{L} = \{\&, \rightarrow, \wedge, \bar{0}\}$$

$$(MTL1) \quad (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$$

$$(MTL2) \quad \varphi \& \psi \rightarrow \varphi$$

$$(MTL3) \quad \varphi \& \psi \rightarrow \psi \& \varphi$$

$$(MTL4) \quad \varphi \wedge \psi \rightarrow \varphi$$

$$(MTL5) \quad \varphi \wedge \psi \rightarrow \psi \wedge \varphi$$

$$(MTL6) \quad \varphi \& (\varphi \rightarrow \psi) \rightarrow \varphi \wedge \psi$$

$$(MTL7a) \quad (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \& \psi \rightarrow \chi)$$

$$(MTL7b) \quad (\varphi \& \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$$

$$(MTL8) \quad (((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi))$$

$$(MTL9) \quad \bar{0} \rightarrow \varphi$$

Inference rule: *modus ponens*

Defined connectives:  $\varphi \vee \psi = ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$ ,  
 $\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi)$ ,  $\neg \varphi = \varphi \rightarrow \bar{0}$ ,  $\bar{1} = \neg \bar{0}$

### Theorem (Local Deduction Theorem)

*For every set of formulae  $\Gamma \cup \{\varphi, \psi\}$ , there is  $n \geq 1$  such that:*

$$\Gamma, \varphi \vdash_{\text{MTL}} \psi \text{ iff } \Gamma \vdash_{\text{MTL}} \varphi \& \dots^n \& \varphi \rightarrow \psi$$

## Definition

An algebra  $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$  is an **MTL-algebra** if

- 1  $\langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$  is a bounded lattice
- 2  $\langle A, \&^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$  is an ordered commutative monoid
- 3  $\rightarrow^{\mathcal{A}}$  is the residuum of  $\&^{\mathcal{A}}$ :
 
$$a \&^{\mathcal{A}} c \leq b \text{ iff } c \leq a \rightarrow^{\mathcal{A}} b$$
- 4  $\mathcal{A}$  satisfies the **prelinearity equation**:  $(x \rightarrow y) \vee (y \rightarrow x) \approx \bar{1}$ .

where the relation  $\leq$  is the lattice ordering. The negation operation is defined as  $\neg^{\mathcal{A}} a = a \rightarrow^{\mathcal{A}} \bar{0}^{\mathcal{A}}$ . If the order is linear, we call it *MTL-chain*.

Let  $*$  be a left-continuous t-norm and  $\Rightarrow_*$  its residuum. Then  $[0, 1]_* = \langle [0, 1], *, \Rightarrow_*, \min, \max, 0, 1 \rangle$  is an MTL-chain.

All MTL-chains over  $[0, 1]$  are of this form and are called *standard chains*.

## Theorem

*The class of all MTL-algebras,  $\text{MTL}$ , is a variety.*

Important subvarieties of  $\text{MTL}$ :

- 1  $\text{BL}$ :  $x \& (x \rightarrow y) \approx x \wedge y$  (**divisibility equation**).
- 2  $\text{MV}$ : **divisibility** +  $\neg \neg x \approx x$  (**involution equation**).
- 3  $\text{P}$ : **divisibility** +  $\neg x \vee ((x \rightarrow x \& y) \rightarrow y) \approx \bar{1}$  (**cancellativity equation**).
- 4  $\text{G}$ :  $x \approx x \& x$  (**contraction equation**).
- 5  $\text{SBL}$ : **divisibility** +  $x \wedge \neg x \approx \bar{0}$  (**pseudocomplementation equation**).
- 6  $\text{BA}$ :  $x \vee \neg x \approx \bar{1}$  (**excluded middle**).

- 1 IMTL:  $\neg\neg x \approx x$  (involution equation).
- 2 PMTL:  $\neg x \vee ((x \rightarrow x \& y) \rightarrow y) \approx \bar{1}$  (cancellativity equation).
- 3 SMTL:  $x \wedge \neg x \approx \bar{0}$  (pseudocomplementation equation).
- 4 WCMTL:  $\neg(x \& y) \vee ((x \rightarrow x \& y) \rightarrow y) \approx \bar{1}$  (weak cancellativity equation).
- 5 WNM:  $\neg(x \& y) \vee (x \wedge y \rightarrow x \& y) \approx \bar{1}$  (weak nilpotent minimum equation).
- 6 NM: involution +  $\neg(x \& y) \vee (x \wedge y \rightarrow x \& y) \approx \bar{1}$  (weak nilpotent minimum equation).
- 7  $C_n$ MTL:  $x^{n-1} \approx x^n$  ( $n$ -contraction equation).
- 8  $C_n$ IMTL: involution +  $x^{n-1} \approx x^n$  ( $n$ -contraction equation).

