PhD course on Mathematical Fuzzy Logic: 6th lesson

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Outline

Definition

Let L be a logic in a language containing that of MTL.

- L enjoys (LDT) iff for every set of formulae $\Gamma \cup \{\varphi, \psi\}$, there is $n \geq 1$ such that: $\Gamma, \varphi \vdash_{\mathrm{L}} \psi$ iff $\Gamma \vdash_{\mathrm{L}} \varphi \& \ldots^{n} \& \varphi \to \psi$.
- L enjoys (Cong) iff for each n-ary $c \in \mathcal{L}$ and each $i \leq n$, $\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \chi_{i-1}, \varphi, \ldots, \chi_n) \leftrightarrow c(\chi_1, \ldots, \chi_{i-1}, \psi, \ldots, \chi_n)$.

Definition

We say that a finitary logic L in a countable language is a core fuzzy logic if

- L expands MTL (if $\Gamma \vdash_{\mathrm{MTL}} \varphi$, then $\Gamma \vdash_{\mathrm{L}} \varphi$),
- L satisfies (Cong),
- L satisfies (LDT).

Proposition

Let L be an expansion of MTL satisfying (Cong). Then, L is a core fuzzy logic iff it is an axiomatic expansion of MTL.

Definition

Let L be a core fuzzy logic and I the set of additional connectives of L. An L-algebra is a structure $\mathcal{A} = \langle A, \&, \rightarrow, \wedge, \vee, \overline{0}, \overline{1}, (c)_{c \in \mathrm{I}} \rangle$ such that $\langle A, \&, \rightarrow, \wedge, \vee, \overline{0}, \overline{1} \rangle$ is an MTL-algebra and for each additional axiom φ of L the identity $\varphi \approx \overline{1}$ holds.

Proposition

Let L be a core fuzzy logic.

- L is an implicative logic in the sense of Rasiowa.
- L is a weakly implicative *fuzzy* logic in the sense of Cintula.
- L is algebraizable with the same translations as MTL.
- $\mathbb{L}$ is an equivalent algebraic semantics of L.
- $\mathbb{L}$ is a variety.
- Every L-algebra is representable as a subdirect product of L-chains.
- For every set of formulae $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{\mathrm{L}} \varphi$ if, and only, $\Gamma \models_{\{\mathrm{L}\text{-chains}\}} \varphi$.

All the logics we have seen so far in the course, with the exception of IPC, are core fuzzy logics.

Adding Baaz's Delta projection: the logic MTL_Δ

Add a unary connective Δ, the rule of necessitation (from φ infer $\Delta\varphi$) and the following axiom schemata:

($\Delta 1$) $\Delta\varphi \vee \neg\Delta\varphi$

($\Delta 2$) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$

($\Delta 3$) $\Delta\varphi \to \varphi$

($\Delta 4$) $\Delta\varphi \to \Delta\Delta\varphi$

($\Delta 5$) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$

MTL_Δ enjoys (Cong) but not (LDT).

Proposition

For each set of formulae $\Sigma \cup \{\varphi, \psi\}$ holds:
$\Sigma, \varphi \vdash_{\mathrm{MTL}_\Delta} \psi$ iff $\Sigma \vdash_{\mathrm{MTL}_\Delta} \Delta\varphi \to \psi$

Definition

An algebra $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}, \Delta^{\mathcal{A}}\rangle$ is an MTL_{Δ}-algebra if

- $\langle A, \&^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\rangle$ is an MTL-algebra,
- $\mathcal{A} \models \alpha \approx \overline{1}$ for each $\alpha \in \{\Delta 1, \ldots, \Delta 5\}$, and
- $\mathcal{A} \models \Delta(\overline{1}) \approx \overline{1}$.

Remark

Let $\mathcal{A}$ be an MTL_{Δ}-chain. Then, $\Delta^{\mathcal{A}}(\overline{1}^{\mathcal{A}}) = \overline{1}^{\mathcal{A}}$, and for every $a \in A \setminus \{\overline{1}^{\mathcal{A}}\}$ $\Delta^{\mathcal{A}}(a) = \overline{0}^{\mathcal{A}}$.

Definition

Let L be a logic in a language containing that of MTL_Δ.

- L enjoys (DT$_\Delta$) iff for every set of formulae $\Gamma \cup \{\varphi, \psi\}$: $\Gamma, \varphi \vdash_{\mathrm{L}} \psi$ iff $\Gamma \vdash_{\mathrm{L}} \Delta\varphi \to \psi$.

