

- A. General classes of fuzzy logics
- B. General completeness properties
- C. Distinguished semantics
- D. Open problems

PhD course on Mathematical Fuzzy Logic: 6th lesson

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Outline

- 1 General classes of fuzzy logics
- 2 General completeness properties
- 3 Distinguished semantics
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Definition

Let L be a logic in a language containing that of MTL.

- L enjoys (LDT) iff for every set of formulae $\Gamma \cup \{\varphi, \psi\}$, there is $n \geq 1$ such that: $\Gamma, \varphi \vdash_L \psi$ iff $\Gamma \vdash_L \varphi \& \dots^n \& \varphi \rightarrow \psi$.
- L enjoys (Cong) iff for each n -ary $c \in \mathcal{L}$ and each $i \leq n$,
 $\varphi \leftrightarrow \psi \vdash_L c(\chi_1, \dots, \chi_{i-1}, \varphi, \dots, \chi_n) \leftrightarrow$
 $c(\chi_1, \dots, \chi_{i-1}, \psi, \dots, \chi_n)$.

Definition

We say that a finitary logic L in a countable language is a **core fuzzy logic** if

- L expands MTL (if $\Gamma \vdash_{\text{MTL}} \varphi$, then $\Gamma \vdash_L \varphi$),
- L satisfies (Cong),
- L satisfies (LDT).

Proposition

Let L be an expansion of MTL satisfying (Cong). Then, L is a core fuzzy logic iff it is an **axiomatic expansion** of MTL.

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Definition

Let L be a core fuzzy logic and I the set of additional connectives of L . An **L-algebra** is a structure $\mathcal{A} = \langle A, \&, \rightarrow, \wedge, \vee, \bar{0}, \bar{1}, (c)_{c \in I} \rangle$ such that $\langle A, \&, \rightarrow, \wedge, \vee, \bar{0}, \bar{1} \rangle$ is an MTL-algebra and for each additional axiom φ of L the identity $\varphi \approx \bar{1}$ holds.

Proposition

Let L be a core fuzzy logic.

- L is an implicative logic in the sense of Rasiowa.
- L is a weakly implicative *fuzzy logic* in the sense of Cintula.
- L is algebraizable with the same translations as MTL.
- \mathbb{L} is an equivalent algebraic semantics of L .
- \mathbb{L} is a variety.
- Every L -algebra is representable as a subdirect product of L -chains.
- For every set of formulae $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_L \varphi$ if, and only, $\Gamma \models_{\{\mathbb{L}\text{-chains}\}} \varphi$.

All the logics we have seen so far in the course, with the exception of IPC, are core fuzzy logics.

Adding Baaz's Delta projection: the logic MTL_Δ

Add a unary connective Δ , the rule of necessitation (from φ infer $\Delta\varphi$) and the following axiom schemata:

$$(\Delta 1) \quad \Delta\varphi \vee \neg\Delta\varphi$$

$$(\Delta 2) \quad \Delta(\varphi \vee \psi) \rightarrow (\Delta\varphi \vee \Delta\psi)$$

$$(\Delta 3) \quad \Delta\varphi \rightarrow \varphi$$

$$(\Delta 4) \quad \Delta\varphi \rightarrow \Delta\Delta\varphi$$

$$(\Delta 5) \quad \Delta(\varphi \rightarrow \psi) \rightarrow (\Delta\varphi \rightarrow \Delta\psi)$$

MTL_Δ enjoys (Cong) but not (LDT).

Proposition

For each set of formulae $\Sigma \cup \{\varphi, \psi\}$ holds:

$$\Sigma, \varphi \vdash_{\text{MTL}_\Delta} \psi \text{ iff } \Sigma \vdash_{\text{MTL}_\Delta} \Delta\varphi \rightarrow \psi$$

Definition

An algebra $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}}, \Delta^{\mathcal{A}} \rangle$ is an **MTL $_{\Delta}$ -algebra** if

- $\langle A, \&^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$ is an MTL-algebra,
- $\mathcal{A} \models \alpha \approx \bar{1}$ for each $\alpha \in \{\Delta 1, \dots, \Delta 5\}$, and
- $\mathcal{A} \models \Delta(\bar{1}) \approx \bar{1}$.