## Definition

Given  $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$ , we define the consequence relation  $\Gamma \models_{\text{MTL}} \varphi$  iff for every  $\mathcal{A} \in \text{MTL}$  and every  $\mathcal{A}$ -evaluation  $e$ : if  $e[\Gamma] \subseteq \{\bar{1}^{\mathcal{A}}\}$ , then  $e(\varphi) = \bar{1}^{\mathcal{A}}$ .

$\langle \mathcal{L}, \models_{\text{MTL}} \rangle$  is a finitary logic:

- 1  $\varphi \models_{\text{MTL}} \varphi$  (Reflexivity)
- 2 If  $\Gamma \subseteq \Delta$  and  $\Gamma \models_{\text{MTL}} \varphi$ , then  $\Delta \models_{\text{MTL}} \varphi$  (Monotonicity)
- 3 If  $\Gamma \models_{\text{MTL}} \varphi$  and for all  $\psi \in \Gamma$ ,  $\Delta \models_{\text{MTL}} \psi$ , then  $\Delta \models_{\text{MTL}} \varphi$  (Cut)
- 4 If  $\Gamma \models_{\text{MTL}} \varphi$  and  $\sigma$  is a substitution, then  $\sigma[\Gamma] \models_{\text{MTL}} \sigma(\varphi)$  (Structurality)
- 5 If  $\Gamma \models_{\text{MTL}} \varphi$ , then there is a finite  $\Delta \subseteq \Gamma$  such that  $\Delta \models_{\text{MTL}} \varphi$  (Finitarity)

# Algebraic completeness

## Theorem

*For every  $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$ ,  $\Gamma \vdash_{\text{MTL}} \varphi$  iff  $\Gamma \models_{\text{MTL}} \varphi$ .*

- 1 For every  $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$ ,  
 $\Gamma \vdash_{\text{MTL}} \varphi$  iff  $\{\psi \approx \bar{1} \mid \psi \in \Gamma\} \models_{\text{MTL}} \varphi \approx \bar{1}$
- 2 For every  $\Pi \cup \{\varphi \approx \psi\} \subseteq \text{Eq}_{\mathcal{L}}$ ,  
 $\Pi \models_{\text{MTL}} \varphi \approx \psi$  iff  $\{\alpha \leftrightarrow \beta \mid \alpha \approx \beta \in \Pi\} \vdash_{\text{MTL}} \varphi \leftrightarrow \psi$
- 3 For every  $\varphi \in \text{Fm}_{\mathcal{L}}$ ,  
 $\varphi \vdash_{\text{MTL}} \varphi \leftrightarrow \bar{1}$  and  $\varphi \leftrightarrow \bar{1} \vdash_{\text{MTL}} \varphi$
- 4 For every  $\varphi, \psi \in \text{Fm}_{\mathcal{L}}$ ,  
 $\varphi \approx \psi \models_{\text{MTL}} \varphi \leftrightarrow \psi \approx \bar{1}$  and  $\varphi \leftrightarrow \psi \approx \bar{1} \models_{\text{MTL}} \varphi \approx \psi$

### Translations:

- $\tau : \varphi \mapsto \varphi \approx \bar{1}$
- $\rho : \alpha \approx \beta \mapsto \alpha \leftrightarrow \beta$

MTL-algebras are the **equivalent algebraic semantics** of MTL.

# Filters

## Definition

Let  $\mathcal{A}$  be an MTL-algebra. A set  $F \subseteq A$  is a **filter** iff:

- 1  $\bar{1} \in F$
- 2 if  $a, b \in F$ , then  $a \& b \in F$
- 3 if  $a \in F$  and  $a \leq b$ , then  $b \in F$

## Definition

Let  $\mathcal{A}$  be an MTL-algebra. A set  $F \subseteq A$  is an **implicative filter** iff:

- 1  $\bar{1} \in F$
- 2 if  $a, a \rightarrow b \in F$ , then  $b \in F$

## Definition

Let  $\mathcal{A}$  be an MTL-algebra. A set  $F \subseteq A$  is a **logical filter** iff for every  $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$  for every  $e$   $\mathcal{A}$ -evaluation, if  $\Gamma \vdash_{\text{MTL}} \varphi$  and  $e[\Gamma] \subseteq F$ , then  $e(\varphi) \in F$ .