Definition

We say that a finitary logic L in a countable language is a Δ-core fuzzy logic if

- L expands MTL_Δ (if $\Gamma \vdash_{\mathrm{MTL}_\Delta} \varphi$, then $\Gamma \vdash_{\mathrm{L}} \varphi$),
- L satisfies (Cong),
- L satisfies (DT_Δ).

Proposition

Let L be an expansion of MTL_Δ satisfying (Cong). Then, L is a Δ-core fuzzy logic iff it is an axiomatic expansion of MTL_Δ.

Definition

Let L be a Δ-core fuzzy logic and I the set of additional connectives of L. An L-algebra is a structure $\mathcal{A} = \langle A, \&, \to, \wedge, \vee, \overline{0}, \overline{1}, (c)_{c\in \mathrm{I}}\rangle$ such that $\langle A, \&, \to, \wedge, \vee, \overline{0}, \overline{1}, \Delta\rangle$ is an MTL_Δ-algebra and for each additional axiom φ of L the identity $\varphi \approx \overline{1}$ holds.

Proposition

Let L be a Δ-core fuzzy logic.

- L is an implicative logic in the sense of Rasiowa.
- L is a weakly implicative *fuzzy* logic in the sense of Cintula.
- L is algebraizable with the same translations as MTL.
- $\mathbb{L}$ is an equivalent algebraic semantics of L.
- $\mathbb{L}$ is a variety.
- Every L-algebra is representable as a subdirect product of L-chains.
- For every set of formulae $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{\mathrm{L}} \varphi$ if, and only, $\Gamma \models_{\{\mathrm{L}\text{-chains}\}} \varphi$.

Outline

Usual methods for standard completeness

First method: (partial embedding method)

- Suppose that $\nvdash_{\mathrm{L}} \varphi$.
- There are an L-chain $\mathcal{A}$ and an $\mathcal{A}$-evaluation e such that $e(\varphi) < \overline{1}^{\mathcal{A}}$.
- Consider the finite set $S = \{e(\psi) \mid \psi$ subformula of $\varphi\}$.
- S is partially embeddable into a standard L-chain $\mathcal{B}$ by f.
- $f(e(\varphi)) < 1$.
- Therefore, we have a standard counterexample for φ.

In fact, we could do the same starting with $\Gamma \nvdash_{\mathrm{L}} \varphi$, for some finite Γ.

Second method: (embedding method)

- Suppose that $\nvdash_{\mathrm{L}} \varphi$.
- There are an L-chain $\mathcal{A}$ and an $\mathcal{A}$-evaluation e such that $e(\varphi) < \overline{1}^{\mathcal{A}}$.
- Let $\mathcal{B}$ be the countable subalgebra generated by $S = \{e(\psi) \mid \psi$ subformula of $\varphi\}$.
- $\mathcal{B}$ is embeddable into a standard L-chain $\mathcal{C}$ by f.
- $f(e(\varphi)) < 1$.
- Therefore, we have a standard counterexample for φ.

In fact, we could do the same starting with $\Gamma \nvdash_{\mathrm{L}} \varphi$, for an arbitrary Γ.

Definition

Let L be a ($\Delta-$)core fuzzy in a language $\mathcal{L}$, and let $\mathbb{K}$ be a class of L-chains. We define:

- L has the property of strong $\mathbb{K}$-completeness, $\mathrm{S}\mathbb{K}\mathrm{C}$ for short, when for every set of formulae $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ and every formula φ, $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$.
- L has the property of finite strong $\mathbb{K}$-completeness, $\mathrm{FS}\mathbb{K}\mathrm{C}$ for short, when for every finite set of formulae $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ and every formula φ, $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$.
- L has the property of $\mathbb{K}$-completeness, $\mathbb{K}\mathrm{C}$ for short, when for every formula φ, $\vdash_{\mathrm{L}} \varphi$ iff $\models_{\mathbb{K}} \varphi$.

1. $\models_{\mathbb{K}} \varphi$ iff $\models_{\mathbf{V}(\mathbb{K})} \varphi$.
2. $\Gamma \models_{\mathbb{K}} \varphi$ iff $\Gamma \models_{\mathbf{Q}(\mathbb{K})} \varphi$ for finite Γ.
3. $\Gamma \models_{\mathbb{K}} \varphi$ iff $\Gamma \models_{\mathbf{ISP}_{\sigma-f}(\mathbb{K})} \varphi$ for arbitrary Γ, where $\mathbf{P}_{\sigma-f}$ denotes the operator of reduced products over countably complete filters.