Remark

Let \mathcal{A} be an MTL $_{\Delta}$ -chain. Then, $\Delta^{\mathcal{A}}(\bar{1}^{\mathcal{A}}) = \bar{1}^{\mathcal{A}}$, and for every $a \in A \setminus \{\bar{1}^{\mathcal{A}}\}$ $\Delta^{\mathcal{A}}(a) = \bar{0}^{\mathcal{A}}$.

Definition

Let L be a logic in a language containing that of MTL_{Δ} .

- L enjoys (DT_{Δ}) iff for every set of formulae $\Gamma \cup \{\varphi, \psi\}$:
 $\Gamma, \varphi \vdash_L \psi$ iff $\Gamma \vdash_L \Delta\varphi \rightarrow \psi$.

Definition

We say that a finitary logic L in a countable language is a Δ -core fuzzy logic if

- L expands MTL_{Δ} (if $\Gamma \vdash_{MTL_{\Delta}} \varphi$, then $\Gamma \vdash_L \varphi$),
- L satisfies (Cong),
- L satisfies (DT_{Δ}) .

Proposition

Let L be an expansion of MTL_{Δ} satisfying (Cong). Then, L is a Δ -core fuzzy logic iff it is an **axiomatic expansion** of MTL_{Δ} .

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Definition

Let L be a Δ -core fuzzy logic and I the set of additional connectives of L . An **L-algebra** is a structure

$\mathcal{A} = \langle A, \&, \rightarrow, \wedge, \vee, \bar{0}, \bar{1}, (c)_{c \in I} \rangle$ such that $\langle A, \&, \rightarrow, \wedge, \vee, \bar{0}, \bar{1}, \Delta \rangle$ is an MTL_{Δ} -algebra and for each additional axiom φ of L the identity $\varphi \approx \bar{1}$ holds.

Proposition

Let L be a Δ -core fuzzy logic.

- L is an implicative logic in the sense of Rasiowa.
- L is a weakly implicative *fuzzy* logic in the sense of Cintula.
- L is algebraizable with the same translations as MTL.
- \mathbb{L} is an equivalent algebraic semantics of L .
- \mathbb{L} is a variety.
- Every L -algebra is representable as a subdirect product of L -chains.
- For every set of formulae $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_L \varphi$ if, and only, $\Gamma \models_{\{\text{L-chains}\}} \varphi$.

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Usual methods for standard completeness

First method: (partial embedding method)

- Suppose that $\not\vdash_L \varphi$.
- There are an L-chain \mathcal{A} and an \mathcal{A} -evaluation e such that $e(\varphi) < \bar{1}^{\mathcal{A}}$.
- Consider the finite set $S = \{e(\psi) \mid \psi \text{ subformula of } \varphi\}$.
- S is **partially embeddable** into a standard L-chain \mathcal{B} by f .
- $f(e(\varphi)) < 1$.
- Therefore, we have a standard counterexample for φ .

In fact, we could do the same starting with $\Gamma \not\vdash_L \varphi$, for some **finite** Γ .

Second method: (embedding method)

- Suppose that $\not\vdash_L \varphi$.
- There are an L-chain \mathcal{A} and an \mathcal{A} -evaluation e such that $e(\varphi) < \bar{1}^{\mathcal{A}}$.
- Let \mathcal{B} be the countable subalgebra generated by $S = \{e(\psi) \mid \psi \text{ subformula of } \varphi\}$.
- \mathcal{B} is **embeddable** into a standard L-chain \mathcal{C} by f .
- $f(e(\varphi)) < 1$.
- Therefore, we have a standard counterexample for φ .

In fact, we could do the same starting with $\Gamma \not\vdash_L \varphi$, for an **arbitrary** Γ .