filters = implicative filters = logical filters

$\mathcal{A} \in \text{MTL}$ ,  $F \subseteq A$  filter. A congruence  $\theta$  of  $\mathcal{A}$  is **compatible** with  $F$  iff:

if  $\langle a, b \rangle \in \theta$  and  $a \in F$ , then  $b \in F$

### Proposition

Given  $\mathcal{A} \in \text{MTL}$  and a filter  $F \subseteq A$ , we define a relation:

$$\Omega_{\mathcal{A}}(F) = \{ \langle a, b \rangle \in A^2 \mid a \rightarrow b, b \rightarrow a \in F \}$$

Then  $\Omega_{\mathcal{A}}(F)$  is a congruence of  $\mathcal{A}$ ,  $F = \bar{1} / \Omega_{\mathcal{A}}(F)$ . Moreover  $\Omega_{\mathcal{A}}(F)$  is compatible with  $F$  and it is the greatest one with this property.

### Proposition

Let  $\mathcal{A}$  be an MTL-algebra and  $\theta$  a congruence of  $\mathcal{A}$ . Then  $\bar{1}/\theta$  is a filter of  $\mathcal{A}$ .

### Proposition

Let  $\mathcal{A}$  be an MTL-algebra and  $\theta$  a congruence of  $\mathcal{A}$ . Then  $\Omega_{\mathcal{A}}(\bar{1}/\theta) = \theta$ .

### Theorem

$\mathcal{A} \in \text{MTL}$ .  $\Omega_{\mathcal{A}}$  is an isomorphism between the lattice of filters and the lattice of congruences of  $\mathcal{A}$ .

## Lemma

*Given  $\mathcal{A} \in \text{MTL}$  and  $a \in A$ , let  $Fi(a)$  be the filter generated by  $a$ . Then  $Fi(a) = \{x \in A \mid a^n \leq x \text{ for some } n \geq 1\}$ .*

## Theorem

*Let  $\mathcal{A} \in \text{MTL}$ . Then:  $\mathcal{A}$  is finitely subdirectly irreducible iff it is linearly ordered (an MTL-chain).*

### Corollary

*Every MTL-algebra is representable as a subdirect product of MTL-chains.*

### Corollary

*For every  $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$ ,  $\Gamma \vdash_{\text{MTL}} \varphi$  iff  $\Gamma \models_{\{\text{MTL-chains}\}} \varphi$ .*

### Corollary

*For every  $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$ ,  $\Gamma \vdash_{\text{MTL}} \varphi$  iff  $\Gamma \models_{\{\text{countable MTL-chains}\}} \varphi$ .*

### Remark

All these properties remain valid for axiomatic extensions as well.

# Some axiomatic extensions of MTL

BL	=	MTL	+	$\varphi \wedge \psi \rightarrow \varphi \& (\varphi \rightarrow \psi)$	(Div)
$\mathcal{L}$	=	BL	+	$\neg\neg\varphi \rightarrow \varphi$	(Inv)
G	=	BL	+	$\varphi \rightarrow \varphi \& \varphi$	(C)
$\Pi$	=	BL	+	$\neg\psi \vee ((\psi \rightarrow \varphi \& \psi) \rightarrow \varphi)$	(Can)
SBL	=	BL	+	$\varphi \wedge \neg\varphi \rightarrow \bar{0}$	(PC)
IMTL	=	MTL	+	$\neg\neg\varphi \rightarrow \varphi$	(Inv)
SMTL	=	MTL	+	$\varphi \wedge \neg\varphi \rightarrow \bar{0}$	(PC)
$\Pi$ IMTL	=	MTL	+	$\neg\psi \vee ((\psi \rightarrow \varphi \& \psi) \rightarrow \varphi)$	(Can)
WCMTL	=	MTL	+	$\neg(\varphi \& \psi) \vee ((\psi \rightarrow \varphi \& \psi) \rightarrow \varphi)$	(WCan)
WNM	=	MTL	+	$(\varphi \& \psi \rightarrow \bar{0}) \vee (\varphi \wedge \psi \rightarrow \varphi \& \psi)$	(WNM)
NM	=	IMTL	+	$(\varphi \& \psi \rightarrow \bar{0}) \vee (\varphi \wedge \psi \rightarrow \varphi \& \psi)$	(WNM)
$C_n$ MTL	=	MTL	+	$\varphi^{n-1} \rightarrow \varphi^n$	( $C_n$ )
$C_n$ IMTL	=	IMTL	+	$\varphi^{n-1} \rightarrow \varphi^n$	( $C_n$ )
CPC	=	MTL	+	$\varphi \vee \neg\varphi$	(EM)

# Completion of countable MTL-chains [Jenei-Montagna-Ono]

Let  $\mathcal{A}$  be a non-trivial countable MTL-chain.