Theorem

Let L *be* ($\Delta-$)*core fuzzy logic. Then:*

1. L *has the* $\mathbb{K}\mathrm{C}$ *if, and only if,* $\mathbb{L} = \mathbf{V}(\mathbb{K})$.
2. L *has the* $\mathrm{FS}\mathbb{K}\mathrm{C}$ *if, and only if,* $\mathbb{L} = \mathbf{Q}(\mathbb{K})$.
3. L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$ *if, and only if,* $\mathbb{L} = \mathbf{ISP}_{\sigma-f}(\mathbb{K})$.

Theorem

Let L *be a* ($\Delta-$)*core fuzzy logic in a propositional language* $\mathcal{L}$ *and let* $\mathbb{K}$ *be a class of* L*-chains. Then the following are equivalent:*

- (i) L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$.
- (ii) *Every countable* L*-chain belongs to* $\mathbf{IS}(\mathbb{K})$.
- (iii) *Every countable subdirectly irreducible* L*-chain belongs to* $\mathbf{IS}(\mathbb{K})$.

Corollary

Let L *be a* ($\Delta-$)*core fuzzy logic. Then, each countable* L*-chain is embeddable into a countable subdirectly irreducible* L*-chain.*

Definition

Given two algebras $\mathcal{A}$ and $\mathcal{B}$ of the same language we say that $\mathcal{A}$ is partially embeddable into $\mathcal{B}$ when every finite partial subalgebra of $\mathcal{A}$ is embeddable into $\mathcal{B}$.

Definition

We say that a class $\mathbb{K}$ of algebras is partially embeddable into a class $\mathbb{M}$ if every finite partial subalgebra of a member of $\mathbb{K}$ is embeddable into a member of $\mathbb{M}$.

Theorem

Let L *be a* $(\Delta-)$*core fuzzy logic and let* $\mathbb{K}$ *be a class of* L*-chains. Then the following are equivalent:*

(i) L *has the* $\mathrm{FS}\mathbb{K}\mathrm{C}$*.*

(ii) *Every* L*-chain belongs to* $\mathbf{ISP}_U(\mathbb{K})$*.*
and if the language is finite we can add:

(iii) *Every (subdirectly irreducible) countable* L*-chain is partially embeddable into* $\mathbb{K}$*.*

(iv) *Every (subdirectly irreducible)* L*-chain is partially embeddable into* $\mathbb{K}$*.*

Corollary

Let L *be a* $(\Delta$*-)core fuzzy logic and let* $\mathbb{K}$ *be a class of* L*-chains such that* L *enjoys the* $\mathrm{FS}\mathbb{K}\mathrm{C}$*. Then* L *has the* $\mathbf{SP}_U(\mathbb{K})\mathrm{C}$*.*

Proposition

Let L be a (Δ-)core fuzzy logic in a language $\mathcal{L}$ and L' a conservative expansion of L. Let further $\mathbb{K}'$ be a class of L'-chains and $\mathbb{K}$ the class of their $\mathcal{L}$-reducts. Then:

- If L' enjoys the $\mathbb{K}'\mathrm{C}$, then L enjoys the $\mathbb{K}\mathrm{C}$.
- If L' enjoys the $\mathrm{FS}\mathbb{K}'\mathrm{C}$, then L enjoys the $\mathrm{FS}\mathbb{K}\mathrm{C}$.
- If L' enjoys the $\mathrm{S}\mathbb{K}'\mathrm{C}$, then L enjoys the $\mathrm{S}\mathbb{K}\mathrm{C}$.

Proposition

Let L be a core fuzzy logic. We have: L has the $\mathrm{S}\mathbb{K}\mathrm{C}$ (resp. $\mathrm{FS}\mathbb{K}\mathrm{C}$) with respect to a class of L-chains $\mathbb{K}$ if and only if $\mathrm{L}_\triangle$ has the $\mathrm{S}\mathbb{K}_\triangle\mathrm{C}$ (resp. $\mathrm{FS}\mathbb{K}_\triangle\mathrm{C}$), where $\mathbb{K}_\triangle$ is the class of $\triangle$-expansions of chains in $\mathbb{K}$.

Proposition

Let L be a $\triangle$-core fuzzy logic and $\mathbb{K}$ a class of L-chains. Then L has the $\mathbb{K}\mathrm{C}$ if and only if L has the $\mathrm{FS}\mathbb{K}\mathrm{C}$.

Corollary

Let L be a core fuzzy logic and $\mathbb{K}$ a class of L-chains. Then L has the $\mathrm{FS}\mathbb{K}\mathrm{C}$ if and only if $\mathrm{L}_\triangle$ has the $\mathbb{K}_\triangle\mathrm{C}$.