Definition

Let L be a $(\Delta-)$ core fuzzy in a language \mathcal{L} , and let \mathbb{K} be a class of L -chains. We define:

- L has the property of **strong \mathbb{K} -completeness**, **SKC** for short, when for **every set** of formulae $\Gamma \subseteq \text{Fm}_{\mathcal{L}}$ and every formula φ , $\Gamma \vdash_L \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$.
- L has the property of **finite strong \mathbb{K} -completeness**, **FSKC** for short, when for **every finite set** of formulae $\Gamma \subseteq \text{Fm}_{\mathcal{L}}$ and every formula φ , $\Gamma \vdash_L \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$.
- L has the property of **\mathbb{K} -completeness**, **KC** for short, when for every formula φ , $\vdash_L \varphi$ iff $\models_{\mathbb{K}} \varphi$.

- 1 $\models_{\mathbb{K}} \varphi$ iff $\models_{\mathbf{V}(\mathbb{K})} \varphi$.
- 2 $\Gamma \models_{\mathbb{K}} \varphi$ iff $\Gamma \models_{\mathbf{Q}(\mathbb{K})} \varphi$ for finite Γ .
- 3 $\Gamma \models_{\mathbb{K}} \varphi$ iff $\Gamma \models_{\mathbf{ISP}_{\sigma-f}(\mathbb{K})} \varphi$ for arbitrary Γ , where $\mathbf{P}_{\sigma-f}$ denotes the operator of reduced products over countably complete filters.

Theorem

Let L be $(\Delta-)$ core fuzzy logic. Then:

- 1 L has the $\mathbb{K}C$ if, and only if, $\mathbb{L} = \mathbf{V}(\mathbb{K})$.
- 2 L has the $FS\mathbb{K}C$ if, and only if, $\mathbb{L} = \mathbf{Q}(\mathbb{K})$.
- 3 L has the $S\mathbb{K}C$ if, and only if, $\mathbb{L} = \mathbf{ISP}_{\sigma-f}(\mathbb{K})$.

Theorem

Let L be a $(\Delta-)$ core fuzzy logic in a propositional language \mathcal{L} and let \mathbb{K} be a class of L -chains. Then the following are equivalent:

- (i) L has the SKC.*
- (ii) Every countable L -chain belongs to $\mathbf{IS}(\mathbb{K})$.*
- (iii) Every countable subdirectly irreducible L -chain belongs to $\mathbf{IS}(\mathbb{K})$.*

Corollary

Let L be a $(\Delta-)$ core fuzzy logic. Then, each countable L -chain is embeddable into a countable subdirectly irreducible L -chain.

Definition

Given two algebras \mathcal{A} and \mathcal{B} of the same language we say that \mathcal{A} is **partially embeddable** into \mathcal{B} when every finite partial subalgebra of \mathcal{A} is embeddable into \mathcal{B} .

Definition

We say that a class \mathbb{K} of algebras is **partially embeddable** into a class \mathbb{M} if every finite partial subalgebra of a member of \mathbb{K} is embeddable into a member of \mathbb{M} .

Theorem

Let L be a $(\Delta-)$ core fuzzy logic and let \mathbb{K} be a class of L -chains. Then the following are equivalent:

- (i) L has the $\text{FS}\mathbb{K}C$.
- (ii) Every L -chain belongs to $\text{ISP}_U(\mathbb{K})$.
and if the language is finite we can add:
- (iii) Every (subdirectly irreducible) countable L -chain is partially embeddable into \mathbb{K} .
- (iv) Every (subdirectly irreducible) L -chain is partially embeddable into \mathbb{K} .

Corollary

Let L be a $(\Delta-)$ core fuzzy logic and let \mathbb{K} be a class of L -chains such that L enjoys the $\text{FS}\mathbb{K}C$. Then L has the $\text{SP}_U(\mathbb{K})C$.

Proposition

Let L be a (Δ -)core fuzzy logic in a language \mathcal{L} and L' a conservative expansion of L . Let further \mathbb{K}' be a class of L' -chains and \mathbb{K} the class of their \mathcal{L} -reducts. Then:

- If L' enjoys the $\mathbb{K}'C$, then L enjoys the $\mathbb{K}C$.
- If L' enjoys the $F\mathbb{K}'C$, then L enjoys the $F\mathbb{K}C$.
- If L' enjoys the $S\mathbb{K}'C$, then L enjoys the $S\mathbb{K}C$.