- For every  $a \in A$ ,  $\text{suc}(a)$  is either the successor of  $a$  in the order of  $\mathcal{A}$  if it exists or  $\text{suc}(a) = a$  otherwise.
- $B = \{\langle a, 1 \rangle \mid a \in A\} \cup \{\langle a, q \rangle \mid \exists a' \in A \text{ such that } a \neq a' \text{ and } \text{suc}(a') = a, q \in \mathbb{Q} \cap (0, 1)\}$ .
- Consider the lexicographical order  $\preceq$  on  $B$ .
- Define the following monoidal operation on  $B$ :

$$\langle a, q \rangle \circ \langle b, r \rangle = \begin{cases} \min_{\preceq} \{\langle a, q \rangle, \langle b, r \rangle\} & \text{if } a \&^{\mathcal{A}} b = \min\{a, b\} \\ \langle a \&^{\mathcal{A}} b, 1 \rangle & \text{otherwise.} \end{cases}$$

- The ordered monoid  $\langle A, \&^{\mathcal{A}}, \bar{1}^{\mathcal{A}}, \preceq \rangle$  is embeddable into  $\langle B, \circ, \langle \bar{1}^{\mathcal{A}}, 1 \rangle, \preceq \rangle$  by mapping every  $a \in A$  to  $\langle a, 1 \rangle$ .

## Completion of countable MTL-chains (cont.)

- $\mathcal{B} = \langle B, \circ, \langle \bar{1}^{\mathcal{A}}, 1 \rangle, \preceq \rangle$  is a densely ordered countable monoid with maximum and minimum, so it is isomorphic to a monoid  $\mathcal{B}' = \langle [0, 1]^{\mathbb{Q}}, \circ', 1, \preceq' \rangle$ . Obviously,  $\langle A, \&^{\mathcal{A}}, \bar{1}^{\mathcal{A}}, \preceq \rangle$  is also embeddable into  $\mathcal{B}'$ . Let  $h$  be such embedding. Moreover, restricted to  $h[A]$ , the residuum of  $\circ'$  exists, call it  $\Rightarrow$ , and  $h(a) \Rightarrow h(b) = h(a \rightarrow^{\mathcal{A}} b)$ .
- $\mathcal{B}'$  is completed to  $[0, 1]$  by defining:

$$\forall \alpha, \beta \in [0, 1] \quad \alpha * \beta = \sup \{ x \circ' y \mid x \leq \alpha, y \leq \beta, x, y \in [0, 1]^{\mathbb{Q}} \}.$$

- $*$  is a left-continuous t-norm, so it defines a standard MTL-algebra  $[0, 1]_*$ , and  $h : \mathcal{A} \hookrightarrow [0, 1]_*$  is an embedding.

# Strong standard completeness of MTL

## Theorem

For every  $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$ ,

$\Gamma \vdash_{\text{MTL}} \varphi$  iff  $\Gamma \models_{\{[0,1]_{*} \mid * \text{ left-continuous t-norm}\}} \varphi$ .

## Other logics

- The same construction allows to prove strong standard completeness for G, WNM, SMTL and  $C_n$ MTL.
- However, the method does not preserve divisibility, cancellation, weak cancellation and involution in general.

# Completion of countable IMTL-chains [Esteva-Godo-Gispert-Montagna]

Let  $\mathcal{A}$  be a non-trivial countable IMTL-chain.