Outline

1. General classes of fuzzy logics
2. General completeness properties
3. Distinguished semantics
4. Open problems

We use:

- the term *standard* or *real-chain* completeness, $\mathcal{R}\mathrm{C}$ for short, to denote $\mathbb{K}$-completeness when $\mathbb{K}$ is the class of L-chains whose lattice reduct is $[0,1]$;
- the term *rational-chain* completeness, $\mathcal{Q}\mathrm{C}$ for short, to denote $\mathbb{K}$-completeness when $\mathbb{K}$ is the class of L-chains whose lattice reduct is $[0,1]^{\mathbb{Q}}$;
- the term *hyperreal-chain* completeness, $\mathcal{R}^{\star}\mathrm{C}$ for short, to denote $\mathbb{K}$-completeness when $\mathbb{K}$ is the class of L-chains whose lattice reduct is in $\mathbf{P}_U([0,1])$;
- the term *finite-chain* completeness, $\mathcal{F}\mathrm{C}$ for short, to denote $\mathbb{K}$-completeness when $\mathbb{K}$ is the class of finite L-chains.

- L has the real-chain embedding property ($\mathcal{R}$-E, for short) iff any countable L-chain can be embedded into a standard L-chain.
- the rational-chain, hyperreal-chain, and finite-chain embedding properties are defined accordingly (we use shorthands: $\mathcal{Q}$-E, $\mathcal{R}^{\star}$-E, and $\mathcal{F}$-E).

Let L be a (Δ-)core fuzzy logic, then by $\mathbb{L}_{\mathcal{Q}}$ and $\mathbb{L}_{\mathcal{R}^\star}$ we denote the classes of elements of $\mathbb{L}$ whose lattice reduct is respectively $[0,1]^{\mathbb{Q}}$ and some ultrapower of $[0,1]$.

Lemma

Let L be a ($\Delta-$)core fuzzy logic. Then $\mathbf{ISP}_U(\mathbb{L}_{\mathcal{Q}}) = \mathbf{IS}(\mathbb{L}_{\mathcal{R}^\star})$.

Theorem

Let L be a $(\Delta-)$core fuzzy logic. The following are equivalent:

1. L *has the* $\mathrm{FS}\mathcal{Q}\mathrm{C}$.
2. L *has the* $\mathrm{S}\mathcal{R}^{\star}\mathrm{C}$.
3. L *has the* $\mathrm{S}\mathcal{Q}\mathrm{C}$.
4. L *has the* $\mathrm{FS}\mathcal{R}^{\star}\mathrm{C}$.

Furthermore, L *has the* $\mathcal{Q}\mathrm{C}$ *if and only if* L *has the* $\mathcal{R}^{\star}\mathrm{C}$.

Theorem

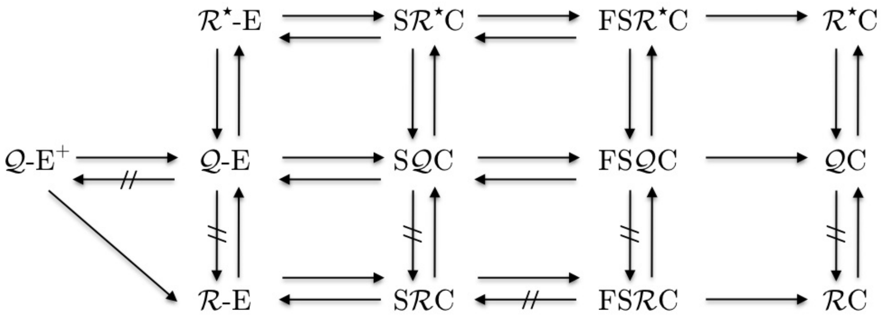
Let L *be a (*Δ*-)core fuzzy logic with* $\mathcal{R}\mathrm{C}$*. Then* L *has the* $\mathcal{Q}\mathrm{C}$*.*

Proposition

Let L be a (Δ-)core fuzzy logic with the $\mathrm{F}S\mathcal{R}\mathrm{C}$. Then L has the $S\mathcal{R}^{\star}\mathrm{C}$ and the $S\mathcal{Q}\mathrm{C}$.

Definition

Let L be an axiomatic extension of MTL. We say that L has the $\mathcal{Q}$-E^{+} iff it has the $\mathcal{Q}$-E and given a rational L-chain $\mathcal{A}$, the extension of $\mathcal{A}$ to a standard chain defined in the last step of the corresponding embedding method (depending on whether $\mathcal{A}$ is or is not involutive) is also an L-chain.