Proposition

Let L be a core fuzzy logic. We have: L has the $\mathbb{K}C$ (resp. $FS\mathbb{K}C$) with respect to a class of L -chains \mathbb{K} if and only if L_Δ has the $\mathbb{K}_\Delta C$ (resp. $FS\mathbb{K}_\Delta C$), where \mathbb{K}_Δ is the class of Δ -expansions of chains in \mathbb{K} .

Proposition

Let L be a Δ -core fuzzy logic and \mathbb{K} a class of L -chains. Then L has the $\mathbb{K}C$ if and only if L has the $FS\mathbb{K}C$.

Corollary

Let L be a core fuzzy logic and \mathbb{K} a class of L -chains. Then L has the $FS\mathbb{K}C$ if and only if L_Δ has the $\mathbb{K}_\Delta C$.

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We use:

- the term *standard* or *real-chain* completeness, \mathcal{RC} for short, to denote \mathbb{K} -completeness when \mathbb{K} is the class of L-chains whose lattice reduct is $[0, 1]$;
- the term *rational-chain* completeness, \mathcal{QC} for short, to denote \mathbb{K} -completeness when \mathbb{K} is the class of L-chains whose lattice reduct is $[0, 1]^{\mathbb{Q}}$;
- the term *hyperreal-chain* completeness, \mathcal{R}^*C for short, to denote \mathbb{K} -completeness when \mathbb{K} is the class of L-chains whose lattice reduct is in $\mathbf{P}_U([0, 1])$;
- the term *finite-chain* completeness, \mathcal{FC} for short, to denote \mathbb{K} -completeness when \mathbb{K} is the class of finite L-chains.

- L has the **real-chain embedding property** (\mathcal{R} -E, for short) iff any countable L-chain can be embedded into a standard L-chain.
- the rational-chain, hyperreal-chain, and finite-chain embedding properties are defined accordingly (we use shorthands: \mathcal{Q} -E, \mathcal{R}^* -E, and \mathcal{F} -E).

Let L be a $(\Delta-)$ core fuzzy logic, then by \mathbb{L}_Q and $\mathbb{L}_{\mathcal{R}^*}$ we denote the classes of elements of \mathbb{L} whose lattice reduct is respectively $[0, 1]^Q$ and some ultrapower of $[0, 1]$.

Lemma

Let L be a $(\Delta-)$ core fuzzy logic. Then $\mathbf{ISP}_U(\mathbb{L}_Q) = \mathbf{IS}(\mathbb{L}_{\mathcal{R}^})$.*

Theorem

Let L be a $(\Delta-)$ core fuzzy logic. The following are equivalent:

- 1 L has the FSQC.
- 2 L has the SR^{*}C.
- 3 L has the SQC.
- 4 L has the FSR^{*}C.

Furthermore, L has the QC if and only if L has the R^{*}C.

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Theorem

Let L be a (Δ) -core fuzzy logic with \mathcal{RC} . Then L has the QC .

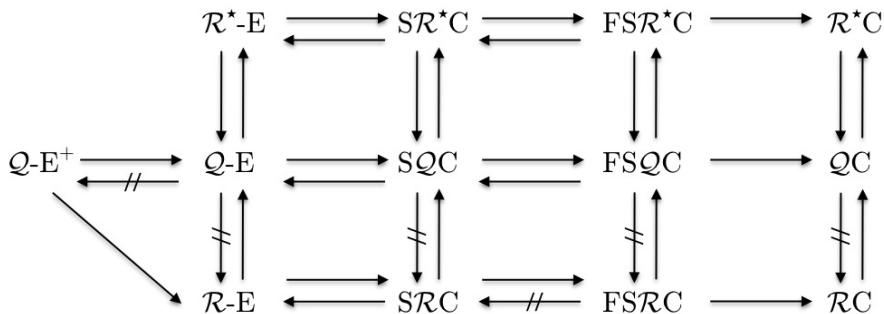
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Proposition

Let L be a (Δ -)core fuzzy logic with the $FSRC$. Then L has the SR^*C and the SQC .