- Define the ordered monoid  $\langle B, \circ, \langle \bar{1}^{\mathcal{A}}, 1 \rangle, \preceq \rangle$  as before.
- Modify the monoidal operation in the following way:

$$\langle a, q \rangle \otimes \langle b, r \rangle = \begin{cases} \langle \bar{0}^{\mathcal{A}}, 1 \rangle & \text{if } a = \text{suc}(\neg b), q + r \leq 1 \\ \langle a, q \rangle \circ \langle b, r \rangle & \text{otherwise.} \end{cases}$$

## Completion of countable IMTL-chains (cont.)

- As before, the ordered monoid  $\langle A, \&^{\mathcal{A}}, \bar{1}^{\mathcal{A}}, \leq \rangle$  is embeddable into  $\langle B, \otimes, \langle \bar{1}^{\mathcal{A}}, 1 \rangle, \preceq \rangle$  which is isomorphic to a monoid  $\mathcal{B}' = \langle [0, 1]^{\mathbb{Q}}, \circ', \preceq' \rangle$ . Again,  $\mathcal{B}'$  is completed to  $[0, 1]$  by defining:

$$\forall \alpha, \beta \in [0, 1] \quad \alpha * \beta = \sup \{ x \circ' y \mid x \leq \alpha, y \leq \beta, x, y \in [0, 1]^{\mathbb{Q}} \}.$$

- $*$  is a left-continuous t-norm with an involutive negation, so it defines a standard IMTL-algebra  $[0, 1]_*$ , and  $h : \mathcal{A} \hookrightarrow [0, 1]_*$  is an embedding.

# Strong standard completeness of IMTL

## Theorem

For every  $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$ ,

$\Gamma \vdash_{\text{IMTL}} \varphi$  iff  $\Gamma \models_{\{[0,1]_{*} \mid * \text{ involutive left-continuous t-norm}\}} \varphi$ .

## Other logics

- The same construction allows to prove strong standard completeness for NM and  $C_n$ IMTL.
- However, the method does not preserve divisibility, cancellation and weak cancellation in general.

## A partial embedding method [Horčík]

- Let  $\mathcal{A} = \langle A, \&, \rightarrow, \wedge, \vee, \bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$  be a  $\Pi$ IMTL-chain and  $G \subseteq A$  a finite subset.
- Let  $S$  be the submonoid of  $\mathcal{A}$  generated by  $G$ .
- By using Dickson's lemma it is proved that  $S$  is residuated and the residuum is given by:  

$$a \rightarrow b = \max\{z \in S \mid a \& z \leq b\}.$$
- $\mathcal{S} = \langle S, \&, \rightarrow, \wedge, \vee, \bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$  is a countable  $\Pi$ IMTL-chain.
- Define a new chain over the set  
 $S' = \{\langle s, r \rangle \mid s \in S \setminus \{\bar{0}^{\mathcal{A}}\}, r \in (0, 1]\} \cup \{\langle \bar{0}^{\mathcal{A}}, 1 \rangle\}$ , with the lexicographical order  $\leq_{lex}$  and the following operations:

$$\langle a, x \rangle \&' \langle b, y \rangle = \langle a \& b, xy \rangle$$

$$\langle a, x \rangle \rightarrow' \langle b, y \rangle = \begin{cases} \langle a \rightarrow b, 1 \rangle & \text{if } a \& (a \rightarrow b) < b, \\ \langle a \rightarrow b, \min\{1, y/x\} \rangle & \text{otherwise.} \end{cases}$$

## A partial embedding method (cont.)

- $S' = \langle S', \&', \rightarrow', \leq_{lex}, \langle \bar{0}^A, 1 \rangle, \langle \bar{1}^A, 1 \rangle \rangle$  is an MTL-chain and there is an embedding  $\Psi : S \rightarrow S'$  defined by  $\Psi(a) = \langle a, 1 \rangle$ . Moreover  $S'$  is cancellative.
- The set  $S'$  is order-isomorphic to the real unit interval  $[0, 1]$ , so there is a standard  $\Pi$ MTL-chain  $\mathcal{B}$  and an isomorphism  $\Phi : S' \rightarrow \mathcal{B}$ . The function  $\Phi \circ \Psi$  is a partial embedding of  $G$  into  $\mathcal{B}$ .

# Finite strong standard completeness of $\Pi$ IMTL

## Theorem

For every finite  $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$ ,

$\Gamma \vdash_{\Pi\text{IMTL}} \varphi$  iff  $\Gamma \models_{\{[0,1]_{*} | * \text{ cancellative left-continuous t-norm}\}} \varphi$ .

## Other logics

- The same construction allows to prove finite strong standard completeness for WCMTL.

Logic	$\mathcal{RC}$	$\mathcal{FSRC}$	$\mathcal{SRC}$
MTL, IMTL, SMTL, G, WNM, NM, $C_n$ MTL, $C_n$ IMTL	Yes	Yes	Yes
WCMTL, $\Pi$ MTL, BL, SBL, $\mathbb{L}$ , $\Pi$	Yes	Yes	No
CPC	No	No	No