$\mathcal{R}^\star$-E
$S\mathcal{R}^\star C$
$FS\mathcal{R}^\star C$
$\mathcal{R}^\star C$
$\mathcal{Q}$-E$^+$
$\mathcal{Q}$-E
$S\mathcal{Q}C$
$FS\mathcal{Q}C$
$\mathcal{Q}C$
$\mathcal{R}$-E
$S\mathcal{R}C$
$FS\mathcal{R}C$
$\mathcal{R}C$

Example

Let $\Pi^\star$ be the axiomatic extension of BL by:

$$(\varphi \wedge \neg\varphi \to \overline{0}) \wedge ((\varphi \to \varphi \& \varphi) \to \neg\varphi \vee \varphi)$$

$\Pi^\star$-chains are SBL-chains that have no idempotents different from the top and the bottom of the chain.

(1) $[0,1]_\Pi$ is the only standard $\Pi^\star$-chain.

(2) There are $\Pi^\star$-chains that are not Π-chains. In fact, the chains of the variety are those obtained by removing the idempotents separating components in any ordinal sum of product chains.

Therefore, in the logic $\Pi^\star$ all the standard completeness properties fail, as well as the $\mathcal{R}$-E and the $\mathcal{Q}$-E$^+$, but it still enjoys the $\mathcal{Q}$-E (and thus, $\mathrm{S}\mathcal{Q}\mathrm{C}$, $\mathrm{FS}\mathcal{Q}\mathrm{C}$ and $\mathcal{Q}\mathrm{C}$).

Definition

Given a class $\mathbb{K}$ of algebras, $\mathbb{K}_{fin}$ will denote the class of its finite members. We say that a class $\mathbb{K}$ of algebras has:

- the finite embeddability property (FEP, for short) if and only if it is partially embeddable into $\mathbb{K}_{fin}$.
- the strong finite model property (SFMP, for short) if and only if every quasiequation that fails to hold in $\mathbb{K}$ can be refuted in some member of $\mathbb{K}_{fin}$.
- the finite model property (FMP, for short) if and only if every equation that fails to hold in $\mathbb{K}$ can be refuted in some member of $\mathbb{K}_{fin}$.

Theorem

Let L be a (Δ-)core fuzzy logic. Then:

(i) L *enjoys the* $\mathcal{F}\mathrm{C}$ *if and only if* $\mathbb{L}$ *enjoys the* FMP*.*

(ii) L *enjoys the* $\mathrm{FS}\mathcal{F}\mathrm{C}$ *if and only if* $\mathbb{L}$ *enjoys the* SFMP*. Moreover, if the language is finite, these properties are also equivalent to the* FEP *for* $\mathbb{L}$*.*

Proposition

Let L be a (Δ-)core fuzzy logic. The following are equivalent:

(i) L enjoys the $S\mathcal{F}C$,

(ii) L enjoys the $\mathcal{F}$-E,

(iii) all L-chains are finite,

(iv) there is a natural number n such that the length of each L-chain is less or equal than n, and

(v) there is a natural number n such that $\vdash_{\mathrm{L}} \bigvee_{i<n}(x_i \to x_{i+1})$.

Corollary

For every (Δ-)core fuzzy logic L *and every natural number* n*, the axiomatic extension* L_n *obtained by adding the schema* $\bigvee_{i<n}(x_i \to x_{i+1})$*, is a ($\Delta$-)core fuzzy logic which is strongly complete with respect the* L*-chains of length less or equal than* n*, and hence enjoys the* $\mathrm{S}\mathcal{F}\mathrm{C}$*.*

Logic	$\mathcal{F}$C	FS$\mathcal{F}$C	S$\mathcal{F}$C
MTL, IMTL, SMTL	Yes	Yes	No
WCMTL, ΠMTL, Π	No	No	No
BL, SBL, Ł, G, NM, WNM, C_nMTL, C_nIMTL	Yes	Yes	No
$Ł_n$, G_n, CPC	Yes	Yes	Yes

Outline

1. General classes of fuzzy logics
2. General completeness properties
3. Distinguished semantics
4. Open problems

Open problems

For which ($\Delta-$)core fuzzy logics do the following implications hold:

- $\mathcal{R}\mathrm{C} \Rightarrow \mathrm{FS}\mathcal{R}\mathrm{C}$
- $\mathcal{Q}\mathrm{C} \Rightarrow \mathrm{FS}\mathcal{Q}\mathrm{C}$
- $\mathcal{F}\mathrm{C} \Rightarrow \mathrm{FS}\mathcal{F}\mathrm{C}$