Definition

Let L be an axiomatic extension of MTL. We say that L has the $\mathcal{Q}\text{-E}^+$ iff it has the $\mathcal{Q}\text{-E}$ and given a rational L -chain \mathcal{A} , the extension of \mathcal{A} to a standard chain defined in the last step of the corresponding embedding method (depending on whether \mathcal{A} is or is not involutive) is also an L -chain.



Example

Let Π^* be the axiomatic extension of BL by:

$$(\varphi \wedge \neg\varphi \rightarrow \bar{0}) \wedge ((\varphi \rightarrow \varphi \& \varphi) \rightarrow \neg\varphi \vee \varphi)$$

Π^* -chains are SBL-chains that have no idempotents different from the top and the bottom of the chain.

- (1) $[0, 1]_{\Pi}$ is the only standard Π^* -chain.
- (2) There are Π^* -chains that are not Π -chains. In fact, the chains of the variety are those obtained by removing the idempotents separating components in any ordinal sum of product chains.

Therefore, in the logic Π^* all the standard completeness properties fail, as well as the \mathcal{R} -E and the \mathcal{Q} -E⁺, but it still enjoys the \mathcal{Q} -E (and thus, SQC, FSQC and QC).

Definition

Given a class \mathbb{K} of algebras, \mathbb{K}_{fin} will denote the class of its finite members. We say that a class \mathbb{K} of algebras has:

- the **finite embeddability property** (FEP, for short) if and only if it is partially embeddable into \mathbb{K}_{fin} .
- the **strong finite model property** (SFMP, for short) if and only if every quasiequation that fails to hold in \mathbb{K} can be refuted in some member of \mathbb{K}_{fin} .
- the **finite model property** (FMP, for short) if and only if every equation that fails to hold in \mathbb{K} can be refuted in some member of \mathbb{K}_{fin} .

Theorem

Let L be a (Δ) -core fuzzy logic. Then:

- (i) L enjoys the \mathcal{FC} if and only if \mathbb{L} enjoys the FMP.*
- (ii) L enjoys the $\text{FS}\mathcal{FC}$ if and only if \mathbb{L} enjoys the SFMP.*
Moreover, if the language is finite, these properties are also equivalent to the FEP for \mathbb{L} .

Proposition

Let L be a (Δ -)core fuzzy logic. The following are equivalent:

- (i) L enjoys the $S\mathcal{F}C$,
- (ii) L enjoys the \mathcal{F} -E,
- (iii) all L -chains are finite,
- (iv) there is a natural number n such that the length of each L -chain is less or equal than n , and
- (v) there is a natural number n such that $\vdash_L \bigvee_{i < n} (x_i \rightarrow x_{i+1})$.

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Corollary

For every (Δ -)core fuzzy logic L and every natural number n , the axiomatic extension L_n obtained by adding the schema $\bigvee_{i < n} (x_i \rightarrow x_{i+1})$, is a (Δ -)core fuzzy logic which is strongly complete with respect the L -chains of length less or equal than n , and hence enjoys the SFC.

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Logic	\mathcal{FC}	$FS\mathcal{FC}$	$S\mathcal{FC}$
MTL, IMTL, SMTL	Yes	Yes	No
WCMTL, IIMTL, II	No	No	No
BL, SBL, \mathfrak{L} , G, NM, WNM, C_n MTL, C_n IMTL	Yes	Yes	No
\mathfrak{L}_n , G_n , CPC	Yes	Yes	Yes

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Open problems

For which $(\Delta-)$ core fuzzy logics do the following implications hold:

- $\mathcal{RC} \Rightarrow \text{FSRC}$
- $\mathcal{QC} \Rightarrow \text{FSQC}$
- $\mathcal{FC} \Rightarrow \text{FSFC}$